On t-filters on Residuated Lattices (AAA88)

Martin Víta, MU Brno

20.6.2014

Definition of a Residuated Lattice

Definition

A *bounded pointed commutative integral residuated lattice* is a structure

$$\mathbf{L} = (L, \&, \rightarrow, \wedge, \vee, \overline{0}, \overline{1})$$

of type $(2, 2, 2, 2, 0, 0)$ which satisfies the following conditions:

(i) $(L, \wedge, \vee, \overline{0}, \overline{1})$ is a bounded lattice.

(ii) $(L, \&, \overline{1})$ is a monoid.

(iii) $(\&, \rightarrow)$ form an adjoint pair, i.e. $x \,\&\, z \leq y$ if and only if $z \leq x \rightarrow y$ for all $x, y, z \in L$.

Definition of a Filter

Definition

A non-empty subset F of L is called a *filter* on $\mathbf{L}$ if following conditions hold for all $x, y \in L$:

(i) if $x, y \in F$, then $x \,\&\, y \in F$,

(ii) if $x \in F$, $x \leq y$, then $y \in F$.

Special Types of Filters

Definition

A nonempty subset F of a BL-algebra $\mathbf{L}$ called a *fantastic* filter if it satisfies:

1. $\overline{1} \in F$
2. $z \to (y \to x) \in F$ and $z \in F$ imply $((x \to y) \to y) \to x \in F$ for all $x, y, z \in A$.

Other types of filters such as implicative, positive implicative, . . . filters are defined similarly by replacing the second condition by some different one.

Summary of Some Existing Results - Example I.

Theorem (Haveshki, Eslami, Saeid (2006))

On BL*-algebra* **L***, the following statements are equivalent:*

1. $\{\overline{1}\}$ *is a fantastic filter.*
2. *Every filter on* **L** *is a fantastic filter.*
3. **L** *is an* MV*-algebra.*

MV-algebras are just BL-algebras satisfying $\neg\neg x = x$.

Motivation – Example I'

Theorem

On BL-algebra **L**, *the following statements are equivalent:*

1. $\{\overline{1}\}$ *is an implicative filter.*
2. *Every filter on* **L** *is an implicative filter.*
3. **L** *is a Gödel algebra.*

Gödel algebras are just BL-algebras satisfying $x \,\&\, x = x$.

Motivation – Example II

Theorem

Let F, G be filters on BL-algebra $\mathbf{L}$ such that $F \subseteq G$. If F is a fantastic filter, then G is a fantastic filter.

Motivation – Example II'

Theorem

Let F, G be filters on BL-algebra $\mathbf{L}$ such that $F \subseteq G$. If F is an implicative filter, then G is an implicative filter.

Motivation – Example III

Theorem

Let F be a filter of (a BL-algebra) $\mathbf{L}$. Then F is a fantastic filter if and only if every filter of the quotient algebra $\mathbf{L}/F$ is a fantastic filter.

Motivation – Example III'

Theorem

Let F be a filter of (a BL-algebra) $\mathbf{L}$. Then F is an implicative filter if and only if every filter of the quotient algebra $\mathbf{L}/F$ is an implicative filter.

Alternative Definitions of Special Types of Filters

Theorem (Kondo and Dudek (2008))

Let $\mathbf{L}$ *be a* BL*-algebra,* $F \subseteq L$ *a filter on* $\mathbf{L}$*. Then* F *is a fantastic filter iff for all* $x \in L$*,* $\neg\neg x \to x \in F$ *and* F *is an implicative filter iff for all* $x \in L$*,* $x \to x \,\&\, x \in F$*.*

Starting now, $\mathbf{L}$ is a residuated lattice.

Generalization: t-filters

Definition

Let t be an arbitrary term. A filter F on $\mathbf{L}$ is a t-filter if $t(\overline{x}) \in F$ for all $\overline{x} \in L$.

$\overline{x}$ is an abbreviation for a list $x, y, \ldots$. Since now, t is a fixed term.

Generalization of the Extension Theorems

Theorem

Let F and G be filters on a residuated lattice $\mathbf{L}$ such that $F \subseteq G$. If F is a t-filter, then so is G.

Generalization of the Triple of Equivalent Characteristics

Theorem

Let $\mathbb{B}$ be a variety of residuated lattices and $\mathbf{L} \in \mathbb{B}$. Moreover let $\mathbb{C}$ be a subvariety of $\mathbb{B}$ which we get by adding the equation in the form $t = \overline{1}$. Then the following statements are equivalent:

1. *$\{\overline{1}\}$ is a t-filter.*
2. *Every filter on $\mathbf{L}$ is a t-filter.*
3. *$\mathbf{L}$ is in $\mathbb{C}$.*

Generalization of the Quotient Characteristics

Theorem

Let F be a filter on a residuated lattice $\mathbf{L}$. Then F is a t-filter if and only if every filter of the quotient algebra $\mathbf{A}/F$ is a t-filter.

Simple Observations

- $\overline{1}$-filters are just filters on $\mathbf{L}$, x-filters are just trivial filters.
- If $t_1(x) \leq t_2(x)$ for all $x \in L$, then $\{F \subseteq L \mid F \text{ is a } t_1\text{-filter}\} \subseteq \{F \subseteq L \mid F \text{ is a } t_2\text{-filter}\}$.

t-filters and Extended Filters

Definition (Kondo (2013))

Let B be an arbitrary nonempty subset of L, F filter on $\mathbf{L}$. The set $E_F(B) = \{x \in L \mid \forall b \in B(x \vee b \in F)\}$ is called *extended filter* associated with B.

Theorem (Kondo (2013))

Let F be a filter on $\mathbf{L}$. Then:

- *F is an implicative filter if and only if $E_F(x \to x^2) = L$ for all $x \in L$*
- *F is a fantastic filter if and only if $E_F(\neg\neg x \to x) = L$ for all $x \in L$*

Generalization for t-filters

Theorem

Let F be a filter on $\mathbf{L}$, t a term. Then F is a t-filter if and only if $E_F(t(x)) = L$ for all $x \in L$.

Proof.

Let x be an arbitrary element of L, F be a t-filter. Since F is a t-filter, then $t(x) \in F$, thus for every element y of L is $y \vee t(x) \in F$, thus $y \in E_F(t(x))$, i.e., $E_F(t(x)) = L$. Conversely, if $E_F(t(x)) = L$, then $\overline{0} \in E_F(t(x))$, therefore $\overline{0} \vee t(x) = t(x) \in F$. QED

□

I-filters and Possible Generalizations

I-filters defined by Z. M. Ma and B. Q. Hu (2014) are just special cases of t-filters (...)
Possible Generalizations: replace the condition $t(\overline{x}) \in F$ by condition in form if $t_1(\overline{x}) \in F$ and $t_2(\overline{x}) \in F$ and ..., then $t(\overline{x}) \in F$ and start dealing with quasivarieties.

Acknowledgement

Thank you for your attention!