ON THE UNIVERSAL ALGEBRAS WITH IDENTICAL DERIVED STRUCTURES

A.G.Pinus

The derived structures of universal and classical algebras are some effective instruments for study of structures and for the classifications of these algebras. Usually derived structures are some classical algebras (groups, semigroups, lattices and so other) which are naturally constructed from initial algebras and such that they give some essential information on these algebras. The extreme situation is as following: for some class $\mathfrak{K}$ for any algebra $\mathfrak{A} \in \mathfrak{K}$ some derived structure $S(\mathfrak{A})$ determines a unique algebra $\mathfrak{A}$ in this class. That means that for $\mathfrak{A}_0, \mathfrak{A}_1 \in \mathfrak{K}$ if the structures $S(\mathfrak{A}_0)$ and $S(\mathfrak{A}_1)$ are isomorphic, then so are the algebras $\mathfrak{A}_0$ and $\mathfrak{A}_1$.

The role of the isomorphism relation for the algebras of some different signatures plays the relation of *rational equivalence* of algebras (which was introduced by A.I.Malcev). Let the algebras $\mathfrak{A}_0 =< A; \sigma_0 >$, $\mathfrak{A}_1 =< A; \sigma_1 >$ have the identical basic sets. Then these algebras are rationally equivalent if the signature functions of the algebra $\mathfrak{A}_i$ are the termal functions of the algebra $\mathfrak{A}_{i-1}$ (for $i = 0, 1$).

Let $\mathfrak{A} =< A; \sigma >$ be a universal algebra. By $Tr\mathfrak{A}$ denote the collection of all termal functions of the algebra $\mathfrak{A}$. Remind that the Galois-closure $LocF$ of a functional clone F on a set A is the collection of functions $\{f | f : A^m \to A$ and for any finite $B \subseteq A^m$ there exists $g \in F$ such that $f \upharpoonright B = g \upharpoonright B, m \in \omega\}$. Let F be functional clone on set A.By $InvF$ denote the collection of all relations on A, such that these relations are stable relatively the functions from F.

The algebras $\mathfrak{A}_0 =< A; \sigma_0 >$ and $\mathfrak{A}_1 =< A; \sigma_1 >$ are *locally rationally equivalent* if rationally equivalent are its Galois enrichments $\mathfrak{A}'_0 =< A; LocTr\mathfrak{A}_0 >$ and $\mathfrak{A}'_1 =< A; LocTr\mathfrak{A}_1 >$.

It is well known that the model uniquely defining the Galois enrichment $\mathfrak{A}' =< A; LocTr\mathfrak{A} >$ of an algebra $\mathfrak{A}$ up to the rational equivalence (and the algebra $\mathfrak{A}$ itself up to the local rational equivalence) is the model $< A; InvTr\mathfrak{A} >$.

The family $InvTr\mathfrak{A}$ of relations contains all relations in $Sub\mathfrak{A}$, $Con\mathfrak{A}$, $Tol\mathfrak{A}$ and the graphs of the maps in $Aut\mathfrak{A}$, $End\mathfrak{A}$, $Iso\mathfrak{A}$ and so on. But this model is not a classical algebras (such as group, a semigroup, a lattice and so on) relatively to the natural operations defined on it.

When we consider some relations or maps on A instead the model $< A; InvTr\mathfrak{A} >$ we obtain some new equivalence relations among algebras with the identical basic sets.

Let L be a some logical language. Then a *function* $f(x_1,...,x_n)$ on the basic set A of the algebra $\mathfrak{A} =< A;\sigma >$ is *L-definable on* $\mathfrak{A}$ if there exists an L-formula $\Phi(x_1,...,x_n,y)$ of the signature σ such that for any $a_1,...,a_n,b \in A$.

$$f(a_1,...,a_n) = b \Leftrightarrow \mathfrak{A} \models \Phi(a_1,...,a_n,b).$$

Let a function f be L-definable on an algebra $\mathfrak{A}$. Then f commutes with any automorphism of the algebra $\mathfrak{A}$. By $StabAut\mathfrak{A}$ denote the collection of all functions on $\mathfrak{A}$ such that these functions commute with any automorphism of the algebra $\mathfrak{A}$, and by $L-Def\mathfrak{A}$ denote the collection of all L-definable functions on the algebra $\mathfrak{A}$.

From the Scott-theorem on countable categoricity of $L_{\omega_1\omega}$-formulas we have

THEOREM 1. Let $\mathfrak{A}$ be countable (finite) algebra with not more than countable signature. Then we have the equality $L_{\omega_1\omega} - Def\mathfrak{A} = StabAut\mathfrak{A}$ ($L_{\omega\omega} - Def\mathfrak{A} = StabAut\mathfrak{A}$).

So we have the following

COROLLARY 1. Let $\mathfrak{A}_0 =< A; \sigma_0 >$, $\mathfrak{A}_1 =< A; \sigma_1 >$ are countable (finite) algebras, such that their signatures not more that countable and their basic sets ure coincide. Then the following conditions are equivalent: 1)$Aut\mathfrak{A}_0 = Aut\mathfrak{A}_1$; 2)all signature functions of the algebra $\mathfrak{A}_i$ are $L_{\omega_1\omega}$-definable ($L_{\omega\omega}$-definable) functions for the algebra $\mathfrak{A}_{1-i}$ (for $i = 0, 1$).

By an *explicit L-schema of the signature* σ (a *disjunctive explicit L-schema*) we denote some L-formula

(*) $\underset{i\in I}{\&}(\Phi_i(x_1,...,x_n) \rightarrow y = t_i(x_1,...,x_n))$,

here t_i are some terms of the signature σ, if the formula

$$\forall x_1,...,x_n(\bigvee_{i\in I} \Phi(x_1,...,x_n))$$

true on for $\mathfrak{A}$ and for any $i \neq j$ for I are true the formulas

$$\forall x_1,...,x_n(\Phi_i(x_1,...,x_n) \,\&\, \Phi_j(x_1,...,x_n) \rightarrow t_i(x_1,...,x_n) = t_j(x_1,...,x_n)).$$
$$(\forall x_1,...,x_n,x,y(\Phi_i(x_1,...,x_n) \,\&\, \Phi_j(x_1,...,x_n) \rightarrow x = y)).$$

A function $f(x_1,...,x_n)$ on the algebra $\mathfrak{A} =< A;\sigma >$ is called as *explicit L-definable*, if its is L-definable on $\mathfrak{A}$ by some explicit L-schema. By $EST_L\mathfrak{A}$ we denote the collection of all explicit L - definable functions on $\mathfrak{A}$. By $StabSub\mathfrak{A}$ denote the collection of all functions on $\mathfrak{A}$ such that all subalgebras of the algebra $\mathfrak{A}$ are closed relatively these functions.

THEOREM 2. Let $\mathfrak{A}$ be countable (finite) algebra having not more that countable signature. Then the following equalities hold.

$$EST_{L_{\omega_1\omega}} = StabAut\mathfrak{A} \bigcap StabSub\mathfrak{A}.\ (EST_{L_{\omega\omega}} = StabAut\mathfrak{A} \bigcap StabSub\mathfrak{A}).$$

COROLLARY 2. Let $\mathfrak{A}_0 =< A;\sigma_0 >$, $\mathfrak{A}_1 =< A;\sigma_1 >$ are countable (finite) algebras having not more that countable signatures and common basic sets. Then the following conditions are equivalent:

1)$Aut\mathfrak{A}_0 = Aut\mathfrak{A}_1$, $Sub\mathfrak{A}_0 = Sub\mathfrak{A}_1$;

2)all signature functions of the algebra $\mathfrak{A}_i$ are explicit $L_{\omega_1\omega}$-definable (explicit $L_{\omega\omega}$-definable) functions for the algebra $\mathfrak{A}_{1-i}$ (for $i = 0, 1$).

Some explicit L-schema $(*)$ is called as *positive explicit L-schema* if the formulas Φ_i are positive (that is – not contain the symbols $\rightarrow$, $\neg$). Denote by $PST_L\mathfrak{A}$ collection of all functions which are defined on the algebra $\mathfrak{A}$ by some positive explicit L-schema. By $StabEpi\mathfrak{A}$ we denote the collection of all functions on $\mathfrak{A}$ such that each commutes with all endomorphisms of the algebra $\mathfrak{A}$ on its itself.

THEOREM 3. Let $\mathfrak{A}$ be a countable algebra having not more that countable signature. Then the following equality holds

$$PST_{L_{\omega_1\omega}} = StabEpi\mathfrak{A} \bigcap StabSub\mathfrak{A}.$$

COROLLARY 3. Let $\mathfrak{A} =< A; \sigma_0 >$, $\mathfrak{A} =< A; \sigma_1 >$ are countable algebras having not more that countable signatures and common basic sets. Then the following conditions are equivalent:
1)$Epi\mathfrak{A}_0 = Epi\mathfrak{A}_1$ and $Sub\mathfrak{A}_0 = Sub\mathfrak{A}_1$;
2)all signature functions of the algebra $\mathfrak{A}_i$ are positive explicit $L_{\omega_1\omega}$-definable functions for the algebra $\mathfrak{A}_{1-i}$ (for $i = 0, 1$).

An explicit (positive explicit) L-schema $(*)$ we define as *explicit (positive explicit) congruence* L-schema on $\mathfrak{A}$ if the following formulas are true the on $\mathfrak{A}$

$$\forall x_1, ..., x_n, x'_k(\Phi_i(x_1, ..., x_k, ..., x_n) \,\&\, \Phi_j(x_1, ..., x'_k, ..., x_n) \rightarrow \theta(t_i(x_1, ..., x_k, ..., x_n), t_j(x_1, ..., x'_k, ..., x_n), x_k, x'_k)).$$

for any $i, j \in I$ and $1 \leqslant k \leqslant n$. Here $\theta(x, y, u, v)$ is the $L_{\omega_1\omega}$-formula such that for any $a, b, c, d \in \mathfrak{A}$

$$\mathfrak{A} \models \theta(a, b, c, d) \Leftrightarrow < a, b > \in \theta^{\mathfrak{A}}_{c,d}.$$

and $\theta^{\mathfrak{A}}_{c,d}$ is the principal congruence on the algebra $\mathfrak{A}$ such that this congruence is generated by the pair $< c, d >$.

Denote by $ESCT_L\mathfrak{A}$ ($PSCT_L\mathfrak{A}$) the collection of all functions on the algebra $\mathfrak{A}$ that are defined on $\mathfrak{A}$ by some explicit (positive explicit) congruence L-schema.

THEOREM 4. Let $\mathfrak{A}$ be a countable algebra having not more that countable signature. Then the following equalities hold

a)$ESCT_{L_{\omega_1\omega}}\mathfrak{A} = StabAut\mathfrak{A} \bigcap StabSub\mathfrak{A} \bigcap StabCon\mathfrak{A}$;

b)$PSCT_{L_{\omega_1\omega}}\mathfrak{A} = StabEpi\mathfrak{A} \bigcap StabSub\mathfrak{A} \bigcap StabCon\mathfrak{A}$.

COROLLARY 4.

a)Let $\mathfrak{A}_0 =< A; \sigma_0 >$, $\mathfrak{A}_1 =< A; \sigma_1 >$ are countable algebras having not more that countable signatures and common basic sets. Then the following conditions are equivalent:

1)$Aut\mathfrak{A}_0 = Aut\mathfrak{A}_1$, $Sub\mathfrak{A}_0 = Sub\mathfrak{A}_1$, $Con\mathfrak{A}_0 = Con\mathfrak{A}_1$;

2)all signature functions of the algebra $\mathfrak{A}_i$ are explicit $L_{\omega_1\omega}$-definable congruence functions for the algebra $\mathfrak{A}_{1-i}$ (for $i = 0, 1$).

b)Let $\mathfrak{A}_0 =< A; \sigma_0 >$, $\mathfrak{A}_1 =< A; \sigma_1 >$ are countable algebras having not more that countable signatures and common basic sets. Then the following conditions are equivalent:

1)$Epi\mathfrak{A}_0 = Epi\mathfrak{A}_1$, $Sub\mathfrak{A}_0 = Sub\mathfrak{A}_1$, $Con\mathfrak{A}_0 = Con\mathfrak{A}_1$;

2)all signature functions of the algebra $\mathfrak{A}_i$ are $PST_{L_{\omega_1\omega}}$-functions for the algebra $\mathfrak{A}_{1-i}$ (for $i = 0, 1$).

Is cold a *conditional (positive conditional) term* for the algebra any explicit (positive explicit) $L_{\omega\omega}$-schema $(*)$ if the set I is finite and Φ_i are quantifier free and disjunctive (are positive quantifier free) $L_{\omega\omega}$-formulas. Denote by $CT\mathfrak{A}$ $(PCT\mathfrak{A})$ the collection of all conditionally termal (positive conditionally termal) functions of the algebra $\mathfrak{A}$.

Remind that an inner isomorphism (inner homomorphism) of the algebra $\mathfrak{A}$ is any isomorphism (homomorphism) between some subalgebras of $\mathfrak{A}$. Denote by $Iso\mathfrak{A}$ $(Ihm\mathfrak{A})$ the semigroup of all inner isomorphisms (inner homomorphisms) of the algebra $\mathfrak{A}$ and denote by $StabIso\mathfrak{A}$ $(StabIhm\mathfrak{A})$ the collection of all functions on the algebra $\mathfrak{A}$ such that these functions are stable relatively all inner isomorphism (inner homomorphism) on the algebra $\mathfrak{A}$.

Remind that the algebra $\mathfrak{A}$ is uniformly locally finite if there exists some function $h : \omega \to \omega$ such that for any $B \subseteq A$ and $n \in \omega$ the inequality $|B| \leqslant n$ implies the inequality $| < B >_{\mathfrak{A}} | \leqslant h(n)$. Here $< B >_{\mathfrak{A}}$ is the subalgebra of $\mathfrak{A}$ generated by the set B.

THEOREM 5. For any finite (uniformly locally finite algebra of the finite signature) algebra $\mathfrak{A}$ the following equalities holds:

a)$CT\mathfrak{A} = StabIso\mathfrak{A}$;

b)$PCT\mathfrak{A} = StabIhm\mathfrak{A}$.

COROLLARY 5. Let $\mathfrak{A}_0 =< A; \sigma_0 >$, $\mathfrak{A}_1 =< A; \sigma_1 >$ are finite (uniform locally finite) algebras (of finite signatures) having the common basic sets. Then the following conditions are equivalent:

1)$Iso\mathfrak{A}_0 = Iso\mathfrak{A}_1$ ($Ihm\mathfrak{A}_0 = Ihm\mathfrak{A}_1$);

2)all signature functions of the algebra $\mathfrak{A}_i$ are conditional termal (positive conditional termal) functions for the algebra $\mathfrak{A}_{1-i}$ (for $i = 0, 1$).

Is cold function $f(x_1, ..., x_n)$ on the basic set of the algebra $\mathfrak{A} =< A; \sigma >$ by *point-termal* if for any $\bar{a} =< a_1, ..., a_n >$ in A^n there exist some term $t_{\bar{a}}(x_1, ..., x_n)$ of the signature σ such that $f(a_1, ..., a_n) = t_{\bar{a}}(a_1, ..., a_n)$.

Denote by $Ptr\mathfrak{A}$ the collection of all point-termal functions for the algebra $\mathfrak{A}$.

THEOREM 6. For any algebra $\mathfrak{A}$ the following equality holds

$$Ptr\mathfrak{A} = StabSub\mathfrak{A}$$

COROLLARY 6. Let are $\mathfrak{A}_0 =< A; \sigma_0 >$, $\mathfrak{A}_1 =< A; \sigma_1 >$ algebras having common basic sets. Then the following conditions are equivalent:

1)$Sub\mathfrak{A}_0 = Sub\mathfrak{A}_1$;

2)all signature functions of the algebra $\mathfrak{A}_i$ are point-termal functions for the algebra $\mathfrak{A}_{1-i}$ (for $i = 0, 1$).

Let L_2 be the language of the second order logic (formulas with the quantifiers $\forall$ and $\exists$ over predicate variables) and $\wedge L_2$-formulas be some (possible infinite) conjunctions of the L_2-formulas.

A function $f(x_1, ..., x_n)$ on the basic set of the algebra $\mathfrak{A} =< A; \sigma >$ is called *automorphicly point wise L-definable* (L be some logical language) if for any $\bar{a} =< a_1, ..., a_n >\in A^n$ there exists an L-formula $\Phi_{\bar{a}}(\bar{x}, y)$ such that the following condition holds: for any $\bar{a}' \in A^n$, $b' \in A$ if for some $\phi \in Aut\mathfrak{A}$, $\bar{a}' = \phi(\bar{a})$ then $\mathfrak{A} \models \Phi_{\bar{a}}(\bar{a}', b') \Leftrightarrow b' = \phi(f(\bar{a}))$. Denote by $L - PDef\mathfrak{A}$ the collection of all automorphicly pointwise L-definable functions for the algebra $\mathfrak{A}$.

From Marek-theorem on the countable categoricity of the L_2-theories of countable models of finite signatures we obtain

THEOREM 7 ($V = L$). For any not more that countable algebra $\mathfrak{A} =< A; \sigma >$ of the finite signature the following equality holds

$$\wedge L_2 - PDef\mathfrak{A} = StabAut\mathfrak{A}.$$

COROLLARY 7 ($V = L$). Let $\mathfrak{A}_0 =< A; \sigma_0 >$, $\mathfrak{A}_1 =< A; \sigma_1 >$ are countable algebras of the finite signatures having common basic sets. Then the following conditions are equivalent:
1)$Aut\mathfrak{A}_0 = Aut\mathfrak{A}_1$;
2)any signature function of the algebra $\mathfrak{A}_i$ is some automorphicly pointwise $\wedge L_2$-definable functions for the algebra $\mathfrak{A}_{1-i}$ (for $i = 0, 1$).

Let as remind of the following definitions. An $\mathfrak{A}$-0-translation of the algebra $\mathfrak{A} =< A;\sigma >$ is any constant map or identical map of A in A. An $\mathfrak{A}$-1-translations is a $\mathfrak{A}$-0-translations or a maps $h(x)$ of A in A such that $h(x) = f(a_1, ..., a_{i-1}, x, a_{i+1}, ..., a_n)$ for some $f \in \sigma$ and $i \leqslant n$ $a_1, ..., a_{i-1}, a_{i+1}, ..., a_n$ in A. (such maps we called by f-1-translations) For any $n > 1$ an $\mathfrak{A}$-n-translation is a some superpositions of some n $\mathfrak{A}$-1-translations. A map of A to A is $\mathfrak{A}$-translation if it is an $\mathfrak{A}$-n-translation for some $n \in \omega$. A function $f(x_1, ..., x_n)$ on the set A is *Con-connected* by $\mathfrak{A}$-translations if for any $a, b \in A$ and any f-1-translation $g(x)$ there exist a natural n and $\mathfrak{A}$-translations $h_1(x), ..., h_n(x)$ such that $g(a) = h(e_1^1), ..., h_i(e_i^2) = h_{i+1}(e_{i+1}^1), ..., h_n(e_n^2) = g(b)$ for some $e_1^1, ..., e_n^1, e_1^2, ..., e_n^2 \in A$ such that $\{e_i^1, e_i^2\} = \{a, b\}$ for $i = 1, ..., n$.

Denote by $Contr\mathfrak{A}$ the collection of all functions which are *Con*-connected by $\mathfrak{A}$-translations functions for the algebra $\mathfrak{A}$.

THEOREM 8. For any universal algebra $\mathfrak{A}$ the following equality holds

$$Contr\mathfrak{A} = StabCon\mathfrak{A}.$$

COROLLARY 8. Let $\mathfrak{A}_0 =< A; \sigma_0 >$, $\mathfrak{A}_1 =< A; \sigma_1 >$ are universal algebras having common basic sets. Then the following conditions are equivalent:
1)$Con\mathfrak{A}_0 = Con\mathfrak{A}_1$;
2)all signature functions of the algebra $\mathfrak{A}_i$ are Con-connected by $\mathfrak{A}_{1-i}$-translations functions (for $i = 0, 1$).