Logics associated with a quasi-primal algebra

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June 26, 2014

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Abstract Algebraic Logic

- A logic $\mathcal{L}$ is a substitution invariant closure operator $C_{\mathcal{L}} \colon \mathcal{P}(Fm) \to \mathcal{P}(Fm)$.
- Pick an algebra $\boldsymbol{A}$. The subsets of A closed under the rules of $\mathcal{L}$ are the filters $\mathcal{F}i_{\mathcal{L}}\boldsymbol{A}$ of $\mathcal{L}$ over $\boldsymbol{A}$.
- Pick any $F \subseteq A$. The Leibniz congruence $\boldsymbol{\Omega}^{\boldsymbol{A}}F$ is the greatest congruence on $\boldsymbol{A}$ compatible with F.
- The class of reduced models of $\mathcal{L}$ is

$$\text{Mod}^*\mathcal{L} = \{\langle \boldsymbol{A}, F\rangle : F \in \mathcal{F}i_{\mathcal{L}}\boldsymbol{A} \text{ and } \boldsymbol{\Omega}^{\boldsymbol{A}}F = \text{Id}_{\boldsymbol{A}}\}.$$

  $\mathcal{L}$ is complete w.r.t. $\text{Mod}^*\mathcal{L}$.

Example: $\text{Mod}^*\mathcal{IPC} = \{\langle \boldsymbol{A}, \{1\}\rangle : \boldsymbol{A} \text{ is an Heyting algebra}\}$.

Abstract Algebraic Logic

Fix a logic $\mathcal{L}$. Two things may happen:

- The Leibniz congruence admits a nice description.
  $\mathcal{L}$ is protoalgebraic if there is a set of formulas $\Delta(x, y, \overline{z})$ s.t. for every algebra $\boldsymbol{A}$, $F \in \mathcal{F}i_{\mathcal{L}}\boldsymbol{A}$ and $a, b \in A$:

$$\langle a, b\rangle \in \boldsymbol{\Omega}^{\boldsymbol{A}}F \Longleftrightarrow \Delta^{\boldsymbol{A}}(a, b, \overline{c}) \subseteq F \text{ for every } \overline{c} \in A.$$

  For $\mathcal{IPC}$ pick $\Delta(x, y, \overline{z}) = \{x \to y, y \to x\}$.
- Truth predicates in $\text{Mod}^*\mathcal{L}$ have a nice description.
  $\mathcal{L}$ is truth-equational if there is a set of equations $\boldsymbol{\tau}(x)$ s.t.

$$F = \{a \in A : \boldsymbol{A} \models \boldsymbol{\tau}(a)\}$$

  for every $\langle \boldsymbol{A}, F\rangle \in \text{Mod}^*\mathcal{L}$.
  For $\mathcal{IPC}$ pick $\boldsymbol{\tau}(x) = \{x \approx 1\}$.

Abstract Algebraic Logic

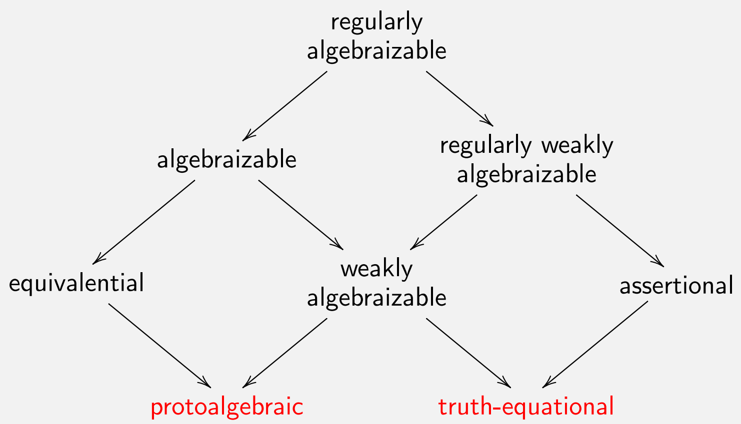

Figure : Some classes of the Leibniz hierarchy.

General problem

- What can we say about the logic of a (finite) matrix $\langle \boldsymbol{A}, F \rangle$?
- Can we classify it within the Leibniz hierarchy somehow?
  Yes, mechanically.
- Can we classify it within the Leibniz hierarchy in a nicer way?
  Yes, for $\boldsymbol{A}$ being a quasi-primal algebra.
- How do algebraizable logics of a variety V look like? Are they determined by a finite matrix?
  For varieties generated by a (finite) quasi-primal algebra.

Quasi-primal algebras

Given a set A, the ternary discriminator function on A is the map $t\colon A^3 \to A$ such that

$$t(a,b,c) = \begin{cases} a & \text{if } a \neq b \\ c & \text{otherwise} \end{cases}$$

for every $a, b, c \in A$.

Definition

An algebra $\boldsymbol{A}$ is quasi-primal if there is a term $t(x,y,z)$ which represents the ternary discriminator term on A, i.e., such that $t^{\boldsymbol{A}}(x,y,z)$ is the ternary discriminator function of A.

Protoalgebraic logics

When is a logic of $\langle \boldsymbol{A}, F \rangle$ protoalgebraic?

Lemma

Let $\boldsymbol{A}$ be a quasi-primal algebra and $\mathcal{C}$ a non-almost inconsistent closure system over A. The logic determined by $\langle \boldsymbol{A}, \mathcal{C} \rangle$ is protoalgebraic if and only if it has theorems.

Proof.

- Pick a theorem $\varphi(x)$ at most in variable x.
- Check that
  $$\Delta(x,y) := \{t(y,x,\varphi(x))\}.$$
  is a set of protoimplication formulas for $\mathcal{L}$.

□

Truth-equational logics

When is a logic of $\langle \boldsymbol{A}, F\rangle$ truth-equational?

Theorem

Let $\boldsymbol{A}$ be a quasi-primal algebra, $\boldsymbol{\tau}(x)$ a set of equations, $F \in \mathcal{P}(A) \smallsetminus \{A\}$ and $\mathcal{L}$ the logic determined by $\langle \boldsymbol{A}, F\rangle$. The following conditions are equivalent:

(i) $\boldsymbol{\tau}(x)$ defines truth in $\langle \boldsymbol{A}, F\rangle$ and $\mathcal{L}$ has theorems.

(ii) $\mathcal{L}$ is truth-equational via $\boldsymbol{\tau}(x)$.

(iii) $\mathcal{L}$ is weakly-algebraizable via $\boldsymbol{\tau}(x)$.

The equivalence of (i) and (ii) is not true in general!

A counterexample

A counterexample to direction (i)⇒(ii). Let $\boldsymbol{A} = \langle \{a, b, \top\}, \Box, \Diamond, \top\rangle$ be the algebra with unary-operations $\Box$ and $\Diamond$ defined as

$$\Box a = \Box b = b \quad \Box\top = \top$$

$$\Diamond b = \Diamond\top = \top \quad \Diamond a = b.$$

Let $\mathcal{L}$ be the logic of $\langle \boldsymbol{A}, \{a, \top\}\rangle$. We have that:

- $\mathcal{L}$ has theorems.
- $\{a, \top\}$ is equationally definable by $\{\Box x \approx \Diamond x\}$.
- $\langle \boldsymbol{A}, \{a, \top\}\rangle \in \mathrm{Mod}^{*}\mathcal{L}$.

It is possible to prove that also $\langle \boldsymbol{A}, \{\top\}\rangle \in \mathrm{Mod}^{*}\mathcal{L}$. Hence truth is not not implicitly definable in $\mathrm{Mod}^{*}\mathcal{L}$.

Truth-equational logics

Corollary

Let $\boldsymbol{A}$ be quasi-primal. The following conditions are equivalent:

(i) There is a closure system $\mathcal{C} \subseteq \mathcal{P}(A)$ s.t. $\langle \boldsymbol{A}, \mathcal{C}\rangle$ determines a non-trivial protoalgebraic logic.

(ii) There is an unary term-function $\neg^{\boldsymbol{A}} : \boldsymbol{A} \to \boldsymbol{A}$ that is not surjective.

For $\boldsymbol{A}$ finite we can strengthen (i) as:

(i') There is a closure system $\mathcal{C} \subseteq \mathcal{P}(A)$ s.t. $\langle \boldsymbol{A}, \mathcal{C}\rangle$ determines a non-trivial weakly-algebraizable logic.

Algebraizable logics

Theorem

Let $\boldsymbol{A}$ be a non-trivial finite quasi-primal algebra. The following conditions are equivalent:

(i) $\mathcal{L}$ is algebraizable with equivalent algebraic semantics $\mathbb{V}(\boldsymbol{A})$.

(ii) $\mathcal{L}$ has theorems and is the logic determined by $\langle \boldsymbol{A}, F\rangle$, for some $F \subseteq A$ such that F is equationally definable and $B \cap F \neq B$ for every non-trivial $\boldsymbol{B} \in \mathbb{S}(\boldsymbol{A})$.

Algebraizable logics

Some sufficient and necessary conditions (or a normal form for algebraizable logics of finite quasi-primal algebras, up to deductive equivalence):

Corollary

Let $\boldsymbol{A}$ finite, non-trivial and quasi-primal. TFAE:

(i) There is an algebraizable logic of $\mathbb{V}(\boldsymbol{A})$.

(ii) There is an algebraizable logic of $\mathbb{V}(\boldsymbol{A})$ with $\boldsymbol{\rho}(x, y) = \{x \leftrightarrow y\}$ and $\boldsymbol{\tau}(x) = \{x \leftrightarrow x \approx x\}$ for some term $x \leftrightarrow y$.

(iii) There is a term $x \leftrightarrow y$ s.t. $x \leftrightarrow x \colon \boldsymbol{A} \to \boldsymbol{A}$ is idempotent and non-surjective and for every $a, b \in A$

$$a \leftrightarrow b \in \{c \leftrightarrow c : c \in A\} \Longrightarrow a = b.$$

Logics of quasi-primal algebras

Summary. For the logic of a g-matrix $\langle \boldsymbol{A}, \mathcal{C} \rangle$ with $\mathcal{C} \subseteq A$ non-trivial closure system and $\boldsymbol{A}$ non-trivial, finite and quasi-primal we have:

protoalgebraic $\longleftrightarrow$ having theorems

truth-equational $\longleftrightarrow$ having theorems + $\mathcal{C} = \{F, A\}$ for some F equationally definable

algebraizable $\longleftrightarrow$ truth-equational + $F \cap B \neq B$ for every non-trivial $\boldsymbol{B} \in \mathbb{S}(\boldsymbol{A})$.

Moreover:

- truth-equational $\longleftrightarrow$ weakly-algebraizable.
- Every algebraizable logic of $\mathbb{V}(\boldsymbol{A})$ is of the kind above.

Logics of quasi-primal algebras

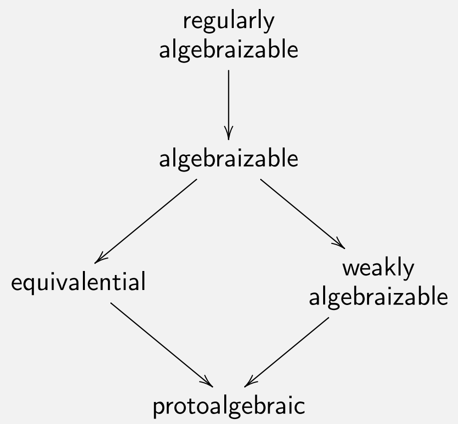

Figure : Logics of finite quasi-primal algebras.

An example

There are 6 matrices on Ł_3, which do not determine a trivial logic. The truth predicate of each of them is equationally definable as follows:

$$\langle \text{Ł}_3, \{1\} \rangle \longmapsto \{x \approx 1\}$$
$$\langle \text{Ł}_3, \{\tfrac{1}{2}\} \rangle \longmapsto \{x \oplus x \approx 1, x \odot x \approx 0\}$$
$$\langle \text{Ł}_3, \{0\} \rangle \longmapsto \{x \approx 0\}$$
$$\langle \text{Ł}_3, \{\tfrac{1}{2}, 1\} \rangle \longmapsto \{x \oplus x \approx 1\}$$
$$\langle \text{Ł}_3, \{0, \tfrac{1}{2}\} \rangle \longmapsto \{x \odot x \approx 0\}$$
$$\langle \text{Ł}_3, \{0, 1\} \rangle \longmapsto \{x \oplus x \approx x\}.$$

An example

- Each of these matrices, except $\langle Ł_3, \{\frac{1}{2}\}\rangle$, determines a truth-equational logic.
- The unique matrices which determine an algebraizable logic are $\langle Ł_3, \{1\}\rangle, \langle Ł_3, \{0\}\rangle, \langle Ł_3, \{\frac{1}{2}, 1\}\rangle$ and $\langle Ł_3, \{0, \frac{1}{2}\}\rangle$.
- These 4 matrices determine the unique algebraizable logics of $\mathbb{V}(Ł_3)$.

Primal algebras

Definition

A finite algebra $\boldsymbol{A}$ is primal if every n-ary function $f : A^n \to A$, with $n \geq 1$, can be represented by a term $\varphi(x_1, \ldots, x_n)$.

Post n-valued chains. Given $n \in \omega$, we let

$$\boldsymbol{P}_n = \langle \{0, \ldots, n-1\}, \wedge, \vee, \neg, 0, 1 \rangle$$

be the algebra where $\wedge$ and $\vee$ are the lattice operations relative to the order

$$0 < n-1 < n-2 < \cdots < 2 < 1$$

and for every $a \in P_n$

$$\neg(a) = \begin{cases} a+1 & \text{if } a \neq n-1 \\ 0 & \text{otherwise.} \end{cases}$$

Logics of $\mathbb{V}(\boldsymbol{A})$

How to associate algebras with a logic $\mathcal{L}$?
First idea:

$$\text{Mod}^*\mathcal{L} \longmapsto \text{Alg}^*\mathcal{L}.$$

This is not always a good idea: $\text{Alg}^*\mathcal{L}$ can fail to be a generalised quasi-variety also for nice logics.
New kind of models yield a nicer class $\text{Alg}\,\mathcal{L}$.

$$\text{Alg}\,\mathcal{L} = \mathbb{P}_{\text{sd}}\text{Alg}^*\mathcal{L}$$

Then, given $\boldsymbol{A}$, let

$$\boldsymbol{\mathcal{L}og}(\boldsymbol{A}) = \langle \{\mathcal{L} : \text{Alg}\,\mathcal{L} = \mathbb{V}(\boldsymbol{A})\}, \leq \rangle.$$

Logics of $\mathbb{V}(\boldsymbol{A})$

Lemma

Let $\boldsymbol{A}$ be primal algebra. $\vdash_{(\cdot)} : \mathcal{C}(A) \to \boldsymbol{\mathcal{L}og}(\boldsymbol{A})$ is a well-defined order reversing embedding.

- If $\boldsymbol{A}$ has at least three elements, there are logics of $\mathbb{V}(\boldsymbol{A})$ which are not determined by a g-matrix of the form $\langle \boldsymbol{A}, \mathcal{C} \rangle$.

Protoalgebraic logics

When is a logic of $\langle \boldsymbol{A}, F\rangle$ protoalgebraic/equivalential?

Lemma

Let $\boldsymbol{A}$ be a primal algebra and $\mathcal{C}$ a non-almost inconsistent closure system over A. The logic $\mathcal{L}$ determined by $\langle \boldsymbol{A}, \mathcal{C}\rangle$ the following conditions are equivalent:

(i) $\mathcal{L}$ finitely equivalential.

(ii) $\mathcal{L}$ protoalgebraic.

(iii) $\mathcal{L}$ has theorems.

(iv) $\emptyset \notin \mathcal{C}$.

Protoalgebraic logics

Proof.

- Enumerate $A = \{a_1, \ldots, a_n\}$.
- Assume w.l.o.g. $a_1 \in C(\emptyset)$.
- Given $1 \leq k \leq n$, let $g_k \colon A^2 \to A$ be the function defined as

$$g_k(b, c) = \begin{cases} a_1 & \text{if } b = c \\ a_k & \text{otherwise} \end{cases}$$

  for every $b, c \in A$.
- Pick a term $x \leftrightarrow_k y$ which represents g_k on $\boldsymbol{A}$.
- The set

$$\Delta(x, y) := \{x \leftrightarrow_k y : 1 \leq k \leq n\}$$

  is a set of congruence formulas for $\mathcal{L}$.

□

Algebraizable logics

Theorem

Let $\boldsymbol{A}$ be a non-trivial primal algebra. The following conditions are equivalent:

(i) $\mathcal{L}$ is algebraizable with equivalent algebraic semantics $\mathbb{V}(\boldsymbol{A})$.

(ii) $\mathcal{L}$ is maximal in $\boldsymbol{\mathcal{L}og}(\boldsymbol{A})$.

(iii) $\mathcal{L}$ is the logic determined by $\langle \boldsymbol{A}, F\rangle$, for some $F \in \mathcal{P}(A) \smallsetminus \{\emptyset, A\}$.

Corollary

Let $\boldsymbol{A}$ be a primal algebra. There are exactly $|\mathcal{P}(A)| - 2$ algebraizable logics whose equivalent algebraic semantics is $\mathbb{V}(\boldsymbol{A})$.

Logics of primal algebras

Summary. For the logic of a g-matrix $\langle \boldsymbol{A}, \mathcal{C}\rangle$ with $\mathcal{C} \subseteq A$ non-trivial closure system and $\boldsymbol{A}$ non-trivial primal, the Leibniz hierarchy reduces to:

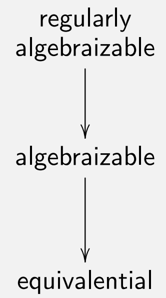

Figure : Logics of finite quasi-primal algebras.

An example

- There are exactly $2^n - 2$ algebraizable logics of $\mathbb{V}(\boldsymbol{P}_n)$.

For $n = 3$ we have that:

- There are 61 different logics determined by a g-matrix whose algebraic reduct is $\boldsymbol{P}_3$.
- 15 of these are equivalential.
- 6 of them are algebraizable and coincide with the algebraizable logics of $\mathbb{V}(\boldsymbol{P}_3)$.
- There are other equivalential logics of $\mathbb{V}(\boldsymbol{P}_3)$.

Definition

Definition

A finite algebra $\boldsymbol{A}$ is ubiquitous algebraizable if the matrix $\langle \boldsymbol{A}, F \rangle$ determines an algebraizable logic of $\mathbb{V}(\boldsymbol{A})$ for every $F \in \mathcal{P}(A) \smallsetminus \{\emptyset, A\}$.

Primal algebras are ubiquitous algebraizable. Is the converse true? Recall that:

Theorem (Foster-Pixley)

Let $\boldsymbol{A}$ be a finite algebra. $\boldsymbol{A}$ is primal if and only if it is simple, has no subalgebra except itself, its only automorphism is the identity map and generates an arithmetical variety.

Some results

This is work-in-progress. For the moment:

Theorem

Let $\boldsymbol{A}$ be a finite algebra in a congruence permutable variety. $\boldsymbol{A}$ is primal if and only if it is ubiquitous algebraizable.

and

Lemma

Let $\boldsymbol{A}$ be a two-element algebra. $\boldsymbol{A}$ is primal if and only if it is ubiquitous algebraizable.

Conclusion

Thank you!