Lattice-valued bornological vector spaces and systems

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Outline

1. Introduction
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3. Lattice-valued bornological vector systems
4. Future work

L-bornological spaces

- There exist well-known concepts of functional analysis, namely, *bornological space* and *bounded map*, which provide a convenient tool to study the notion of "boundedness".
- The construct **Born** of bornological spaces and bounded maps has already found applications in Functional Analysis.
- In 2011, M. Abel and A. Šostak introduced the category L-**Born** of L-bornological spaces over a complete lattice L, in order to start the development of the theory of lattice-valued bornology as an extension of the theory of crisp bornological spaces.
- M. Abel and A. Šostak showed that L-**Born** is a topological construct provided that the lattice L is infinitely distributive.

Topological properties of L-bornological spaces

- J. Paseka, S. Solovjovs, and M. Stehlík gave the necessary and sufficient condition on the lattice L for the category L-**Born** to be topological (correcting an error of M. Abel and A. Šostak).
- J. Paseka, S. Solovjovs, and M. Stehlík introduced the category **L**-**Born** of variable-basis lattice-valued bornological spaces (in the sense of S. E. Rodabaugh) over a subcategory **L** of the category **Sup** of $\bigvee$-semilattices, showing the necessary and sufficient condition on **L** for the category **L**-**Born** to be topological.
- These topologicity conditions for the categories **L**-**Born** and L-**Born** make a striking difference with the case of the categories for lattice-valued topology, which are topological provided that their respective underlying lattices are complete.

Topos-theoretic properties of L-bornological spaces

- J. Paseka, S. Solovjovs, and M. Stehlík showed that for a frame L (satisfying one additional condition), the full subcategory L-$\mathbf{Born}_s$ of L-$\mathbf{Born}$ of strict L-bornological spaces (in the sense of M. Abel and A. Šostak) is a topological construct, which is a well-fibred quasitopos, i.e., provides a topological universe.
- This result makes another striking difference with the categories of lattice-valued topological spaces, which fail both to be cartesian closed and to have representable extremal partial morphisms, and which, therefore, do not provide quasitopoi.

Lattice-valued bornological vector spaces

- Motivated by the theory of bornological *vector* spaces, this talk introduces the category L-**VBorn** of L-bornological *vector* spaces over a complete lattice L.
- We show that for some complete lattices L, L-**VBorn** is topological over the category **Vec** of vector spaces, coming to the conclusion that L-**VBorn** is a topologically algebraic construct.
- The category L-**VBorn** and its respective results are additionally extended to variable-basis (in the sense of S. E. Rodabaugh) over subcategories of the category **Sup** of $\bigvee$-semilattices.

Lattice-valued bornological vector systems

- Stimulated by the concept of *topological system* of S. Vickers (a common setting for both point-set and point-free topologies), J. Paseka, S. Solovjovs, and M. Stehlík introduced the category L-**BornSys** of L-bornological systems, and showed that the category L-**Born** is isomorphic to its full reflective subcategory.
- This talk introduces the category L-**VBornSys** of L-bornological *vector* systems, and shows that the category L-**VBorn** is isomorphic to a full reflective subcategory of the category L-**VBornSys**.
- The category L-**VBornSys** and its respective results are extended to variable-basis over subcategories of the category **Sup**.

Lattice-valued bornology and cancer research

- The purpose of our investigation is the development of a lattice-valued analogue of the Hausdorff dimension in convex bornological spaces of J. Almeida and L. Barreira, to provide a new technique for the study of cancer-related diseases in humans.
- Our machinery will take into account the presence of fuzziness in the real world phenomena (in the sense of L. A. Zadeh).

Bornological spaces and bounded maps

Every map $X \xrightarrow{f} Y$ gives rise to the *forward powerset operator* $\mathcal{P}X \xrightarrow{f^{\rightarrow}} \mathcal{P}Y$, which is defined by $f^{\rightarrow}(S) = \{f(s) \mid s \in S\}$.

Definition 1

A *bornological space* is a pair $(X, \mathcal{B})$, where X is a set, and $\mathcal{B}$ (a *bornology* on X) is a subfamily of $\mathcal{P}X$ (the elements of which are called *bounded sets*), which satisfy the following axioms:

1. $X = \bigcup \mathcal{B} (= \bigcup_{B \in \mathcal{B}} B)$;
2. if $B \in \mathcal{B}$ and $D \subseteq B$, then $D \in \mathcal{B}$;
3. if $\mathcal{S} \subseteq \mathcal{B}$ is finite, then $\bigcup \mathcal{S} \in \mathcal{B}$.

Given bornological spaces $(X_1, \mathcal{B}_1)$, $(X_2, \mathcal{B}_2)$, a map $X_1 \xrightarrow{f} X_2$ is *bounded* provided that $f^{\rightarrow}(B_1) \in \mathcal{B}_2$ for every $B_1 \in \mathcal{B}_1$. **Born** is the construct of bornological spaces and bounded maps.

L-bornological spaces and L-bounded maps

Given a complete lattice L, every map $X \xrightarrow{f} Y$ provides the *forward L-powerset operator* $L^X \xrightarrow{f_L^{\rightarrow}} L^Y$ with $(f_L^{\rightarrow}(B))(y) = \bigvee_{f(x)=y} B(x)$.

Definition 2 (M. Abel and A. Šostak)

An *L-bornological space* is a pair $(X, \mathcal{B})$, where X is a set, and $\mathcal{B}$ (an *L-bornology* on X) is a subfamily of L^X (the elements of which are called *bounded L-sets*), which satisfy the following axioms:

1. $\bigvee_{B \in \mathcal{B}} B(x) = \top_L$ for every $x \in X$;
2. if $B \in \mathcal{B}$ and $D \leqslant B$, then $D \in \mathcal{B}$;
3. if $\mathcal{S} \subseteq \mathcal{B}$ is finite, then $\bigvee \mathcal{S} \in \mathcal{B}$.

Given L-bornological spaces $(X_1, \mathcal{B}_1)$, $(X_2, \mathcal{B}_2)$, a map $X_1 \xrightarrow{f} X_2$ is *L-bounded* provided that $f_L^{\rightarrow}(B_1) \in \mathcal{B}_2$ for every $B_1 \in \mathcal{B}_1$. L-**Born** is the construct of L-bornological spaces and L-bounded maps.

Crisp bornological spaces as L-bornological spaces

Remark 3

The construct 2-**Born** is concretely isomorphic to the construct **Born**, where $2 = \{\bot, \top\}$ is the two-element complete lattice.

Every subset B of a set X can be considered as an L-characteristic map $X \xrightarrow{B_L} L$, which is defined by

$$B_L(x) = \begin{cases} \top_L, & x \in B \\ \bot_L, & \text{otherwise.} \end{cases}$$

Proposition 4

Given a complete lattice L with at least two elements, there exists a full concrete embedding 2-**Born** $\xhookrightarrow{E}$ L-**Born** *defined by* $E(X, \mathcal{B}) = (X, \mathcal{B}_L)$ *with* $\mathcal{B}_L = \{C \in L^X \mid C \leqslant B_L \text{ for some } B \in \mathcal{B}\}$.

Ideal complete distributivity

Definition 5

Given a complete lattice L, a subset $S \subseteq L$ is called a *lattice ideal* of L provided that

1. if $a \in L$ and $a \leqslant b$ for some $b \in S$, then $a \in S$;
2. if $T \subseteq S$ is finite, then $\bigvee T \in S$.

Definition 6

A complete lattice L is called *ideally completely distributive at* $\top_L$ provided that for every non-empty family $\{S_i \mid i \in I\}$ of lattice ideals of L, $\bigwedge_{i \in I}(\bigvee S_i) = \top_L$ implies $\bigvee_{h \in H}(\bigwedge_{i \in I} h(i)) = \top_L$, where H is the set of choice functions on $\bigcup_{i \in I} S_i$, which are maps $I \xrightarrow{h} \bigcup_{i \in I} S_i$ such that $h(i) \in S_i$ for every $i \in I$.

L-**Born** is a topological construct

Theorem 7

The construct (L-**Born**, $|-|$) *is topological iff* L *is ideally completely distributive at* $\top_L$.

- By the above theorem, the constructs 2-**Born** (crisp approach) and $[0,1]$-**Born** (fuzzy approach) are topological.
- From now on, let L be ideally completely distributive at $\top_L$.

Products of L-bornological spaces

Given a set-indexed family of maps $(X_i \xrightarrow{B_i} L)_{i\in I}$, define a map $\prod_{i\in I} X_i \xrightarrow{\otimes_{i\in I} B_i} L$ by the formula $(\otimes_{i\in I} B_i)((x_i)_{i\in I}) = \bigwedge_{i\in I} B_i(x_i)$.

Proposition 8

The product of a family $((X_i, \mathcal{B}_i))_{i\in I}$ of L-bornological spaces is given by the source $((\prod_{i\in I} X_i, \mathcal{B}^{\otimes}) \xrightarrow{\pi_j} (X_j, \mathcal{B}_j))_{j\in I}$, where $\prod_{i\in I} X_i \xrightarrow{\pi_j} X_j$ is the j-th projection map and $\mathcal{B}^{\otimes} = \{B \in L^X \mid B \leqslant \otimes_{i\in I} B_i$ such that $B_i \in \mathcal{B}_i$ for every $i \in I\}$.

L-bornological vector spaces

Definition 9

An *L-bornological vector space* is a tuple $(X, +, *, \mathcal{B})$, where $(X, +, *)$ is a vector space over the field of real numbers $\mathbb{R}$, and $(X, \mathcal{B})$ is an L-bornological space such that:

1. the map $(X, \mathcal{B}) \times (X, \mathcal{B}) \xrightarrow{+} (X, \mathcal{B})$ is L-bounded;
2. the map $(\mathbb{R}, \mathcal{B}_{\mathbb{R}L}) \times (X, \mathcal{B}) \xrightarrow{*} (X, \mathcal{B})$ is L-bounded, where $\mathcal{B}_{\mathbb{R}L}$ is the image under the embedding E of Proposition 4 of the bornology $\mathcal{B}_{\mathbb{R}}$ of bounded sets (in the classical sense) on $\mathbb{R}$.

Given L-bornological vector spaces $(X_1, +, *, \mathcal{B}_1)$, $(X_2, +, *, \mathcal{B}_2)$, a map $X_1 \xrightarrow{f} X_2$ is *L-bounded linear* provided that f is L-bounded and linear (i.e., $f(x + y) = f(x) + f(y)$ and $f(r * x) = r * f(x)$ for every $x, y \in X_1$ and every $r \in \mathbb{R}$). L-**VBorn** is the construct of L-bornological vector spaces and L-bounded linear maps.

L-**VBorn** is topologically algebraic

Vec is the category of vector spaces and linear maps.

Theorem 10

The concrete category $(L\text{-}\mathbf{VBorn}, |-|)$ *over* **Vec** *is topological.*

Theorem 11

The construct $(L\text{-}\mathbf{VBorn}, |-|)$ *is topologically algebraic.*

Proof.

It is enough to show the following two properties.

1. The forgetful functor $L\text{-}\mathbf{VBorn} \xrightarrow{|-|} \mathbf{Set}$ has a left adjoint.
2. The category L-**VBorn** is (Epi, Initial Source)-factorizable.

L-convex L-bornological vector spaces

Definition 12

Given a vector space $(X, +, *)$ and an L-set $A \in L^X$, A is called

1. *L-circled* provided that for every $x \in X$ and every $r \in \mathbb{R}$ such that $|r| \leqslant 1$, $A(x) \leqslant A(r * x)$;
2. *L-convex* provided that for every $x_1, x_2 \in X$ and every positive $r_1, r_2 \in \mathbb{R}$ with $r_1 + r_2 = 1$, $A(x_1) \wedge A(x_2) \leqslant A(r_1 * x_1 + r_2 * x_2)$;
3. *L-disk* provided that A is both L-convex and L-circled.

An *L-disked hull* A_{dh} of A is defined by $A_{dh} = \bigwedge\{B \in L^X \mid B$ is an L-disk and $A \leqslant B\}$. The disked hull A_{dh} of A is an L-disk.

Definition 13

An L-bornological vector space $(X, +, *, \mathcal{B})$ is called *L-convex* provided that $B_{dh} \in \mathcal{B}$ for every $B \in \mathcal{B}$. L-**CVBorn** is the full subcategory of L-**VBorn** of L-convex L-bornological vector spaces.

Variable-basis lattice-valued forward powerset operator

- **Set** is the category of sets and maps.
- **Sup** is the category of $\bigvee$-semilattices and $\bigvee$-preserving maps.

Proposition 14

Every subcategory **L** *of* **Sup** *provides a functor* $\mathbf{Set} \times \mathbf{L} \xrightarrow{(-)^{\to}} \mathbf{Sup}$, *which is defined by* $((X_1, L_1) \xrightarrow{(f,\psi)} (X_2, L_2))^{\to} = L_1^{X_1} \xrightarrow{(f,\psi)^{\to}} L_2^{X_2}$, *where* $((f,\psi)^{\to}(B))(x_2) = \bigvee_{f(x_1)=x_2} \psi \circ B(x_1)$.

Variable-basis lattice-valued bornology

Definition 15

Given a subcategory **L** of **Sup**, **L-Born** is the category, which is concrete over the product category **Set** $\times$ **L**, whose

objects are triples $(X, L, \mathcal{B})$, where L is an **L**-object, and $(X, \mathcal{B})$ is an L-bornological space; and whose

morphisms $(X_1, L_1, \mathcal{B}_1) \xrightarrow{(f,\psi)} (X_2, L_2, \mathcal{B}_2)$ (called **L**-*bounded maps*) consist of a map $X_1 \xrightarrow{f} X_2$ and an **L**-morphism $L_1 \xrightarrow{\psi} L_2$ such that $(f, \psi)^{\rightarrow}(B_1) \in \mathcal{B}_2$ for every $B_1 \in \mathcal{B}_1$.

Example 16

Every complete lattice L is a $\bigvee$-semilattice, which provides a subcategory L of **Sup**, whose only morphism is the identity map $L \xrightarrow{1_L} L$. The categories L-**Born** and L-**Born** are then concretely isomorphic.

A particular category of $\bigvee$-semilattices

Given a $\bigvee$-semilattice homomorphism $L_1 \xrightarrow{\psi} L_2$, there exists a $\bigwedge$-preserving map $L_2 \xrightarrow{\psi^\vdash} L_1$ with $\psi^\vdash(b) = \bigvee\{a \in L_1 \mid \psi(a) \leqslant b\}$.

Definition 17

$\mathbf{L}^\vdash$ is the subcategory of **Sup**, whose objects L are ideally completely distributive at $\top_L$, and whose morphisms $L_1 \xrightarrow{\psi} L_2$ are such that the map $L_2 \xrightarrow{\psi^\vdash} L_1$ has the following property:

$$\psi^\vdash(\bigvee S) = \bigvee_{s \in S} \psi^\vdash(s) \text{ for every } S \subseteq L_2 \text{ such that } \bigvee S = \top_{L_2}.$$

L-Born is a topological category

Theorem 18

The concrete category $(\mathbf{L}\text{-}\mathbf{Born}, |-|)$ *is topological over the product category* $\mathbf{Set} \times \mathbf{L}$ *iff* $\mathbf{L}$ *is a subcategory of* $\mathbf{L}^{\vdash}$.

Example 19

An example of $\mathbf{L}$, which gives a topological category $\mathbf{L}$-$\mathbf{Born}$, is the subcategory of $\mathbf{Sup}$, whose objects satisfy the condition of the category $\mathbf{L}^{\vdash}$, and whose morphisms are $\bigvee$-semilattice isomorphisms.

Variable-basis lattice-valued bornological vector spaces

Definition 20

Given a subcategory **L** of **Sup**, **L-VBorn** is the category, which is concrete over the product category **Vec** $\times$ **L**, whose

objects (**L**-*bornological vector spaces*) are L-bornological vector spaces $(X, +, *, L, \mathcal{B})$ such that L is an **L**-object; and whose

morphisms (**L**-*bounded linear maps*) $(X_1, +, *, L_1, \mathcal{B}_1) \xrightarrow{(f,\psi)} (X_2, +, *, L_2, \mathcal{B}_2)$ are **L**-bounded maps $(X_1, L_1, \mathcal{B}_1) \xrightarrow{(f,\psi)} (X_2, L_2, \mathcal{B}_2)$ such that the map $(X_1, +, *) \xrightarrow{f} (X_2, +, *)$ is linear.

From now on, let **L** be a subcategory of the category $\mathbf{L}^{\vdash}$.

L-VBorn is topologically algebraic over **Set** $\times$ **L**

Theorem 21

The concrete category $(\mathbf{L\text{-}VBorn}, |-|)$ *over* $\mathbf{Vec} \times \mathbf{L}$ *is topological.*

Theorem 22

The concrete category $(\mathbf{L\text{-}VBorn}, |-|)$ *over* $\mathbf{Set} \times \mathbf{L}$ *is topologically algebraic.*

b_e-lattices and their homomorphisms

Definition 23

- A *b_e-lattice* is a poset C, which has finite $\bigvee$ and non-empty $\bigwedge$.
- Given b_e-lattices C_1 and C_2, a *b_e-lattice homomorphism* is a map $C_1 \xrightarrow{\varphi} C_2$, which preserves finite $\bigvee$.
- b_e-**Lat** stands for the construct of b_e-lattices and b_e-lattice homomorphisms.

L-bornological systems

Definition 24

An *L-bornological system* is a triple $(X, C, \models)$, where X is a set, C is a b_e-lattice, and $X \times C \xrightarrow{\models} L$ is a map (*L-satisfaction relation* on (X, C)), which satisfies the following axioms:

1. for every $x \in X$, $\bigvee_{c \in C} \models(x, c) = \top_L$;
2. if $c \in C$ and $D \in L^X$ are such that $D \leqslant \models(-, c)$, then there exists $c' \in C$ such that $D = \models(-, c')$;
3. for every $x \in X$, the map $C \xrightarrow{\models(x,-)} L$ preserves finite $\bigvee$.

L-bornological system morphisms

Definition 25

Given L-bornological systems $(X_1, C_1, \models_1)$ and $(X_2, C_2, \models_2)$, an *L-bornological system morphism* (also called *L-bounded map*) $(X_1, C_1, \models_1) \xrightarrow{(f,\varphi)} (X_2, C_2, \models_2)$ consists of a map $X_1 \xrightarrow{f} X_2$ and a b_e-lattice homomorphism $C_1 \xrightarrow{\varphi} C_2$ such that for every $x_2 \in X_2$ and every $c_1 \in C_1$, $\models_2(x_2, \varphi(c_1)) = \bigvee_{f(x_1)=x_2} \models_1(x_1, c_1)$.

Definition 26

L-**BornSys** is the category of L-bornological systems and L-bounded maps, which is concrete over the product category **Set** $\times$ b_e-**Lat**.

Spaces versus systems

Theorem 27

1. *There exists a full embedding* $L\text{-}\mathbf{Born} \xhookrightarrow{E} L\text{-}\mathbf{BornSys}$, *which is defined by* $E((X_1, \mathcal{B}_1) \xrightarrow{f} (X_2, \mathcal{B}_2)) = (X_1, \mathcal{B}_1, \models_1) \xrightarrow{(f, \bar{f}_L^{\rightarrow})} (X_2, \mathcal{B}_2, \models_2)$, *where* $\models_i(x, B) = B(x)$, *and* $\bar{f}_L^{\rightarrow}$ *is the restriction of* $f_L^{\rightarrow}$ *to* $\mathcal{B}_1$, $\mathcal{B}_2$.
2. *There exists a functor* $L\text{-}\mathbf{BornSys} \xrightarrow{Spat} L\text{-}\mathbf{Born}$ *given by* $Spat((X_1, C_1, \models_1) \xrightarrow{(f,\varphi)} (X_2, C_2, \models_2)) = (X_1, \{Ext_1(c) \mid c \in C_1\}) \xrightarrow{f} (X_2, \{Ext_2(c) \mid c \in C_2\})$ *with* $(Ext_i(c))(x) = \models_i(x, c)$.
3. *Spat is a left-adjoint-left-inverse to* E.

L-bornological vector systems

Definition 28

An *L-bornological vector system* is a quintuple $(X, +, *, C, \models)$, where $(X, +, *)$ is a vector space over $\mathbb{R}$, and $(X, C, \models)$ is an L-bornological system, which satisfy the next axioms:

1. for every $c_1, c_2 \in C$, there exists $c \in C$ with $\models(x_1, c_1) \wedge \models(x_2, c_2) \leqslant \models(x_1 + x_2, c)$ for every $x_1, x_2 \in X$;
2. for every $S \in \mathcal{B}_{\mathbb{R}}$, and every $c \in C$, there exists $c' \in C$ such that $\bigvee\{\models(x', c) \mid s * x' = x \text{ for some } s \in S\} \leqslant \models(x, c')$ for every $x \in X$.

L-bornological vector system morphisms

Definition 29

Given L-bornological vector systems $(X_1, +, *, C_1, \models_1)$ and $(X_2, +, *, C_2, \models_2)$, an *$L$-bornological vector system morphism* (also called *L-bounded linear map*) $(X_1, +, *, C_1, \models_1) \xrightarrow{(f,\varphi)} (X_2, +, *, C_2, \models_2)$ is an L-bounded map $(X_1, C_1, \models_1) \xrightarrow{(f,\varphi)} (X_2, C_2, \models_2)$ such that $(X_1, +, *) \xrightarrow{f} (X_2, +, *)$ is a linear map.

Definition 30

L-**VBornSys** is the category of L-bornological vector systems and L-bounded linear maps, concrete over the category **Vec** $\times$ b_e-**Lat**.

Spaces versus systems

Theorem 31

1. *There exists a full embedding* L-**VBorn** $\xhookrightarrow{E}$ L-**VBornSys**, *which is defined by* $E((X_1, +, *, \mathcal{B}_1) \xrightarrow{f} (X_2, +, *, \mathcal{B}_2)) = (X_1, +, *, \mathcal{B}_1, \models_1) \xrightarrow{(f, \bar{f}_L^{\rightarrow})} (X_2, +, *, \mathcal{B}_2, \models_2)$, *with* $\models_i(x, B) = B(x)$, *and* $\bar{f}_L^{\rightarrow}$ *the restriction of* $f_L^{\rightarrow}$ *to* $\mathcal{B}_1$, $\mathcal{B}_2$.
2. *There exists a functor* L-**VBornSys** $\xrightarrow{Spat}$ L-**VBorn** *given by* $Spat((X_1, +, *, C_1, \models_1) \xrightarrow{(f, \varphi)} (X_2, +, *, C_2, \models_2)) = (X_1, +, *, \{Ext_1(c) \mid c \in C_1\}) \xrightarrow{f} (X_2, +, *, \{Ext_2(c) \mid c \in C_2\})$ *with* $(Ext_i(c))(x) = \models_i(x, c)$.
3. *Spat is a left-adjoint-left-inverse to* E.

L-convex L-bornological vector systems

Definition 32

An L-bornological vector system $(X, +, *, C, \models)$ is called *L-convex* provided that for very $c \in C$, there exists $c' \in C$ such that $(Ext(c))_{dh} = Ext(c')$. L-**CVBornSys** is the full subcategory of L-**VBornSys** of L-convex L-bornological vector systems.

Theorem 33

1. *There exists the restriction* $L\text{-}\mathbf{CVBorn} \xhookrightarrow{CE} L\text{-}\mathbf{CVBornSys}$ *of the full embedding* $L\text{-}\mathbf{VBorn} \xhookrightarrow{E} L\text{-}\mathbf{VBornSys}$ *.*
2. *There exists the restriction* $L\text{-}\mathbf{CVBornSys} \xrightarrow{CSpat} L\text{-}\mathbf{CVBorn}$ *of the functor* $L\text{-}\mathbf{VBornSys} \xrightarrow{Spat} L\text{-}\mathbf{VBorn}$*.*
3. *CSpat is a left-adjoint-left-inverse to CE.*

Variable-basis lattice-valued bornological systems

Definition 34

Given a subcategory **L** of **Sup**, **L-BornSys** is the category, concrete over the product category **Set** $\times$ **L** $\times$ b_e-**Lat**, whose

objects are tuples $(X, L, C, \models)$, where L is an **L**-object, and $(X, C, \models)$ is an L-bornological system; and whose

morphisms $(X_1, L_1, C_1, \models_1) \xrightarrow{(f,\psi,\varphi)} (X_2, L_2, C_2, \models_2)$ (also called ***L-bounded maps***) have a map $X_1 \xrightarrow{f} X_2$, an **L**-morphism $L_1 \xrightarrow{\psi} L_2$, and a b_e-lattice homomorphism $C_1 \xrightarrow{\varphi} C_2$ such that for every $x_2 \in X_2$ and every $c_1 \in C_1$, $\models_2(x_2, \varphi(c_1)) = \bigvee_{f(x_1)=x_2} \psi \circ \models_1(x_1, c_1)$.

Spaces versus systems

Theorem 35

1. *There exists a full embedding* $\mathbf{L\text{-}Born} \xhookrightarrow{E} \mathbf{L\text{-}BornSys}$ *given by* $E((X_1, L_1, \mathcal{B}_1) \xrightarrow{(f,\psi)} (X_2, L_2, \mathcal{B}_2)) = (X_1, L_1, \mathcal{B}_1, \models_1) \xrightarrow{(f,\psi,\overline{(f,\psi)}^{\to})} (X_2, L_2, \mathcal{B}_2, \models_2)$, $\models_i(x, B) = B(x)$, *and* $\overline{(f,\psi)}^{\to}$ *is the restriction of* $(f,\psi)^{\to}$ *to* $\mathcal{B}_1$, $\mathcal{B}_2$.
2. *There exists a functor* $\mathbf{L\text{-}BornSys} \xrightarrow{Spat} \mathbf{L\text{-}Born}$ *given by* $Spat((X_1, L_1, C_1, \models_1) \xrightarrow{(f,\psi,\varphi)} (X_2, L_2, C_2, \models_2)) = (X_1, L_1, \{Ext_1(c) \mid c \in C_1\}) \xrightarrow{(f,\psi)} (X_2, L_2, \{Ext_2(c) \mid c \in C_2\})$, *where* $(Ext_i(c))(x) = \models_i(x, c)$.
3. *Spat is a left-adjoint-left-inverse to* E.

Variable-basis lattice-valued bornological vector systems

Definition 36

Given a subcategory **L** of **Sup**, **L-VBornSys** is the category, concrete over the product category **Vec** $\times$ **L** $\times$ b_e-**Lat**, whose

objects (**L**-*bornological vector systems*) are L-bornological vector systems $(X, +, *, L, C, \models)$ such that L is an **L**-object; and whose

morphisms (**L**-*bounded linear maps*) $(X_1, +, *, L_1, C_1, \models_1) \xrightarrow{(f,\psi,\varphi)} (X_2, +, *, L_2, C_2, \models_2)$ are **L**-bounded maps $(X_1, L_1, \mathcal{B}_1) \xrightarrow{(f,\psi,\varphi)} (X_2, L_2, \mathcal{B}_2)$ such that the map $X_1 \xrightarrow{f} X_2$ is linear.

Spaces versus systems

Theorem 37

1. *There exists a full embedding* **L-VBorn** $\xhookrightarrow{E}$ **L-VBornSys** *defined by* $E((X_1,+,*,L_1,\mathcal{B}_1) \xrightarrow{(f,\psi)} (X_2,+,*,L_2,\mathcal{B}_2)) = (X_1,+,*,L_1,\mathcal{B}_1,\models_1) \xrightarrow{(f,\psi,\overline{(f,\psi)}^{\rightarrow})} (X_2,+,*,L_2,\mathcal{B}_2,\models_2)$, $\models_i(x,B)=B(x)$, $\overline{(f,\psi)}^{\rightarrow}$ *is the restriction of* $(f,\psi)^{\rightarrow}$ *to* $\mathcal{B}_1,\mathcal{B}_2$.
2. *There exists a functor* **L-BornSys** $\xrightarrow{Spat}$ **L-Born** *defined by* $Spat((X_1,+,*,L_1,C_1,\models_1) \xrightarrow{(f,\psi,\varphi)} (X_2,+,*,L_2,C_2,\models_2)) = (X_1,+,*,L_1,\{Ext_1(c)\,|\,c \in C_1\}) \xrightarrow{(f,\psi)} (X_2,+,*,L_2,\{Ext_2(c)\,|\,c \in C_2\})$, $(Ext_i(c))(x) = \models_i(x,c)$.
3. *Spat is a left-adjoint-left-inverse to the embedding* E.

Conclusion

- Following the idea of L-bornological space over a complete lattice L of M. Abel and A. Šostak, this talk has introduced the category L-**VBorn** of L-bornological vector spaces, which appeared to be a topologically algebraic construct.
- We have also introduced a variable-basis modification **L-VBorn** of L-**VBorn** (in the sense of S. E. Rodabaugh) over a subcategory **L** of the category **Sup** of $\bigvee$-semilattices, and showed that **L-VBorn** is topologically algebraic over the category **Set** $\times$ **L**.
- Moreover, we have introduced the category L-**VBornSys** of L-bornological vector systems, and showed that the category L-**VBorn** is isomorphic to its full reflective subcategory. This result has been additionally extended variable-basis.

Open problems

Problem 38

What will be a lattice-valued (possibly, variable-basis) analogue of the crisp "topology – bornology" dualities?

Problem 39

What will be the analogue of lattice-valued (possibly, variable-basis) bornological vector spaces, which is based in *lattice-valued* vector spaces in the sense of *lattice-valued* algebra?

Problem 40

What is the nature of the category **L-VBornSys** of lattice-valued bornological vector systems, namely, whether it is essentially algebraic over its ground category?

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Thank you for your attention!