On SI-groups

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Preliminaries

For an arbitrary abelian group A and a prime number p we define a p-component A_p of the group A:

$$A_p = \{a \in A : \; p^n a = 0, \text{ for some } n \in \mathbb{N}\} .$$

Often we will use the designation:

$$\mathbb{P}(A) = \{p \in \mathbb{P} : \; o(a) = p, \text{ for some } a \in A\} .$$

The torsion part of A is denoted by $T(A)$.

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Definition 1

Let $(A, +, 0)$ be an abelian group. An operation $* \colon A \times A \to A$ is called a ring multiplication, if for all $a, b, c \in A$ there holds

$$a * (b + c) = a * b + a * c \ \text{ and } (b + c) * a = b * a + c * a.$$

The algebraic system $(A, +, *, 0)$ is called a ring.

Preliminaries

Definition 2

Let A be an abelian group. If on A there does not exist any nonzero ring multiplication, then A is called a nil-group. If on A there does not exist any nonzero associative ring multiplication, then we say that A is a nil_a-group.

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Remark

Every nil-group is a nil_a-group. Every torsion nil_a-group is a nil-group, by Theorem 3.4 in [1] and Theorem 120.3 in [4]. By Theorem 4.1 in [1], a mixed nil_a-group does not exist. It is easily seen that every ring multiplication on arbitrary subgroup of the group $\mathbb{Q}^+$ is associative and that every abelian torsion-free group of the rank 1 can be embedded in the group $\mathbb{Q}^+$. Thus, the concepts of nil_a-group and nil-group are equivalent also in the class of abelian torsion-free groups of rank 1. In the light of the available literature, it is not known whether there exists a torsion-free nil_a-group A of rank more than 1 such that A is not a nil-group.

Definition 3

A ring in which every subring is an ideal (two-sided) is called an SI-ring. An abelian group A is called an SI-group, if every ring R with $R^+ = A$ is an SI-ring.

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An abelian group A is called an SI_H-group, if every associative ring R with $R^+ = A$ is an H-ring.

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Definition 4

An abelian group A is called an SI_H-group, if every associative ring R with $R^+ = A$ is an H-ring.

Remark

Obviously, every SI-group is an SI_H-group. There exist mixed SI_H-groups which are not SI-groups.

Examples

Example 5

It is easily seen that $Z(p^n)$ is an SI_H-group for all prime numbers p and positive integers n.

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Lemma 6

A direct summand of an SI_H-group is an SI_H-group.

Proof

Let A and B be abelian groups and let $G = A \oplus B$. Suppose that A is not an SI_H-group. Then there exists an associative ring S with $S^+ = A$ such that S is not an H-ring. Since every subring of an H-ring is an H-ring, $R = S \oplus B^0$ is not an H-ring, and consequently, G is not an SI_H-group. □

Examples

Corollary 7

Every torsion-free SI_H-group is reduced.

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Lemma 8

Let R be an associative ring with $R^+ = A$ and let M be a left-sided R-module. If $R \circ M \neq \{0\}$, then $A \oplus M$ is not an SI_H-group.

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Every torsion-free SI_H-group is reduced.

Lemma 8

Let R be an associative ring with $R^+ = A$ and let M be a left-sided R-module. If $R \circ M \neq \{0\}$, then $A \oplus M$ is not an SI_H-group.

Proof

Let $S = \begin{pmatrix} R & M \\ 0 & 0 \end{pmatrix}$. *Then S is an associative ring with $S^+ \cong A \oplus M$ and* $T = \begin{pmatrix} R & 0 \\ 0 & 0 \end{pmatrix}$ *is a subring of S satisfying $TS \not\subseteq T$. Thus $T \ntriangleleft S$.* □

Examples/Torsion SI_H-groups

Corollary 9

From the above lemma we obtain at once that:

- $\mathbb{Z}^+ \oplus A$ *is not an* SI_H*-group for an arbitrary abelian group* A*;*
- $Z(p^m) \oplus Z(p^n)$ *is not an* SI_H*-group for all positive integers* m*,* n*.*

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Theorem 10 (S. Feigelstock)

A nontrivial torsion abelian group A *is an* SI_H*-group if and only if each of its nontrivial p-components* A_p *satisfies one of the following conditions:*

(i) $A_p = Z(p^n)$, *n a positive integer;*

(ii) $A_p = Z(p^n) \oplus D$, *with* D *a divisible p-group and* $n = 0$ *or* $n = 1$.

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Theorem 11 (S. Feigelstock)

The torsion part of an SI_H*-group is an* SI_H*-group.*

Mixed SI_H-groups

Theorem 12 (S. Feigelstock)

If A is mixed SI_H-group, then $T(A) = \bigoplus_{p \in \mathbb{P}(A)} Z(p^{n_p})$, where n_p is a positive integer for all $p \in \mathbb{P}(A)$.

Theorem 12 (S. Feigelstock)

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Remark

In [2] we proved a more accurate version of the above theorem. Namely, we obtained $n_p = 1$, for all $p \in \mathbb{P}(A)$.

Mixed SI_H-groups

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Remark

In [2] we proved a more accurate version of the above theorem. Namely, we obtained $n_p = 1$, for all $p \in \mathbb{P}(A)$.

The following proposition was useful in our proof:

Proposition 13 (R. R. Andruszkiewicz and M. Woronowicz)

Let p and n be a prime number and a positive integer, respectively. Let A be an abelian group such that $A_p \neq \{0\}$ or $T(A) \neq A$. If $n \geq 2$, then $R = \mathbb{Z}_{p^n} \oplus A^0$ is not an H-ring.

Mixed SI_H-groups

Proof

Take any $a \in A$. *Let* $\alpha = (p^{n-1}, a)$. *Then* $\alpha^2 = 0$, *because* $n \geq 2$. *Therefore* $[\alpha] = \langle \alpha \rangle$. *Suppose, contrary to our claim, that* $(1,0)\alpha \in [\alpha]$. *Then there exists* $k \in \mathbb{Z}$ *such that* $(p^{n-1}, 0) = k(p^{n-1}, a)$. *Hence* $p^{n-1} = kp^{n-1}$ *and* $0 = ka$. *If* $o(a) = \infty$, *then from the equality* $ka = 0$ *it follows that* $k = 0$. *Thus* $p^{n-1} = 0$ *in* $\mathbb{Z}_{p^n}$, *a contradiction. If* $o(a) = p$, *then* $p \mid k$. *So there exists* $l \in \mathbb{Z}$ *such that* $k = lp$. *Therefore* $p^{n-1} = kp^{n-1} = lp^n = 0$ *in* $\mathbb{Z}_{p^n}$, *a contradiction. Thus* $(1,0)\alpha \notin [\alpha]$. □

Proof

Take any $a \in A$. Let $\alpha = (p^{n-1}, a)$. Then $\alpha^2 = 0$, because $n \geq 2$. Therefore $[\alpha] = \langle \alpha \rangle$. Suppose, contrary to our claim, that $(1,0)\alpha \in [\alpha]$. Then there exists $k \in \mathbb{Z}$ such that $(p^{n-1}, 0) = k(p^{n-1}, a)$. Hence $p^{n-1} = kp^{n-1}$ and $0 = ka$. If $o(a) = \infty$, then from the equality $ka = 0$ it follows that $k = 0$. Thus $p^{n-1} = 0$ in $\mathbb{Z}_{p^n}$, a contradiction. If $o(a) = p$, then $p \mid k$. So there exists $l \in \mathbb{Z}$ such that $k = lp$. Therefore $p^{n-1} = kp^{n-1} = lp^n = 0$ in $\mathbb{Z}_{p^n}$, a contradiction. Thus $(1,0)\alpha \notin [\alpha]$. □

Theorem 14 (R. R. Andruszkiewicz and M. Woronowicz)

Let $\emptyset \neq P \subseteq \mathbb{P}$ and let A be an SI_H-group satisfying $A = pA$ and $A_p = \{0\}$, for all $p \in P$. Then $G = \left(\bigoplus_{p \in P} Z(p)\right) \oplus A$ is an SI_H-group.

Mixed SI_H-groups

Proposition 15 (R. R. Andruszkiewicz and M. Woronowicz)

Let H be an abelian group satisfying $H_p = \{0\}$ and $H \neq pH$ for some $p \in \mathbb{P}$. If $\dim_{\mathbb{Z}_p} H/pH \geq 2$, then $G = \mathbb{Z}_p^+ \oplus H$ is not an SI_H-group.

Mixed SI_H-groups

Proposition 15 (R. R. Andruszkiewicz and M. Woronowicz)

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Lemma 16 (S. Feigelstock)

If A is an SI_H-group, then every p-component A_p of A is a direct summand of A.

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Theorem 17 (R. R. Andruszkiewicz and M. Woronowicz)

Let G be a mixed SI_H-group and let $p \in \mathbb{P}(A)$. Then there exists $H \leq G$ such that $G = G_p \oplus H$, where $H = \langle h \rangle + pH$, for some $h \in H$.

Mixed SI_H-groups

Proof

It follows from Lemma 16 that there exists $H \leq G$ *such that* $G = G_p \oplus H$. *If* $H = pH$, *then* $H = \langle h \rangle + pH$, *for all* $h \in H$. *If* $H \neq pH$, *then* $\dim_{\mathbb{Z}_p} H/pH = 1$, *by Proposition 15. Hence* $H = \langle h \rangle + pH$, *for all* $h \in H \setminus pH$. □

Mixed SI_H-groups

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It follows from Lemma 16 that there exists $H \leq G$ *such that* $G = G_p \oplus H$. *If* $H = pH$, *then* $H = \langle h \rangle + pH$, *for all* $h \in H$. *If* $H \neq pH$, *then* $\dim_{\mathbb{Z}_p} H/pH = 1$, *by Proposition 15. Hence* $H = \langle h \rangle + pH$, *for all* $h \in H \setminus pH$. □

Remark

It follows from Theorem 12 and Remark 8 that $G_p = Z(p)$, *hence* $pG = pH$. *If* $H \neq pH$, *then the subgroup* H *is not uniquely determined. In fact, let* $K = \langle (a, h) \rangle + pH$, *for some* $0 \neq a \in G_p$, $h \in H \setminus pH$. *Then* $G = G_p + K$. *Suppose that* $k(a, h) + (0, ph_1) = l(a, 0)$, *for some* $k, l \in \mathbb{Z}$, $h_1 \in H$. *Then* $ka = la$ *and* $kh + ph_1 = 0$, *hence* $k \equiv l \pmod{p}$ *and* $kh = -ph_1 \in pH$. *As* $h \in H \setminus pH$ *we have* $p \mid k$. *Therefore* $G_p \cap K = \{0\}$. *Moreover* $(a, h) \notin H$, *so* $K \neq H$.

Mixed SI_H-groups

Proposition 18 (R. R. Andruszkiewicz and M. Woronowicz)

Let H be a nil$_a$-group with $H_p = \{0\}$, for some $p \in \mathbb{P}$. If there exists $h_0 \in H$ such that $H = \langle h_0 \rangle + pH$, then $G = Z(p) \oplus H$ is an SI_H-group.

Mixed SI_H-groups

Proposition 18 (R. R. Andruszkiewicz and M. Woronowicz)

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Now, we prove a fact about subgroups of the group $\mathbb{Q}^+$, which will be applied in the next example of mixed SI_H-group:

Lemma 19

If A is a subgroup of the group $\mathbb{Q}^+$ satisfying $A \neq pA$, for some $p \in \mathbb{P}$, then $A/pA \cong Z(p)$. In particular, $A = pA + \langle a \rangle$, for all $a \in A \setminus pA$.

Proof

Suppose, contrary to our claim, that $\dim_{\mathbb{Z}_p} A/pA \geq 2$. *Then there exist* $a_1, a_2 \in A \setminus pA$ *such that* $a_1 + pA$, $a_2 + pA$ *are linearly independent over the field* $\mathbb{Z}_p$. *Let* $n = \min\left\{m \in \mathbb{N} : \ ma_1 \in \langle a_2 \rangle\right\}$. *Then* $na_1 = ka_2$, *for some* $k \in \mathbb{Z}$. *Thus* $n(a_1 + pA) - k(a_2 + pA) = pA$, *hence* $p \mid n$ *and* $p \mid k$. *Therefore* $n = pn_1$ *and* $k = pk_1$, *for some* $n_1 \in \mathbb{N}$, $k_1 \in \mathbb{Z}$. *Moreover,* A *is torsion-free, and consequently,* $n_1 a_1 = k_1 a_2$, *which contradicts the minimality of the number* n. *Therefore* $\dim_{\mathbb{Z}_p} A/pA = 1$, *hence* $A/pA \cong Z(p)$. □

Mixed SI_H-groups

Proof

Suppose, contrary to our claim, that $\dim_{\mathbb{Z}_p} A/pA \geq 2$*. Then there exist* $a_1, a_2 \in A \setminus pA$ *such that* $a_1 + pA$, $a_2 + pA$ *are linearly independent over the field* $\mathbb{Z}_p$*. Let* $n = \min \left\{ m \in \mathbb{N} : \; ma_1 \in \langle a_2 \rangle \right\}$*. Then* $na_1 = ka_2$*, for some* $k \in \mathbb{Z}$*. Thus* $n(a_1 + pA) - k(a_2 + pA) = pA$*, hence* $p \mid n$ *and* $p \mid k$*. Therefore* $n = pn_1$ *and* $k = pk_1$*, for some* $n_1 \in \mathbb{N}$, $k_1 \in \mathbb{Z}$*. Moreover, A is torsion-free, and consequently,* $n_1 a_1 = k_1 a_2$*, which contradicts the minimality of the number n. Therefore* $\dim_{\mathbb{Z}_p} A/pA = 1$*, hence* $A/pA \cong Z(p)$*.* □

Example 20

Let H be a nil-subgroup of the group $\mathbb{Q}^+$ *such that* $H \neq pH$ *for some* $p \in \mathbb{P}$*. Then there exists* $h_0 \in H \setminus qH$*, hence* $H = \langle h_0 \rangle + pH$*, by Lemma 19. Moreover* $H_p = \{0\}$*, so it follows from Proposition 18 that* $Z(p) \oplus H$ *is an* SI_H*-group.*

Torsion-free SI_H-groups

Theorem 21 (R. R. Andruszkiewicz and M. Woronowicz)

Let A be an abelian torsion-free group. Then A is an SI_H-group if and only if either A is a nil$_a$-group or $A \cong \mathbb{Z}^+$.

Torsion-free SI_H-groups

Theorem 21 (R. R. Andruszkiewicz and M. Woronowicz)

Let A be an abelian torsion-free group. Then A is an SI_H-group if and only if either A is a nil$_a$-group or $A \cong \mathbb{Z}^+$.

Remark

The proof of this theorem relies on important results about H-rings obtained by the R. L. Kruse in [5].

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Corollary 22

The converse theorem to Theorem 17 is not true. In fact, let p and q be distinct prime numbers and let $H = \left[\frac{1}{q}\right]^+$. Then $H \neq pH$, so $H = \langle h \rangle + pH$, for some $h \in H \setminus pH$, by Lemma 19. But H is not an SI_H-group by Theorem 21, so it follows from Lemma 6 that $G = Z(p) \oplus H$ is not an SI_H-group.

Lemma 23 (R. R. Andruszkiewicz and M. Woronowicz)

Let both A and H be abelian groups. If A is not a nil-group and A is a homomorphic image of H, then $A \oplus H$ is not an SI-group.

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Proof

Let $f\colon H \to A$ be an epimorphism. Let R be a ring such that $R^+ = A$ and $R^2 \neq \{0\}$. It is easy to check that the operation $*\colon (A \oplus H) \times (A \oplus H) \to (A \oplus H)$ defined by:

$$(a_1, x_1) * (a_2, x_2) = \big(a_1 f(x_2) + a_2 f(x_1), 0\big), \text{ for all } a_1, a_2 \in A,\ x_1, x_2 \in H$$

is a ring multiplication on the group $A \oplus H$. Since $R^2 \neq \{0\}$, there exist $a, b \in A$ such that $a \cdot b \neq 0$. Let $y \in f^{-1}\big(\{b\}\big)$. Then $(0, y)^2 = (0, 0)$, so $\big[(0, y)\big] = \big\langle(0, y)\big\rangle$ and $(a, 0) * (0, y) = (a \cdot f(y), 0) = (a \cdot b, 0) \notin \big[(0, y)\big]$. Therefore $\big[(0, y)\big]$ is not an ideal of the ring R. □

Corollary 24

Let H be an abelian group such that $H \neq pH$, for some $p \in \mathbb{P}$. Then $Z(p) \oplus H$ is not an SI-group.

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Remark

From the above corollary and Example 20 it follows, that the class of all SI-group is a proper subclass of the class of all SI_H-group.

Corollary 24

Let H be an abelian group such that $H \neq pH$, for some $p \in \mathbb{P}$. Then $Z(p) \oplus H$ is not an SI-group.

Remark

From the above corollary and Example 20 it follows, that the class of all SI-group is a proper subclass of the class of all SI_H-group.

Consider the following statement:

Corollary 25 (S. Feigelstock)

Let G be an SI_H-group, and let p be a prime for which $G_p \neq \{0\}$. Then $G = G_p \oplus H$, for some subgroup H of G such that $H = pH$

Remark

It turns out that the ring multiplication constructed by S. Feigelstock in the proof of Corollary 25 (cf. Corollary 11 in [3]) is not associative. Therefore Feigelstock's proof provides the truth for this corollary for SI-groups referred to in the Definition 3. Example 20 shows, that Corollary 25 is false in the class of SI_H-grups. In addition $G_p = Z(p)$, by Remark 4 and Theorem 12, hence $H = pG$.
Also the other Feigelstock's results based on Corollary 25 are proved only for SI-groups and some of them are false in the class of SI_H-grups (cf. [2]).

References

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Thank you for your attention!