EXOTIC BARYCENTRIC ALGEBRAS

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REAL AFFINE SPACES

Given a vector space (a module) A over a field (a subring R of) $\mathbb{R}$.

An **affine space** A **over** R (or **affine** R-**space**) is the algebra

$$\left(A, \sum_{i=1}^{n} x_i r_i \,\middle|\, \sum_{i=1}^{n} r_i = 1\right).$$

This algebra is equivalent to

$(A, P, \underline{R})$, (or to $(A, \underline{R})$ if 2 is invertible in R),

where

$$\underline{R} = \{\underline{r} \mid r \in R\}, \text{ and } xyzP = x - y + z$$

and

$$xy\underline{r} = \underline{r}(x, y) = x(1 - r) + yr.$$

The class $\underline{\underline{R}}$ of all affine R-spaces is a variety.

Abstractly, $\underline{\underline{R}}$ is defined as the class of idempotent entropic Mal'cev algebras $(A, P, \underline{R})$ with a ternary Mal'cev operation P and binary operations $\underline{r}$ for each $r \in R$, satisfying the identities:

$$\begin{aligned} &xy\underline{0} = x = yx\underline{1}, \\ &xy\underline{p}\ xy\underline{q}\ \underline{r} = xy\ \underline{pq\underline{r}}, \\ &xy\underline{p}\ xy\underline{q}\ xy\underline{r}\ P = xy\ \underline{pqrP}. \end{aligned}$$

for all $p, q, r \in R$.

The variety $\underline{\underline{R}}$ satisfies also the **entropic** identities

$$xy\underline{p}\ zt\underline{p}\ \underline{q} = xz\underline{q}\ yt\underline{q}\ \underline{p}$$

for all $p, q \in R$ and the **cancellation laws**

$$(xy\underline{p} = xz\underline{p}) \to y = z$$

for all $p \in R$ with $p \neq 0$.

CONVEX SETS and BARYCENTRIC ALGEBRAS

Let F be a subfield of $\mathbb{R}$,
$I^o(F) :=]0,1[= (0,1) \subset F$ and
$I(F) := [0,1] \subset F$.

Convex subsets of affine F-spaces are $I^o(F)$-subreducts $(A, \underline{I}^o(F))$ of F-spaces.

The class $\mathcal{C}v(F)$ of convex sets generates the variety $\mathcal{B}(F)$ of F-**barycentric algebras** axiomatized by the following:

idempotence: $xx\underline{p} = x$
(I),

skew-commutativity: $xy\underline{p} = xy\underline{1-p} =: xy\underline{p}'$
(SC),

skew-associativity: $xy\underline{p}\,z\,\underline{q} = x\,yz\underline{q/(p\circ q)}\,\underline{p\circ q}$
(SA)
for all $p, q \in I^o$. Note that
$p \circ q = (p'q')' = p + q - pq$.

MODES

An algebra (A, Ω) is a **mode** if it is

- **idempotent**:

$$x...x\omega = x,$$

  for each n-ary $\omega \in \Omega$, and

- **entropic**:

$$(x_{11}...x_{1n}\omega)...(x_{m1}...x_{mn}\omega)\,\varphi$$

$$= (x_{11}...x_{m1}\varphi)...(x_{1n}...x_{mn}\varphi)\,\omega.$$

  for all $\omega, \varphi \in \Omega$.

Affine R-spaces and their subreducts (subalgebras of reducts) are modes. In particular, F-barycentric algebras are modes.

F-BARYCENTRIC ALGEBRAS

Theorem The class $\mathcal{C}v(F)$ and the quasivariety $\mathcal{C}(F)$ of cancellative F-barycentric algebras coincide.

Proposition The following conditions are equivalent for any non-trivial subalgebra $(A, \underline{I}^o(F))$ of $(F, \underline{I}^o(F))$:

(a) $(A, \underline{I}^o(F))$ is a line segment of $(F, \underline{I}^o(F))$;

(b) $(A, \underline{I}^o(F))$ is isomorphic to $(I(F), \underline{I}^o(F))$;

(c) $(A, \underline{I}^o(F))$ is generated by two (different) elements;

(d) $(A, \underline{I}^o(F))$ is a free algebra on two free generators in the quasivariety $\mathcal{C}(F)$ and in the variety $\mathcal{B}(F)$.

NOTE: The algebra $(I(F), \underline{I}^o(F))$ embeds into each non-trivial F-convex set.

Proposition The quasivariety $\mathcal{C}(F) = \mathcal{C}v(F)$ of F-convex sets is a minimal subquasivariety of the variety $\mathcal{B}(F)$.

In particular, $\mathcal{C}(F)$ is generated by any one of $(F, \underline{I}^o(F))$ and $(I(F), \underline{I}^o(F))$.

Proposition Let R be a (unital) subring of $\mathbb{R}$. Then the free algebra over X in the quasivariety of subreducts of a given type τ of affine R-spaces is isomorphic to the τ-subreduct, generated by X, of the free affine R-space over X.

The set of elements of the free $\underline{I}^o(R)$-algebra over $X = \{x_0, \ldots, x_n\}$ coincides with the n-dimensional **simplex** $S_n(R)$ over R:

$$\{x_0 a_0 + \cdots + x_n a_n \mid a_i \in I(R),\ \sum_{i=1}^{n} a_i = 1\}.$$

DYADIC CONVEX SETS

Consider the ring

$$\mathbb{D} = \mathbb{Z}[1/2] = \{m2^{-n} \mid m, n \in \mathbb{Z}\}$$

of **dyadic rational numbers**.

A **dyadic convex set** is the intersection of a real convex set with the space $\mathbb{D}^k$.

• Dyadic convex sets are subreducts $(A, \underline{I}^o(\mathbb{D}))$ of affine $\mathbb{D}$-spaces.

Proposition Each dyadic convex set $(A, \underline{I}^o(\mathbb{D}))$ is equivalent to $(A, \cdot) = (A, \frac{1}{2}(x + y))$.

The operation $\cdot$ is:

idempotent: $x \cdot x = x$,
commutative: $x \cdot y = y \cdot x$,
entropic (medial): $(x \cdot y) \cdot (z \cdot t) = (x \cdot z) \cdot (y \cdot t)$.

Hence the dyadic convex sets are **commutative binary modes** (or CB-modes).

REAL VERSUS DYADIC

• Not each dyadic interval is isomorphic to the interval $I(\mathbb{D})$, and not each is generated by its endpoints.

Example The dyadic interval $[0,3]$ is generated by no less than 3 elements, and is not isomorphic to $I(\mathbb{D})$. The minimal set of generators is given e.g. by the numbers $0,2,3$.

• There are infinitely many isomorphism types of dyadic intervals.

• The $\underline{I}^o(\mathbb{D})$-reduct of an affine $\mathbb{D}$-space $\mathbb{D}$ may not be an interval of $\mathbb{D}$.

• The quasivariety of convex subsets of affine $\mathbb{D}$-spaces forms a proper subclass of the quasivariety of cancellative commutative binary modes (barycentric algebras over $\mathbb{D}$).

• The dyadic unit interval $I(\mathbb{D})$ does not embed into each $\underline{I}^o(\mathbb{D})$-subreduct of an affine $\mathbb{D}$-space.

CONVEX SETS OVER SUBRINGS OF $\mathbb{R}$

Natural requirements for "convex subsets" of affine R-spaces, where $R \leq \mathbb{R}$:

• $R \neq \mathbb{Z}$ (we need non-trivial unit interval),
• excluding non-faithful affine R-spaces (so that "convex sets" embed into affine spaces over R but not over homomorphic images of R).

Suitable candidates:
principal ideal subdomains R of $\mathbb{R}$.

Advantages:
• well developed theory of such R-modules (and hence also affine R-spaces)
• nice characterizations of quasivarieties (Belkin) (In particular faithful affine R-spaces form a (minimal) quasivariety.)

R-CONVEX SETS

Definition Let R be a principal ideal subdomain of the ring $\mathbb{R}$ containing the ring $\mathbb{Z}$ of integers but different from $\mathbb{Z}$. Then a subset C of an affine R-space $(A, P, \underline{R})$ is called an R-**convex set** if the affine space is faithful and C is an $\underline{I}^o(R)$-subreduct of $(A, P, \underline{R})$.

• The class of $\underline{I}^o(R)$-subreducts of faithful affine R-spaces is a quasivariety, denoted as $\mathcal{C}v(R)$.

• Two distinct points of an R-convex subset of an affine R-space may belong to more than one of its one-dimensional subspaces. (E.g. the subalgebra of $(\mathbb{D}, \underline{I}^o(\mathbb{D}))$ generated by 0 and 3 is a $\mathbb{D}$-convex set but it does not contain all points of the $\mathbb{D}$-line $(\mathbb{D}, \underline{\mathbb{D}})$ lying between 0 and 3.)

GEOMETRIC R-CONVEX SETS

Proposition The quasivariety $\mathcal{C}v(R)$ is generated by the algebra $(R, \underline{I}^o(R))$, and hence, it is minimal.

Definition For $a \neq b$ in an affine R-space A, let
$\ell(a,b) = \{ab\underline{r} \mid r \in R\}$.
For $c, d \in \ell(a,b)$, with $c \leq d$, the **segment** of $\ell(a,b)$ joining c and d is the set

$$[c,d]_{\ell(a,b)} := \{x \in \ell(a,b) \mid c \leq x \leq d\}.$$

An R-convex subset C of a faithful affine R-space A is **geometric**, if for all $a, b \in A$ with $a \neq b$ and $c, d \in C$, if $c, d \in \ell(a,b)$, then $[c,d]_{\ell(a,b)} \subseteq C$.

Proposition Let C be an R-convex subset of the affine space R^k. Then the following condition are equivalent.

• C is a geometric convex subset of R^k;

• C is the intersection of R^k and the convex hull $ch_{\mathbb{R}}(C)$ of C in $\mathbb{R}^k$, in fact

$$(C, \underline{I}^o(R)) = (ch_{\mathbb{R}}(C), \underline{I}^o(R)) \cap (R^k, \underline{I}^o(R));$$

• C is the intersection of R^k and some convex subset of $\mathbb{R}^k$.

Corollary The class of geometric R-convex sets generates the quasivariety $\mathcal{C}v(R)$ of R-convex sets.

BARYCENTRIC ALGEBRAS (B, I)

Let $I = I(\mathbb{R})$ and $I^o = I^o(\mathbb{R})$.
Convex sets as algebras $(B, \underline{I})$:
(I) and (SC) hold, but (SA) is not defined for $p \circ q = 0$.

Define a new binary operation $\rightarrow$ on I:

$$p \rightarrow q = \begin{cases} q/p & \text{if } p > q; \\ 1 & \text{otherwise.} \end{cases}$$

Then for all $p, q \in I^o$, $q < q1\underline{p} = p \circ q$ and

$$q/(p \circ q) = p \circ q \rightarrow q.$$

And for all $p, q \in I$, (SA) can be written as:

$$xy\underline{p}\, z\underline{q} = x\, (yz\underline{p \circ q \rightarrow q})\, \underline{p \circ q} \qquad \text{(SA}'\text{)}$$

HOW TO AXIOMATIZE I^o or I ?

Barycentric algebras were defined as algebras $(B, \underline{I}^o)$ or $(B, \underline{I})$ satisfying certain identities. However the intervals I^o and I were not axiomatized in an abstract way. Hence the following two questions.

1. How to axiomatize I^o or I?

2. How to extend the definition of barycentric algebras to include barycentric algebras over algebras axiomatizing I?

Note that the operations of I needed in the axiomatization of barycentric algebras are the arithmetical operations $+, \cdot, ', /$ and linear ordering restricted to I.

ŁΠ-ALGEBRAS

ŁΠ-algebras were introduced by F. Montagna, F. Esteva and L. Godo as an algebraization of the so-called ŁΠ-logic. This logic results from the combination of Łukasiewicz and product logics, two of the main fuzzy logics.

An **ŁΠ-algebra** is an algebra

$$(A, \oplus, \neg, \cdot_\pi, \to_\pi, 0, 1),$$

where $(A, \oplus, \neg, 0, 1)$ is an MV-**algebra**, and $(A, \cdot_\pi, \to_\pi, 1)$, is a **product algebra** (a commutative monoid with residuation), satisfying certain additional identities.

Recall: MV-algebras are algebras of infinitely-valued Łukasiewicz logic and product algebras are algebras of product logic.

Each ŁΠ-algebra has also a structure of a distributive lattice and satisfies

$$x \cdot_\pi (x \to_\pi y) = x \wedge y.$$

Typical examples are given by

- Boolean algebras, where
  $\vee = \oplus$, $\wedge = \cdot_\pi$, $x \to y = x \to_\pi y$, $\neg ='$,

- interval ŁΠ-algebras $(I, \oplus, \neg, \cdot_\pi, \to_\pi, 0, 1)$, where

$$\begin{aligned} \neg x &:= 1 - x\,; \\ x \oplus y &:= 1 \wedge (x + y)\,; \\ x \cdot_\pi y &:= x \cdot y\,; \\ x \to_\pi y &:= \ \mathbf{if}\, x \le y\, \mathbf{then}\, 1\, \mathbf{else}\, y/x\,. \end{aligned}$$

ŁΠ-algebras form a variety, generated by the interval ŁΠ-algebras and the Boolean algebra $\mathbf{2}$.

ABSTRACT BARYCENTRIC ALGEBRAS

An **abstract barycentric algebra** is a two sorted algebra $(A, J, F \sqcup \{t\})$ with two sorts A and J, the set

$$F = \{\oplus, \neg, \cdot_\pi, \rightarrow_\pi, 0, 1\}$$

of operations defined on J with values in J, and one ternary operation

$$t : A \times A \times J \to A; (x, y, p) \mapsto xy\underline{p} =: \underline{p}(y, x)$$

such that:

(A) (J, F) is an ŁΠ-algebra;

(B) the operation t satisfies the following identities for $x, y \in A$ and $p \in J$:

$$\underline{0}\,(x, y) = y = \underline{1}(y, x)\,;$$
$$\underline{p}\,(x, x) = x\,;$$
$$\underline{p}\,(x, y) = \underline{\neg p}\,(y, x)\,;$$
$$\underline{p}\,(x, \underline{q}\,(y, z)) = \underline{p \circ q}\,((\underline{p \circ q \rightarrow_\pi q})(x, y), z)\,.$$

The derived operation $\circ$ is defined by $p \circ q := \neg((\neg p) \cdot_\pi (\neg q))$.

MAIN EXAMPLES

Each barycentric algebra $(A, \underline{I})$ can be considered as abstract barycentric algebra $(A, J = I)$, where (I, F) is an interval ŁΠ-algebra described before.

Each abstract barycentric algebra (A, J) has a subalgebra $(A, \mathbf{2})$, where $\mathbf{2}$ is a two element Boolean algebra.

Proposition Each barycentric algebra $(A, \underline{I})$ (a homomorphic image of a convex set) has an abstract counterpart (a homomorphic image of the counterpart of this convex set.)

Much of the theory of barycentric algebras $(A, \underline{I})$ curry over to abstract barycentric algebras (A, I).

FURTHER EXAMPLES

Proposition Binary operations of Boolean affine spaces satisfy the identities (I), (SC) and (SA) defining barycentric algebras.

Proposition The binary reducts $(A, J = B)$ of affine spaces over a Boolean ring B are abstract barycentric algebras.

Subalgebras of such binary reducts form the variety of so-called B-**sets** investigated by G. Bergman and T. Stokes. They all can be viewed as abstract barycentric algebras.

Certain B-sets extended by a semilattice operation form modes equivalent to **if-then-else-algebras** of E. G. Manes.

Finally, **rectangular modes** (investigated by R. Pöschel and M. Reichel) can be shown to be equivalent to some B-sets, whence they also can be viewed as abstract barycentric algebras.

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