Segmentation of an Image Using Free Distributive Lattices

Jan Pavlík

Brno University of Technology
Brno, Czech Republic

June 20, 2014

Outline

Outline

Outline

Digital image

Definition

A *digital space* is a directed graph without loops. The nodes are called *pixels* and the binary relation is called an *adjacency*.

Digital image

Definition

A *digital space* is a directed graph without loops. The nodes are called *pixels* and the binary relation is called an *adjacency*. Here we admit only symmetric adjacencies.

Digital image

Definition

A *digital space* is a directed graph without loops. The nodes are called *pixels* and the binary relation is called an *adjacency*. Here we admit only symmetric adjacencies. A *digital image* is triple (V, π, f) where (V, π) is a digital space and $f : V \to (C, \leq)$ is an assignment of *colors*. $(C, \leq)$ is a poset with the greatest element $\top$.

Digital image

Definition

A *digital space* is a directed graph without loops. The nodes are called *pixels* and the binary relation is called an *adjacency*. Here we admit only symmetric adjacencies. A *digital image* is triple (V, π, f) where (V, π) is a digital space and $f : V \to (C, \leq)$ is an assignment of *colors*. $(C, \leq)$ is a poset with the greatest element $\top$.

A paradigmatic digital space is a digitization of an Euclidean space. Each pixel represents a subset of the space and the adjacency reflects a property of being zero-distant.

Digital image

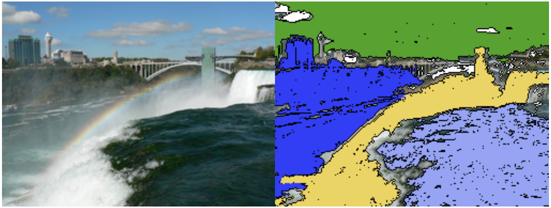
Definition

A *digital space* is a directed graph without loops. The nodes are called *pixels* and the binary relation is called an *adjacency*. Here we admit only symmetric adjacencies. A *digital image* is triple (V, π, f) where (V, π) is a digital space and $f : V \to (C, \leq)$ is an assignment of *colors*. $(C, \leq)$ is a poset with the greatest element $\top$.

A paradigmatic digital space is a digitization of an Euclidean space. Each pixel represents a subset of the space and the adjacency reflects a property of being zero-distant.The digital image is supposed to represent a distribution of a physical quantity over a real or virtual digitized space.

Image segmentation

Image segmentation aims to decompose the image into meaningful parts (called here **admissible sets**) which represent **objects** in the original space.



Thresholding

Thresholding

Given an image $\mathcal{I} = (V, \pi, f)$ and a color $c \in C$, then the set $f_c = f^{-1}(\uparrow c) = \{x \in V | f(x) \geq c\}$ is called a c-cut of $\mathcal{I}$. It yields a segmentation where each admissible set is either a singleton or a π-connected component of f_c.

Thresholding

Thresholding

Given an image $\mathcal{I} = (V, \pi, f)$ and a color $c \in C$, then the set $f_c = f^{-1}(\uparrow c) = \{x \in V | f(x) \geq c\}$ is called a c-cut of $\mathcal{I}$. It yields a segmentation where each admissible set is either a singleton or a π-connected component of f_c.

The corresponding equivalence relation is

$$(x, y) \in \rho_c \Leftrightarrow \text{exists a } \pi\text{-path } \gamma : x \rightsquigarrow_\pi y \text{ with } s(\gamma) \geq c.$$

Here $s(\gamma) = \min\{f(z) | z \in \gamma\}$ for a nontrivial path and $s(\gamma) = \top$ for a trivial path (*total connectedness of a path*).

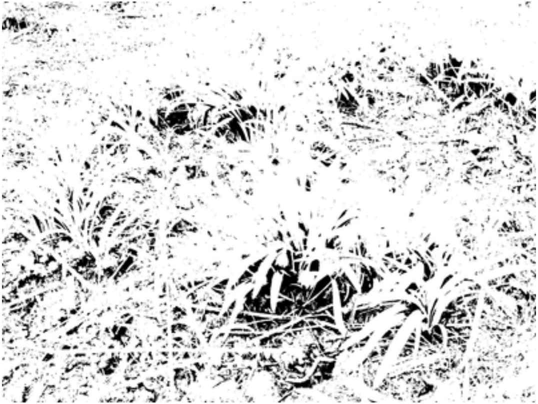
Segmentation by thresholding

One can find object within the image if the threshold is selected

some clever way.

Segmentation by thresholding

One can find object within the image if the threshold is selected



some clever way.

Segmentation by thresholding

One can find object within the image if the threshold is selected

some clever way. 

Segmentation by thresholding

One can find object within the image if the threshold is selected

some clever way.

Segmentation by thresholding

One can find object within the image if the threshold is selected

some clever way.

Segmentation by thresholding

One can find object within the image if the threshold is selected some clever way.

Segmentation by thresholding

One can find object within the image if the threshold is selected

some clever way.

Similarity criterion

A collection of quantities determining inclusion of a pixel into an object can be merged into a single mapping – *a criterion* $\xi : V^2 \to (P, \leq)$ to some (finite) poset.

Similarity criterion

A collection of quantities determining inclusion of a pixel into an object can be merged into a single mapping – *a criterion* $\xi : V^2 \to (P, \leq)$ to some (finite) poset.

Example: consider the graph distance d in the digital space and 3 channels R,G,B as functions $V \to \{0, \dots, 255\}$ (referred to as *initial quantities*), then we can take, e.g.,

$\xi(x, y) = (d(x, y), |R(x) - R(y)|, |G(x) - G(y)|, |B(x) - B(y)|)$

Since the order on $\mathbb{R}^4$ is not linear, the question arises, how to treat the criterion to get admissible sets corresponding to such situation.

Linear affinity

The method of **fuzzy affinity** by Rosenfeld improved by Carvalho, Kong and Herman is based on linearization of the set $(P, \leq)$. We suitably transform the initial quantities, so that their average gives us a function of **affinity** $\psi : V^2 \to \langle 0, 1\rangle$ which measures a *local similarity* between pixels. This function may reflect properties of object we are to find. We add requirements:

- $\psi(x, y) = \psi(y, x)$,
- $\psi(x, x) = 1$,
- $(x, y) \notin \pi \Rightarrow \psi(x, y) = 0$

Linear affinity

The method of **fuzzy affinity** by Rosenfeld improved by Carvalho, Kong and Herman is based on linearization of the set $(P, \leq)$. We suitably transform the initial quantities, so that their average gives us a function of **affinity** $\psi : V^2 \to \langle 0, 1 \rangle$ which measures a *local similarity* between pixels. This function may reflect properties of object we are to find. We add requirements:

- $\psi(x, y) = \psi(y, x)$,
- $\psi(x, x) = 1$,
- $(x, y) \notin \pi \Rightarrow \psi(x, y) = 0$

Example: in case of the previous example we may use $\psi(x, y) = \frac{3}{3+R(x,y)+R(x,y)+B(x,y)}$ if $d(x, y) = 1$.

Connectedness function

If ψ is seen as a measure of how much the pixels are held together, it makes sense to consider such a quantity along a path as a minimum of ψ of all consecutive pairs - this value expresses the connectedness of the whole path.

Connectedness function

If ψ is seen as a measure of how much the pixels are held together, it makes sense to consider such a quantity along a path as a minimum of ψ of all consecutive pairs - this value expresses the connectedness of the whole path. This idea can be used for all paths $\gamma : x \rightsquigarrow y$ hence we can define the **total connectedness** of elements x and y as

$$\mu(x, y) = \max_{\gamma : x \rightsquigarrow y} \min_{(u,v) \in S(\gamma)} \psi(u, v)$$

where $S(\gamma)$ is a set of all pairs of consecutive elements along the path γ.

Connectedness function

If ψ is seen as a measure of how much the pixels are held together, it makes sense to consider such a quantity along a path as a minimum of ψ of all consecutive pairs - this value expresses the connectedness of the whole path. This idea can be used for all paths $\gamma : x \rightsquigarrow y$ hence we can define the **total connectedness** of elements x and y as

$$\mu(x, y) = \max_{\gamma : x \rightsquigarrow y} \min_{(u,v) \in S(\gamma)} \psi(u, v)$$

where $S(\gamma)$ is a set of all pairs of consecutive elements along the path γ.
If the path γ is seen as a chain and $\psi(u, v)$ as a strength of the link (u, v), then $\mu(x, y)$ is the strength of the strongest chain connecting x and y.

Fuzzy segmentation

For each $x_0 \in V$, there is a function $\mu(x_0, -) : V \to \langle 0, 1 \rangle$ which determines a "hope" of x, that if x_0 is in the sought object then x belongs to it as well.

Fuzzy segmentation

For each $x_0 \in V$, there is a function $\mu(x_0, -) : V \to \langle 0, 1\rangle$ which determines a "hope" of x, that if x_0 is in the sought object then x belongs to it as well.
Hence we may choose a threshold $t \in \langle 0, 1\rangle$ and do a thresholding of V.

Theorem

Carvalho, Kong, Herman Given a digital image on a set V with an affinity $\psi : V^2 \to P$ satisfying the properties above, then for each $t \in \langle 0, 1\rangle$ there is a partition of V whose classes are closed under local connectedness at the level t.

Nonlinear extension of fuzzy-affinitial method

The disadvantage of the method is the linearization which necessarily loses information on incomparability. It adds new comparisons which may result in undesired sets.

Nonlinear extension of fuzzy-affinitial method

The disadvantage of the method is the linearization which necessarily loses information on incomparability. It adds new comparisons which may result in undesired sets. To overcome that, we create a new connectedness function directly from ξ. In order to do that we rewrite the essence of the method.

Essence of the fuzzy thresholding

Given an affinity and its connectedness function, then, for a pair of elements x_0, x_1, we can find a minimal object $\Omega(x_0, x_1)$ which contains both of them.

Essence of the fuzzy thresholding

Given an affinity and its connectedness function, then, for a pair of elements x_0, x_1, we can find a minimal object $\Omega(x_0, x_1)$ which contains both of them. One can see that

$$\Omega(x_0, x_1) = \{y | \mu(x_0, y) \geq \mu(x_0, x_1)\} = \mu(x_0, -)_{\mu(x_0,x_1)}.$$

Thus this set contains all elements y which are connected to x_0 at least as strongly as x_0 to x_1. Ternary relation of *tightness*

$$\Phi = \{(x_0, x_1, y) | y \in \Omega(x_0, x_1)\}$$

captures the whole essence of the afinitial method.

Derivation of the essence for a general case

$$(x_0, x_1, y) \in \Phi$$

Derivation of the essence for a general case

$$(x_0, x_1, y) \in \Phi$$

$$\Leftrightarrow \quad y \in \Omega(x_0, x_1)$$

Derivation of the essence for a general case

$$
\begin{aligned}
& (x_0, x_1, y) \in \Phi \\
\Leftrightarrow \quad & y \in \Omega(x_0, x_1) \\
\Leftrightarrow \quad & \mu(x_0, y) \geq \mu(x_0, x_1)
\end{aligned}
$$

Derivation of the essence for a general case

$$
\begin{aligned}
& (x_0, x_1, y) \in \Phi \\
\Leftrightarrow \quad & y \in \Omega(x_0, x_1) \\
\Leftrightarrow \quad & \mu(x_0, y) \geq \mu(x_0, x_1) \\
\Leftrightarrow \quad & \max_{\gamma : x_0 \rightsquigarrow y} \min_{(u,v) \in S(\gamma)} \psi(u, v) \geq \max_{\delta : x_0 \rightsquigarrow x_1} \min_{(p,q) \in S(\delta)} \psi(p, q)
\end{aligned}
$$

Derivation of the essence for a general case

$$\begin{aligned}
& (x_0, x_1, y) \in \Phi \\
\Leftrightarrow \quad & y \in \Omega(x_0, x_1) \\
\Leftrightarrow \quad & \mu(x_0, y) \geq \mu(x_0, x_1) \\
\Leftrightarrow \quad & \max_{\gamma : x_0 \rightsquigarrow y} \min_{(u,v) \in S(\gamma)} \psi(u, v) \geq \max_{\delta : x_0 \rightsquigarrow x_1} \min_{(p,q) \in S(\delta)} \psi(p, q) \\
\Leftrightarrow \quad & \forall \delta \in Q(x_0, x_1) \exists \gamma \in Q(x_0, y) \forall (u, v) \in S(\gamma) \exists (p, q) \in S(\delta) \\
& \qquad \psi(u, v) \geq \psi(p, q)
\end{aligned}$$

where $Q(x, z)$ denotes the set of all paths $x \rightsquigarrow z$ in the complete graph on V (i.e. arbitrary finite injective sequences with given endpoints).

Derivation of the essence for a general case

$$
\begin{aligned}
& (x_0, x_1, y) \in \Phi \\
\Leftrightarrow \quad & y \in \Omega(x_0, x_1) \\
\Leftrightarrow \quad & \mu(x_0, y) \geq \mu(x_0, x_1) \\
\Leftrightarrow \quad & \max_{\gamma : x_0 \rightsquigarrow y} \min_{(u,v) \in S(\gamma)} \psi(u, v) \geq \max_{\delta : x_0 \rightsquigarrow x_1} \min_{(p,q) \in S(\delta)} \psi(p, q) \\
\Leftrightarrow \quad & \forall \delta \in Q(x_0, x_1) \exists \gamma \in Q(x_0, y) \forall (u, v) \in S(\gamma) \exists (p, q) \in S(\delta) \\
& \psi(u, v) \geq \psi(p, q)
\end{aligned}
$$

where $Q(x, z)$ denotes the set of all paths $x \rightsquigarrow z$ in the complete graph on V (i.e. arbitrary finite injective sequences with given endpoints). We have expressed Φ without the use of linearity and ψ can be replaced by ξ.

Function of general connectedness

The obtained condition can be rewriten in terms of upper sets as follows:

$$\uparrow \{\uparrow \{\xi(p,q) | (p,q) \in S(\gamma)\} | \gamma : x_0 \rightsquigarrow y\} \subseteq$$

$$\subseteq \uparrow \{\uparrow \{\xi(u,v) | (u,v) \in S(\delta)\} | \delta : x_0 \rightsquigarrow x_1\}$$

Function of general connectedness

The obtained condition can be rewriten in terms of upper sets as follows:

$$\uparrow\{\uparrow\{\xi(p,q)|(p,q)\in S(\gamma)\}|\gamma : x_0 \rightsquigarrow y\}\subseteq$$

$$\subseteq\uparrow\{\uparrow\{\xi(u,v)|(u,v)\in S(\delta)\}|\delta : x_0 \rightsquigarrow x_1\}$$

This enables to define the **total connectedness** of general elements $x, z \in V$ as
$\kappa(x,z)=\uparrow\{\uparrow\{\xi(p,q)|(p,q)\in S(\gamma)\}|\gamma : x \rightsquigarrow z\}$ which yields

$$(x_0, x_1, y) \in \Phi \Leftrightarrow \kappa(x_0, y) \geq \kappa(x_0, x_1).$$

Lattice of upper sets

As we see, in order to describe the resulting entity κ we need double process of creation of upper sets.

Lattice of upper sets

As we see, in order to describe the resulting entity κ we need double process of creation of upper sets.
Generally, given a poset $(S, \leq)$, let $\mathcal{U}(S, \leq)$ denotes the set of all nonempty upper sets of S.

Lattice of upper sets

As we see, in order to describe the resulting entity κ we need double process of creation of upper sets.
Generally, given a poset $(S, \leq)$, let $\mathcal{U}(S, \leq)$ denotes the set of all nonempty upper sets of S. We may repeat this procedure again to obtain a poset $\mathcal{W}(S, \leq) = \mathcal{U}(\mathcal{U}(S, \leq)$. One can derive, from the properties of $\mathcal{U}$, that $\mathcal{W}(S, \leq)$ is distributive lattice.

Lattice of upper sets

As we see, in order to describe the resulting entity κ we need double process of creation of upper sets.
Generally, given a poset $(S, \leq)$, let $\mathcal{U}(S, \leq)$ denotes the set of all nonempty upper sets of S. We may repeat this procedure again to obtain a poset $\mathcal{W}(S, \leq) = \mathcal{U}(\mathcal{U}(S, \leq)$. One can derive, from the properties of $\mathcal{U}$, that $\mathcal{W}(S, \leq)$ is distributive lattice. Moreover we have isotone injection $\eta : (S, \leq) \to \mathcal{W}(S, \leq)$ given by composition of two antitone injections. It has a universal property which makes $\mathcal{W}(S, \leq)$ a **free distributive lattice over poset** $(S, \leq)$.

Free distributive lattice over a poset

Consider the categories $\mathcal{DL}at$ and $\mathcal{P}os$ of distributive lattices and posets, respectively. The obvious forgetful functor $Z : \mathcal{DL}at \to \mathcal{P}os$ has a left adjoint $\mathcal{W} : \mathcal{P}os \to \mathcal{DL}at$, i.e., for every poset $(S, \leq)$ there exists a distributive lattice $\mathcal{W}(S, \leq)$ and isotone mapping $\eta : (S, \leq) \to Z\mathcal{W}(S, \leq)$ such that for every Q and every $\phi : (S, \leq) \to ZQ$ there exists a unique lattice homomorphism $\widetilde{\phi} : \mathcal{W}(S, \leq) \to Q$ such that $\phi = Z\widetilde{\phi} \circ \eta$. The situation is depicted by:

$$\begin{array}{cc}
\mathcal{P}os & \mathcal{DL}at \\
\begin{tikzcd}
(S, \leq) \arrow[r, "\eta"] \arrow[dr, "\forall_2 \phi"'] & Z\mathcal{W}(S, \leq) \arrow[d, dashed, "Z\widetilde{\phi}"] \\
 & Z(Q)
\end{tikzcd}
&
\begin{tikzcd}
\mathcal{W}(S, \leq) \arrow[r, dashed, "\exists_3 \widetilde{\phi}"] & \forall_1 Q
\end{tikzcd}
\end{array}$$

Thresholding by free terms

We return to the poset $(P, \leq)$ and the derived assignment κ of total connectedness. We obtain a mapping

$$\kappa : V^2 \to \mathcal{W}(P, \leq).$$

Since the elements of $\mathcal{W}(P, \leq)$ can be seen as terms, we can write

$$\kappa(x, y) = \bigvee_{\gamma : x \rightsquigarrow y} \ \bigwedge_{(u,v) \in S(\gamma)} \xi(u, v).$$

Thresholding by free terms

We return to the poset $(P, \leq)$ and the derived assignment κ of total connectedness. We obtain a mapping

$$\kappa : V^2 \to \mathcal{W}(P, \leq).$$

Since the elements of $\mathcal{W}(P, \leq)$ can be seen as terms, we can write

$$\kappa(x, y) = \bigvee_{\gamma : x \rightsquigarrow y} \bigwedge_{(u,v) \in S(\gamma)} \xi(u, v).$$

In order to apply this for thresholding, it is advantageous to employ the theory of L-fuzzy relations.

L-fuzzy equivalence

Given a complete distributive lattice L, then a mapping $\rho : V^2 \to L$ can be seen as *binary L-fuzzy relation* on L. It is *reflexive* if $\rho(x,x) = \top$ and *transitive* if $\rho(x,y) \vee \rho(y,z) \leq \rho(x,z)$. A reflexive, symmetric, transitive L-fuzzy relation is called **L-fuzzy equivalence**.

L-fuzzy equivalence

Given a complete distributive lattice L, then a mapping $\rho : V^2 \to L$ can be seen as *binary L-fuzzy relation* on L. It is *reflexive* if $\rho(x,x) = \top$ and *transitive* if $\rho(x,y) \vee \rho(y,z) \leq \rho(x,z)$. A reflexive, symmetric, transitive L-fuzzy relation is called **L-fuzzy equivalence**.

Properties of L-fuzzy equivalence

- Every reflexive symmetric L-fuzzy relation ρ generates an L-fuzzy equivalence $\overline{\rho}(x,y) = \bigvee_{\gamma: x \rightsquigarrow y} \bigwedge_{(u,v)\in S(\gamma)} \rho(u,v)$.

L-fuzzy equivalence

Given a complete distributive lattice L, then a mapping $\rho : V^2 \to L$ can be seen as *binary L-fuzzy relation* on L. It is *reflexive* if $\rho(x,x) = \top$ and *transitive* if $\rho(x,y) \vee \rho(y,z) \leq \rho(x,z)$. A reflexive, symmetric, transitive L-fuzzy relation is called **L-fuzzy equivalence**.

Properties of L-fuzzy equivalence

- Every reflexive symmetric L-fuzzy relation ρ generates an L-fuzzy equivalence $\overline{\rho}(x,y) = \bigvee_{\gamma:x\rightsquigarrow y} \bigwedge_{(u,v)\in S(\gamma)} \rho(u,v)$.
- Every L-fuzzy equivalence σ gives rise, for every threshold $t \in L$, an equivalence relation $\sigma_t = \{(x,y) \in V^2 | \sigma_t(x,y) \geq t\}$

L-fuzzy equivalence

Given a complete distributive lattice L, then a mapping $\rho : V^2 \to L$ can be seen as *binary L-fuzzy relation* on L. It is *reflexive* if $\rho(x,x) = \top$ and *transitive* if $\rho(x,y) \vee \rho(y,z) \leq \rho(x,z)$. A reflexive, symmetric, transitive L-fuzzy relation is called **L-fuzzy equivalence**.

Properties of L-fuzzy equivalence

- Every reflexive symmetric L-fuzzy relation ρ generates an L-fuzzy equivalence $\overline{\rho}(x,y) = \bigvee_{\gamma : x \rightsquigarrow y} \bigwedge_{(u,v) \in S(\gamma)} \rho(u,v)$.
- Every L-fuzzy equivalence σ gives rise, for every threshold $t \in L$, an equivalence relation $\sigma_t = \{(x,y) \in V^2 | \sigma_t(x,y) \geq t\}$
- Every L-fuzzy equivalence σ induces a collection of L-fuzzy sets (an L-fuzzy partition of V) whose cuts, for any $t \in L$ are partitions of V.

Collective similarity

Given $t \in P$, let $C(t)$ be the relation on V of "being similar at least on the level t" containing all pair (x, y) such that $\xi(x, y) \geq t$.

Collective similarity

Given $t \in P$, let $C(t)$ be the relation on V of "being similar at least on the level t" containing all pair (x, y) such that $\xi(x, y) \geq t$. Given a term $\tau = \bigvee_{i \in \{1,\ldots,n\}} \bigwedge_{j \in \{1,\ldots,m_i\}} s_{i,j} \in \mathcal{W}(P, \leq)$ then for each i there is a set $E_i = \bigcup_{j \in \{1,\ldots,m_i\}} C(s_{i,j})$. Then $\kappa(x, y) \geq \tau$ (written as $(x, y) \in C(\tau)$) iff x and y are connected by a path in graph (V, E_i) for each i. The relation $(x, y) \in C(\tau)$ now means "x is connected to y at on the collective level τ".

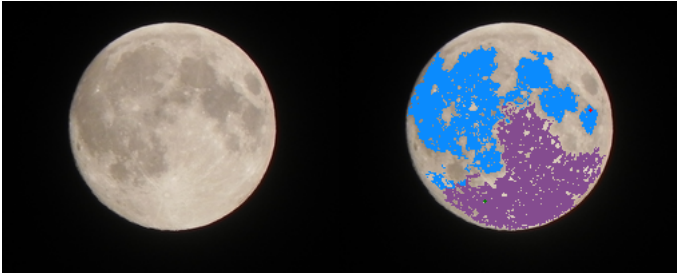
L-fuzzy segmentation

Now we see that κ is exactly the L-fuzzy equivalence generated by the criterion ξ. Then $C(\tau)$ is a cut of κ by a threshold $\tau \in \mathcal{W}(P, \leq)$, thus it is an equivalence relation. Hence κ produces an L-partition and consequently a partition of the image. Its classes are of the form $\theta(x, \tau) = \{y \in V | \kappa(x, y) \geq \tau\}$, which are classes of collective similarity $C(\tau)$.

Theorem

Given a digital image on a set V with a criterion $\xi : V^2 \to P$ satisfying the properties above, then for each term $\tau \in \mathcal{W}(P, \leq)$ there is a partition of V whose classes are closed under connectivity at least at the collective level τ.

Example



Here the criterion $\xi : V^2 \to \mathbb{Z}^2$ consists of a pair of distance and brightness with reversed order. The threshold is here $((1,3) \wedge (2,1)) \vee ((1,3) \wedge (3,0)) \vee (2,2) \vee ((3,1) \wedge (4,0)) \vee ((1,1) \wedge (6,0))$.

Acknowledgement

The author acknowledges support by the project CZ.1.07/2.3.00/30.0005 of Brno University of Technology.

THANK YOU FOR YOUR ATTENTION.