Generalized Płonka Sums

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Operations on algebras

Operations on algebras induced by operations on universes

- $\mathbf{T}$ - equational theory
- $Alg(\mathbf{T})$ - category of algebras of the theory $\mathbf{T}$

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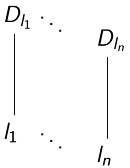

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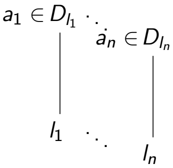

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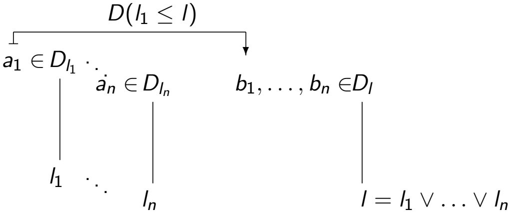

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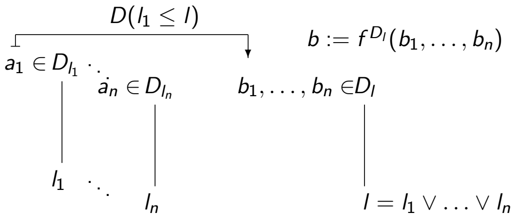

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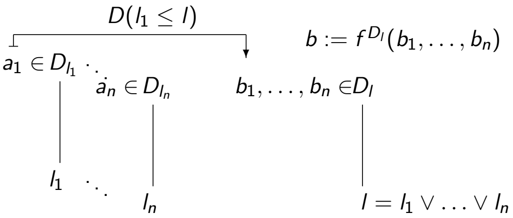

- $D(l_i \leq l) : D_{l_i} \to D_l$
- $b_i := D(l_i \leq l)(a_i) \in D_l$
- $f^{\coprod^P D}(a_1, \ldots, a_n) := b$

Płonka sums

Theorem [J. Płonka 1967]

If $\mathbf{T}$ is regular, then Płonka sum of $\mathbf{T}$ algebras is a $\mathbf{T}$-algebra.

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3. As Płonka sum is induced by an operation on universes of algebras, it is given by a morphism of monads. This allows us for some simplifications: to consider free algebras only and move between algebras over different categories (the rest will be taken care off by 'abstract nonsense').

Plan of the talk

Plan

1. The category of regular equational theories
2. Monads and their algebras
3. The category of semi-analytic monads
4. More on morphisms of monads
5. Category of regular polynomials over an algebra
6. Morphism of monads that induce (generalized) Płonka sums
7. Examples

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- $\mathbf{T} = \langle L, A \rangle$ is a *regular equational theory*, if A is a set of regular equations in contexts over signature L.
- A *regular interpretation of regular equational theories* $\mathbf{I} : \mathbf{T} \to \mathbf{T}'$ sends n-ary symbols f in L to regular terms in contexts $\mathbf{I}(f) : \vec{x}^n$ in $\mathbf{T}'$ so that for any equation $s = t : \vec{x}^n$ in $\mathbf{T}$ we have

$$T \vdash \mathbf{I}(s) = \mathbf{I}(t) : \vec{x}^n$$

Monads (on *Set*)

For any equational theory **T**, the forgetful functor $\mathcal{U}$ has a left adjoint $\mathcal{F}$, the free **T**-algebra functor:

$$Alg(\mathbf{T}) \underset{\mathcal{F}}{\overset{\mathcal{U}}{\rightleftarrows}} Set$$

Thus $\mathcal{F}(X)$ is the free **T**-algebra on the set X. It can be constructed as the set of terms with additional constants from the set X divided by the provable equality in theory T. The unit of the adjunction

$$\eta_X : X \to \mathcal{U}\mathcal{F}(X)$$

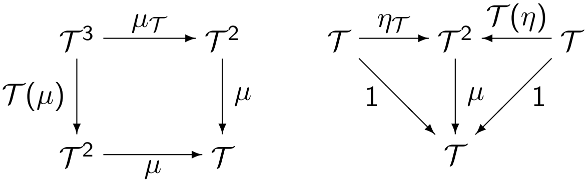
is an embedding of generators, and the counit

$$\varepsilon_A : \mathcal{F}\mathcal{U}(A) \to A$$

is an evaluation of the terms over the universe of A in algebra A.

Monads (on *Set*)
definition

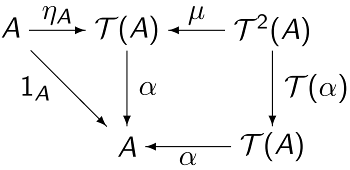
The composed endofunctor $\mathcal{T} = \mathcal{U}\mathcal{F} : Set \to Set$ together with natural transformations $\eta : 1_{Set} \to \mathcal{T}$ and $\mu = \mathcal{U}\varepsilon\mathcal{F} : \mathcal{T}^2 \to \mathcal{T}$ make the diagrams



commute. The left square expresses the fact that 'evaluation commutes with substitution'.
A *monad* on the category *Set* is an endofunctor $\mathcal{T}$ on *Set* together with two natural transformations η and μ as above making the above diagrams commute.

Algebras for monads

An *algebra for a monad* $\mathcal{T}$ or $\mathcal{T}$-algebra is a set A (the universe of the algebra) together with a function (structure map) $\alpha : \mathcal{T}(A) \to A$ such that



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$$\begin{array}{ccccc} A & \xrightarrow{\eta_A} & \mathcal{T}(A) & \xleftarrow{\mu} & \mathcal{T}^2(A) \\ & \searrow^{1_A} & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \mathcal{T}(\alpha)} \\ & & A & \xleftarrow[\alpha]{} & \mathcal{T}(A) \end{array}$$

commutes. A *morphism of $\mathcal{T}$-algebras* $h : (A, \alpha) \to (A', \alpha')$ is a function $h : A \to A'$ compatible with the structure maps.

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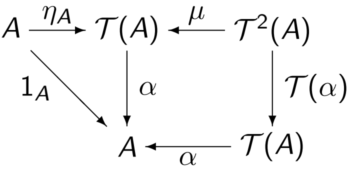


commutes. A *morphism of $\mathcal{T}$-algebras* $h : (A, \alpha) \to (A', \alpha')$ is a function $h : A \to A'$ compatible with the structure maps. So we have the categories of algebras for monad $Alg(\mathcal{T})$ as well.

The monads $(\mathcal{R}, \eta, \mu)$ arising from regular equational theories **T** are more special then arbitrary monads on *Set*. They are characterized by some additional abstract conditions (finitary, preserves pullbacks along monomorphisms). They have much more specific presentations. There is a functor

$$R : \mathbb{S} \to Set$$

where $\mathbb{S}$ of finite sets $\underline{n} = \{1, \ldots, n\}$ and surjections to *Set* ($R(\underline{n})$ *is* the set of regular terms in context $\vec{x}^n$ divided by provable equations), for set X,

$$\mathcal{R}(X) = \sum_{n \in \omega} \left[\begin{array}{c} X \\ n \end{array} \right] \otimes_n R_n$$

where

- $\left[\begin{array}{c} X \\ n \end{array} \right]$ is the set of monomorphisms from $\underline{n}$ to X
- $\otimes_n$ is the tensor over symmetric group S_n.

A *morphism of monads* $\tau : (\mathcal{T}, \eta, \mu) \to (\mathcal{T}', \eta', \mu')$ is a natural transformation $\tau : \mathcal{T} \to \mathcal{T}'$ compatible with η's and μ's, i.e. the diagram

$$\begin{array}{ccccc} 1_{Set} & \xrightarrow{\eta} & \mathcal{T} & \xleftarrow{\mu} & \mathcal{T}^2 \\ & \searrow^{\eta'} & \downarrow{\tau} & & \downarrow{\tau * \tau} \\ & & \mathcal{T}' & \xleftarrow[\mu']{} & \mathcal{T}'^2 \end{array}$$

commutes.

It induces a functor between categories of algebras:

$$Alg(\tau) : Alg(\mathcal{T}') \longrightarrow Alg(\mathcal{T})$$

$$(A, \alpha : \mathcal{T}'(A) \to A) \mapsto (A, \alpha \circ \tau_A : \mathcal{T}(A) \to A)$$

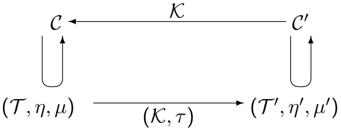
A *morphism of semi-analytic monads* τ is a morphism of monads such that naturality squares for monomorphisms are pullbacks.

Regular theories vs semi-analytic monads

Theorem (S. Szawiel, MZ)

The category **SanMnd** of semi-analytic monads on *Set* is equivalent to the category of regular theories **RegET**. This correspondence respects categories of algebras.

We can consider morphism between monads defined on different categories. They induce functors between categories of algebras.



1. $\mathcal{K} : \mathcal{C}' \to \mathcal{C}$ a functor
2. $\tau : \mathcal{T}\mathcal{K} \to \mathcal{K}\mathcal{T}'$ a natural transformation
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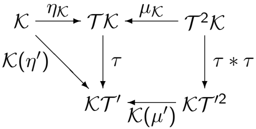

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is given by

$$\overline{\mathcal{K}}(A, \alpha : \mathcal{T}'(A) \to A) = (\mathcal{K}(A), \mathcal{T}\mathcal{K}(A) \stackrel{\tau_A}{\longrightarrow} \mathcal{K}\mathcal{T}'(A) \stackrel{\mathcal{K}(\alpha)}{\longrightarrow} \mathcal{K}(A))$$

Lift of a monad to the category of diagrams

If $(\mathcal{T}, \eta, \mu)$ is a monad on *Set* and C is a small category, then we have a monad $(\hat{\mathcal{T}}, \hat{\eta}, \hat{\mu})$ on Set^C, the *lift of the monad* $\mathcal{T}$ *to* Set^C. It is defined by composition, for a functor $F : C \to Set$:

$$\hat{\mathcal{T}}(F) = \mathcal{T} \circ F, \quad \hat{\eta}_F = \eta_F : F \to \mathcal{T} \circ F, \quad \hat{\mu}_F = \mu_F : \mathcal{T}^2 \circ F \to \mathcal{T} \circ F$$

where $\hat{\eta}_F$ is the component of the natural transformation $\hat{\eta}$ at a functor F and η_F is the wiskering of the natural transformation $\eta : 1_{Set} \to \mathcal{T}$ along the functor F; the same applies to the definition of $\hat{\mu}$.
The category of algebras for the lifted monads $Alg(\hat{\mathcal{T}})$ is equivalent to the category of C-diagrams of algebras for $\mathcal{T}$, i.e. $Alg(\mathcal{T})^C$.

Let $\mathcal{T} = (\mathcal{T}, \eta, \mu)$ be a semi-analytic monad, $T : \mathbb{S} \to Set$ the coefficient functor for $\mathcal{T}$ so that for any set X

$$\mathcal{T}(X) = \sum_{n\in\omega} \left[\begin{array}{c} X \\ n \end{array} \right] \otimes_n T_n$$

$(A, \alpha : \mathcal{T}(A) \to A)$ and $\mathcal{T}$-algebra. We define a *category* **A** *of regular polynomials over* $\mathcal{T}$*-algebra* A.
The objects of **A** are elements of A. A morphism in **A** is an equivalence class of triples

$$[\vec{a}, i, r]_\sim : \vec{a}(i) \to \alpha([\vec{a}, r]_\sim)$$

where $\vec{a} : \underline{n} \to A$ is an injection, $i \in \underline{n}$, $r \in T_n$, for some $n \in \omega$. Note that $[\vec{a}, r]_\sim$ is an element of $\mathcal{T}(A)$. We identify triples

$$\langle \vec{a} \circ \sigma, i, r\rangle \sim \langle \vec{a}, \sigma(i), T(\sigma)(r)\rangle$$

where $\sigma \in S_n$.

Generalized Płonka sum

- $\pi : \mathcal{R} \to \mathcal{T}$ a morphism of semi-analytic monads, defined by a natural transformation $\pi : R \to T$ in $Set^{\mathbb{S}}$
- (A, α) be a $\mathcal{T}$-algebra
- $\mathbf{A}$ the category of regular polynomials over (A, α)
- $\hat{\mathcal{R}}$ is the lift of the monad $\mathcal{R}$ to the category $Set^{\mathbf{A}}$

We shall define a morphism of monads

$$(\bigsqcup_{\mathbf{A}}, \lambda) : \mathcal{R} \to \hat{\mathcal{R}}$$

that gives rise to an operation

$$Alg(\mathcal{R})^{\mathbf{A}} \cong Alg(\hat{\mathcal{R}}) \longrightarrow Alg(\mathcal{R})$$

Let $\mathbf{F} : \mathbf{A} \to Alg(\mathcal{R})$ be a functor, and $F : \mathbf{A} \to Set$ the composition of $\mathbf{F}$ with the forgetful functor. We shall define the component

$$\lambda_F : \mathcal{R}(\coprod_{a\in A} F(a)) \longrightarrow \coprod_{a\in A} \mathcal{R}(F(a))$$

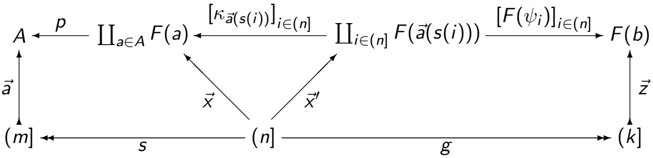
of λ

$$\lambda : \mathcal{R} \circ \bigsqcup{}_{\mathbf{A}} \longrightarrow \bigsqcup{}_{\mathbf{A}} \circ \hat{\mathcal{R}}$$

- $[\vec{x}, r]_\sim \in \mathcal{R}(\coprod_{a\in A} F(a))$
- $\vec{x} : (n] \to \coprod_{a\in A} F(a)$ is an injection
- $r \in R_n$
- $p : \coprod_{a\in A} F(a) \to A$ the projection from the coproduct to the index set

Generalized Płonka sum

definition



- $\vec{a} \circ s$ a surjection-injection factorization of $p \circ \vec{x}$
- $\kappa_a : F(a) \to \coprod_{a\in A} F(a)$ is the injection into coproduct
- $\vec{x}'$ - the unique making the triangle in the middle commute
- We put

$$b = \alpha([\vec{a}, \pi_m(R(s)(r))])$$

and, for $i \in (n]$, we have a morphism

$$\psi_i = [\vec{a}, s(i), \pi_m(R(s)(r))] : \vec{a}(s(i)) \longrightarrow b$$

in the category $\mathbf{A}$.

Generalized Płonka sum

Finally, we put

$$\lambda^{\mathcal{R}}_{F}([\vec{x}, r]_{\sim}) = [\vec{z}, R(g)(r)]_{\sim}$$

Theorem

$(\bigsqcup_A, \lambda^{\mathcal{R}}) : \mathcal{R} \to \hat{\mathcal{R}}$ is a lax morphism of monads. $\Box$

Examples

1. Identity interpretation $1 : \mathcal{R} \to \mathcal{R}$ of a semi-analytic monad $\mathcal{R}$. Płonka sum of identity interpretation over an $\mathcal{R}$-algebra A of a constant diagram F equal to an $\mathcal{R}$-algebra B is the product $A \times B$.
2. The usual Płonka sum comes from the unique semi-analytic interpretation of a $\mathcal{R} \to \mathcal{S}$, of any semi-analytic monad in the monda for sup-semilattices.
3. More sophisticated examples. Let $2\mathcal{S}$ be the monad corresponding to the theory of two theories of sup-semilattices taken together. Let $\mathbf{R}$ be the monad arising from a regular theory that is a 'sum' of two regular equational theories (having nothing to do one with the other). Then we have morphism of semi-analytic monads $\mathcal{R} \to 2\mathcal{S}$ such that the two parts of $\mathcal{R}$ are interpreted in different parts of $2\mathcal{S}$.