A survey of barycentric algebras

Jonathan D.H. Smith

Department of Mathematics,

Iowa State University,

Ames, IA 50011-2064, U.S.A.

email: jdhsmith@iastate.edu

http://orion.math.iastate.edu/jdhsmith/homepage.html

ALGEBRAS

ALGEBRAS

Operator domain Ω, **type** $\tau : \Omega \to \mathbb{N}; \omega \mapsto \omega\tau$

ALGEBRAS

Operator domain Ω, **type** $\tau : \Omega \to \mathbb{N}; \omega \mapsto \omega\tau$

For $\omega \in \Omega$, **algebra** (A, Ω) of type τ

has $\omega\tau$-ary (**basic**) **operation**

$$\omega : A^{\omega\tau} \to A; (x_1, \ldots, x_{\omega\tau}) \mapsto x_1 \ldots x_{\omega\tau}\omega$$

ALGEBRAS

Operator domain Ω, **type** $\tau : \Omega \to \mathbb{N}; \omega \mapsto \omega\tau$

For $\omega \in \Omega$, **algebra** (A, Ω) of type τ

has $\omega\tau$-ary (**basic**) **operation**

$$\omega : A^{\omega\tau} \to A; (x_1, \ldots, x_{\omega\tau}) \mapsto x_1 \ldots x_{\omega\tau}\omega$$

Subset B of A.

ALGEBRAS

Operator domain Ω, **type** $\tau : \Omega \to \mathbb{N}; \omega \mapsto \omega\tau$

For $\omega \in \Omega$, **algebra** (A, Ω) of type τ

has $\omega\tau$-ary (**basic**) **operation**

$\omega : A^{\omega\tau} \to A; (x_1, \ldots, x_{\omega\tau}) \mapsto x_1 \ldots x_{\omega\tau}\omega$

Subset B of A. If $\forall\, \omega \in \Omega,\ \forall\, x_1, \ldots, x_{\omega\tau} \in A,\ \ldots$

ALGEBRAS

Operator domain Ω, **type** $\tau : \Omega \to \mathbb{N}; \omega \mapsto \omega\tau$

For $\omega \in \Omega$, **algebra** (A, Ω) of type τ

has $\omega\tau$-ary (**basic**) **operation**

$\omega : A^{\omega\tau} \to A; (x_1, \ldots, x_{\omega\tau}) \mapsto x_1 \ldots x_{\omega\tau}\omega$

Subset B of A. If $\forall\, \omega \in \Omega,\ \forall\, x_1, \ldots, x_{\omega\tau} \in A,\ \ldots$

$[\forall\, 1 \leq i \leq \omega\tau,\ x_i \in B] \Rightarrow x_1 \ldots x_{\omega\tau}\omega \in B$, then B is a **subalgebra**;

ALGEBRAS

Operator domain Ω, **type** $\tau : \Omega \to \mathbb{N}; \omega \mapsto \omega\tau$

For $\omega \in \Omega$, **algebra** (A, Ω) of type τ

has $\omega\tau$-ary (**basic**) **operation**

$\omega : A^{\omega\tau} \to A; (x_1, \ldots, x_{\omega\tau}) \mapsto x_1 \ldots x_{\omega\tau}\omega$

Subset B of A. If $\forall\, \omega \in \Omega,\ \forall\, x_1, \ldots, x_{\omega\tau} \in A,\ \ldots$

$[\forall\, 1 \leq i \leq \omega\tau,\ x_i \in B] \Rightarrow x_1 \ldots x_{\omega\tau}\omega \in B$, then B is a **subalgebra**;

$[\exists\, 1 \leq i \leq \omega\tau.\ x_i \in B] \Rightarrow x_1 \ldots x_{\omega\tau}\omega \in B$, then B is a **sink**;

ALGEBRAS

Operator domain Ω, **type** $\tau : \Omega \to \mathbb{N}; \omega \mapsto \omega\tau$

For $\omega \in \Omega$, **algebra** (A, Ω) of type τ

has $\omega\tau$-ary (**basic**) **operation**

$\omega : A^{\omega\tau} \to A; (x_1, \ldots, x_{\omega\tau}) \mapsto x_1 \ldots x_{\omega\tau}\omega$

Subset B of A. If $\forall\, \omega \in \Omega,\ \forall\, x_1, \ldots, x_{\omega\tau} \in A,\ \ldots$

$[\forall\, 1 \leq i \leq \omega\tau,\ x_i \in B] \Rightarrow x_1 \ldots x_{\omega\tau}\omega \in B$, then B is a **subalgebra**;

$[\exists\, 1 \leq i \leq \omega\tau.\ x_i \in B] \Rightarrow x_1 \ldots x_{\omega\tau}\omega \in B$, then B is a **sink**;

$[\forall\, 1 \leq i \leq \omega\tau,\ x_i \in B] \Leftrightarrow x_1 \ldots x_{\omega\tau}\omega \in B$, then B is a **wall**.

MODES

MODES

Algebra (A, Ω) is . . .

MODES

Algebra (A, Ω) is ...

... **idempotent** if $\forall\, a \in A$, singleton $\{a\}$ is a subalgebra,

MODES

Algebra (A, Ω) is $\ldots$

$\ldots$ **idempotent** if $\forall\, a \in A$, singleton $\{a\}$ is a subalgebra,

i.e. $\forall\, \omega \in \Omega\,,\ a \ldots a\omega = a$;

MODES

Algebra (A, Ω) is $\ldots$

$\ldots$ **idempotent** if $\forall\, a \in A$, singleton $\{a\}$ is a subalgebra,

i.e. $\forall\, \omega \in \Omega\,,\; a \ldots a\omega = a$;

$\ldots$ **entropic** if $\forall\, \omega \in \Omega\,,\; \omega : (A^{\omega\tau}, \Omega) \to (A, \Omega)$ is a homomorphism;

MODES

Algebra (A,Ω) is $\ldots$

$\ldots$ **idempotent** if $\forall\, a \in A$, singleton $\{a\}$ is a subalgebra,

i.e. $\forall\, \omega \in \Omega\,,\ a \ldots a\omega = a$;

$\ldots$ **entropic** if $\forall\, \omega \in \Omega\,,\ \omega : (A^{\omega\tau}, \Omega) \to (A,\Omega)$ is a homomorphism;

i.e. $\forall\, \omega, \omega' \in \Omega\,,$

$$\big(x_{11} \ldots x_{1(\omega'\tau)}\omega'\big) \ldots \big(x_{(\omega\tau)1} \ldots x_{(\omega\tau)(\omega'\tau)}\omega'\big)\,\omega$$
$$= \big(x_{11} \ldots x_{(\omega\tau)1}\omega\big) \ldots \big(x_{1(\omega'\tau)} \ldots x_{(\omega\tau)(\omega'\tau)}\omega\big)\,\omega'\ ;$$

MODES

Algebra (A,Ω) is ...

... **idempotent** if $\forall\, a \in A$, singleton $\{a\}$ is a subalgebra,

i.e. $\forall\, \omega \in \Omega\,,\ a \dots a\omega = a$;

... **entropic** if $\forall\, \omega \in \Omega\,,\ \omega : (A^{\omega\tau}, \Omega) \to (A,\Omega)$ is a homomorphism;

i.e. $\forall\, \omega, \omega' \in \Omega\,,$

$$\big(x_{11} \dots x_{1(\omega'\tau)}\omega'\big) \dots \big(x_{(\omega\tau)1} \dots x_{(\omega\tau)(\omega'\tau)}\omega'\big)\, \omega$$

$$= \big(x_{11} \dots x_{(\omega\tau)1}\omega\big) \dots \big(x_{1(\omega'\tau)} \dots x_{(\omega\tau)(\omega'\tau)}\omega\big)\, \omega' \ ;$$

... a **mode** if it's both idempotent and entropic.

MODES

Algebra (A,Ω) is ...

... **idempotent** if $\forall\, a \in A$, singleton $\{a\}$ is a subalgebra,

i.e. $\forall\, \omega \in \Omega\,,\; a \dots a\omega = a$;

... **entropic** if $\forall\, \omega \in \Omega\,,\; \omega : (A^{\omega\tau},\Omega) \to (A,\Omega)$ is a homomorphism;

i.e. $\forall\, \omega, \omega' \in \Omega\,,$

$$\big(x_{11} \dots x_{1(\omega'\tau)}\omega'\big) \dots \big(x_{(\omega\tau)1} \dots x_{(\omega\tau)(\omega'\tau)}\omega'\big)\, \omega$$
$$= \big(x_{11} \dots x_{(\omega\tau)1}\omega\big) \dots \big(x_{1(\omega'\tau)} \dots x_{(\omega\tau)(\omega'\tau)}\omega\big)\, \omega' \;;$$

... a **mode** if it's both idempotent and entropic.

REMARK: (A,Ω) is a mode iff each polynomial is a homomorphism.

MODES

Algebra (A, Ω) is ...

... **idempotent** if $\forall\, a \in A$, singleton $\{a\}$ is a subalgebra,

i.e. $\forall\, \omega \in \Omega\,,\ a \dots a\omega = a$;

... **entropic** if $\forall\, \omega \in \Omega\,,\ \omega : (A^{\omega\tau}, \Omega) \to (A, \Omega)$ is a homomorphism;

i.e. $\forall\, \omega, \omega' \in \Omega\,,$

$$\big(x_{11} \dots x_{1(\omega'\tau)}\omega'\big) \dots \big(x_{(\omega\tau)1} \dots x_{(\omega\tau)(\omega'\tau)}\omega'\big)\, \omega$$
$$= \big(x_{11} \dots x_{(\omega\tau)1}\omega\big) \dots \big(x_{1(\omega'\tau)} \dots x_{(\omega\tau)(\omega'\tau)}\omega\big)\, \omega' \ ;$$

... a **mode** if it's both idempotent and entropic.

REMARK: (A, Ω) is a mode iff each polynomial is a homomorphism.

PROPOSITION: Each subreduct of a mode (A, Ω) is a mode.

EXAMPLES OF MODES

EXAMPLES OF MODES

EXAMPLE: Sets are modes (empty Ω).

EXAMPLES OF MODES

EXAMPLE: Sets are modes (empty Ω).

EXAMPLE: A (meet) semilattice $(A, \cdot)$ is a mode.

EXAMPLES OF MODES

EXAMPLE: Sets are modes (empty Ω).

EXAMPLE: A (meet) semilattice $(A, \cdot)$ is a mode.

EXAMPLE: A left trivial semigroup $(A, \cdot)$ with $x \cdot y = x$ is a mode.

EXAMPLES OF MODES

EXAMPLE: Sets are modes (empty Ω).

EXAMPLE: A (meet) semilattice $(A, \cdot)$ is a mode.

EXAMPLE: A left trivial semigroup $(A, \cdot)$ with $x \cdot y = x$ is a mode.

EXAMPLE: Let E be a unital module over a commutative, unital ring R.

An R-linear combination $\sum_{i=1}^{n} x_i r_i$ is **affine** if $\sum_{i=1}^{n} r_i = 1$ in R.

Then E with all affine linear combinations (an **affine** R-**space**) is a mode.

EXAMPLES OF MODES

EXAMPLE: Sets are modes (empty Ω).

EXAMPLE: A (meet) semilattice $(A, \cdot)$ is a mode.

EXAMPLE: A left trivial semigroup $(A, \cdot)$ with $x \cdot y = x$ is a mode.

EXAMPLE: Let E be a unital module over a commutative, unital ring R.

An R-linear combination $\sum_{i=1}^{n} x_i r_i$ is **affine** if $\sum_{i=1}^{n} r_i = 1$ in R.

Then E with all affine linear combinations (an **affine** R-**space**) is a mode.

EXAMPLE: Convex sets are modes (subreducts of affine $\mathbb{R}$-spaces).

NEW MODES FROM OLD

NEW MODES FROM OLD

Modes (A, Ω), (B, Ω), (C, Ω).

NEW MODES FROM OLD

Modes (A, Ω), (B, Ω), (C, Ω).

PROPOSITION: The class of modes of a given type forms a variety.

NEW MODES FROM OLD

Modes (A, Ω), (B, Ω), (C, Ω).

PROPOSITION: The class of modes of a given type forms a variety.

COROLLARY: Products, quotients, limits and colimits

of modes of type τ form modes of type τ.

NEW MODES FROM OLD

Modes (A, Ω), (B, Ω), (C, Ω).

PROPOSITION: The class of modes of a given type forms a variety.

COROLLARY: Products, quotients, limits and colimits

of modes of type τ form modes of type τ.

PROPOSITION: $\mathrm{Hom}(A, B)$ is a submode of the power $(B, \Omega)^A$.

NEW MODES FROM OLD

Modes (A,Ω), (B,Ω), (C,Ω).

PROPOSITION: The class of modes of a given type forms a variety.

COROLLARY: Products, quotients, limits and colimits of modes of type τ form modes of type τ.

PROPOSITION: $\mathrm{Hom}(A,B)$ is a submode of the power $(B,\Omega)^A$.

PROPOSITION: The **tensor product** $A\otimes B$ is a mode, defined by the adjoint relation $\mathrm{Hom}\,(A\otimes B, C)\cong \mathrm{Hom}\,(A, \mathrm{Hom}\,(B,C))$.

SUBALGEBRA MODES: THE FRACTAL PROPERTY

SUBALGEBRA MODES: THE FRACTAL PROPERTY

For mode (A, Ω), consider the set AS of subalgebras of (A, Ω).

SUBALGEBRA MODES: THE FRACTAL PROPERTY

For mode (A, Ω), consider the set AS of subalgebras of (A, Ω).

PROPOSITION: (AS, Ω) is a mode under the **complex operations**

$$X_1 \dots X_{\omega\tau}\omega = \{x_1 \dots x_{\omega\tau}\omega \mid x_i \in X_i\}$$

SUBALGEBRA MODES: THE FRACTAL PROPERTY

For mode (A, Ω), consider the set AS of subalgebras of (A, Ω).

PROPOSITION: (AS, Ω) is a mode under the **complex operations**

$$X_1 \dots X_{\omega\tau}\omega = \{x_1 \dots x_{\omega\tau}\omega \mid x_i \in X_i\}$$

with monomorphism $\eta : (A, \Omega) \to (AS, \Omega); a \mapsto \{a\}$.

SUBALGEBRA MODES: THE FRACTAL PROPERTY

For mode (A, Ω), consider the set AS of subalgebras of (A, Ω).

PROPOSITION: (AS, Ω) is a mode under the **complex operations**

$$X_1 \dots X_{\omega\tau}\omega = \{x_1 \dots x_{\omega\tau}\omega \mid x_i \in X_i\}$$

with monomorphism $\eta : (A, \Omega) \to (AS, \Omega); a \mapsto \{a\}$.

COROLLARY: Set AP of **polytopes** (finitely generated subalgebras)

is a submode (AP, Ω) of (AS, Ω), with corestricted $\eta : A \to AP$.

SUBALGEBRA MODES: THE FRACTAL PROPERTY

For mode (A, Ω), consider the set AS of subalgebras of (A, Ω).

PROPOSITION: (AS, Ω) is a mode under the **complex operations**

$$X_1 \dots X_{\omega\tau}\omega = \{x_1 \dots x_{\omega\tau}\omega \mid x_i \in X_i\}$$

with monomorphism $\eta : (A, \Omega) \to (AS, \Omega); a \mapsto \{a\}$.

COROLLARY: Set AP of **polytopes** (finitely generated subalgebras) is a submode (AP, Ω) of (AS, Ω), with corestricted $\eta : A \to AP$.

PROPOSITION: Each mode (A, Ω) yields limit **fractal modes** (AS^∞, Ω) and (AP^∞, Ω) with endomorphisms $\eta : (AS^\infty, \Omega) \to (AS^\infty, \Omega)$ and $\eta : (AP^\infty, \Omega) \to (AP^\infty, \Omega)$.

MODALS

MODALS

A **modal** $(D, +, \Omega)$ is a (join) semilattice $(D, +)$ and mode (D, Ω)

MODALS

A **modal** $(D, +, \Omega)$ is a (join) semilattice $(D, +)$ and mode (D, Ω) with the distributivity $\forall\, \omega \in \Omega\,,\ \forall\, 1 \leq i \leq \omega\tau\,,$

$x_1 \ldots (x_i + x_i') \ldots x_{\omega\tau}\omega = x_1 \ldots x_i \ldots x_{\omega\tau}\omega + x_1 \ldots x_i' \ldots x_{\omega\tau}\omega$.

MODALS

A **modal** $(D,+,\Omega)$ is a (join) semilattice $(D,+)$ and mode (D,Ω)

with the distributivity $\forall\,\omega\in\Omega\,,\ \forall\,1\leq i\leq\omega\tau\,,$

$x_1\ldots(x_i+x_i')\ldots x_{\omega\tau}\omega=x_1\ldots x_i\ldots x_{\omega\tau}\omega+x_1\ldots x_i'\ldots x_{\omega\tau}\omega$.

EXAMPLE: For each mode (A,Ω), have modals $(AS,+,\Omega)$

and $(AP,+,\Omega)$, with join and complex products.

MODALS

A **modal** $(D,+,\Omega)$ is a (join) semilattice $(D,+)$ and mode (D,Ω) with the distributivity $\forall\,\omega\in\Omega\,,\ \forall\,1\leq i\leq\omega\tau\,,$

$$x_1\ldots(x_i+x_i')\ldots x_{\omega\tau}\omega = x_1\ldots x_i\ldots x_{\omega\tau}\omega + x_1\ldots x_i'\ldots x_{\omega\tau}\omega.$$

EXAMPLE: For each mode (A,Ω), have modals $(AS,+,\Omega)$ and $(AP,+,\Omega)$, with join and complex products.

May approximate elements of AS with elements of AP.

MODALS

A **modal** $(D, +, \Omega)$ is a (join) semilattice $(D, +)$ and mode (D, Ω)

with the distributivity $\forall\, \omega \in \Omega\,,\ \forall\, 1 \leq i \leq \omega\tau\,,$

$x_1 \dots (x_i + x_i') \dots x_{\omega\tau}\omega = x_1 \dots x_i \dots x_{\omega\tau}\omega + x_1 \dots x_i' \dots x_{\omega\tau}\omega$.

EXAMPLE: For each mode (A, Ω), have modals $(AS, +, \Omega)$

and $(AP, +, \Omega)$, with join and complex products.

May approximate elements of AS with elements of AP.

EXAMPLE: Distributive lattice $(L, +, \cdot)$, with absorption $\leq_+ = \leq_\cdot$;

MODALS

A **modal** $(D, +, \Omega)$ is a (join) semilattice $(D, +)$ and mode (D, Ω) with the distributivity $\forall\, \omega \in \Omega\,,\ \forall\, 1 \le i \le \omega\tau\,,$

$$x_1 \ldots (x_i + x_i') \ldots x_{\omega\tau}\omega = x_1 \ldots x_i \ldots x_{\omega\tau}\omega + x_1 \ldots x_i' \ldots x_{\omega\tau}\omega.$$

EXAMPLE: For each mode (A, Ω), have modals $(AS, +, \Omega)$ and $(AP, +, \Omega)$, with join and complex products.

May approximate elements of AS with elements of AP.

EXAMPLE: Distributive lattice $(L, +, \cdot)$, with absorption $\le_+ = \le_\cdot$; more generally **disemilattice** with potentially distinct $\le_+$ and $\le_\cdot$.

MODALS

A **modal** $(D, +, \Omega)$ is a (join) semilattice $(D, +)$ and mode (D, Ω) with the distributivity $\forall\, \omega \in \Omega\,,\ \forall\, 1 \leq i \leq \omega\tau\,,$

$$x_1 \ldots (x_i + x_i') \ldots x_{\omega\tau}\omega = x_1 \ldots x_i \ldots x_{\omega\tau}\omega + x_1 \ldots x_i' \ldots x_{\omega\tau}\omega.$$

EXAMPLE: For each mode (A, Ω), have modals $(AS, +, \Omega)$ and $(AP, +, \Omega)$, with join and complex products.

May approximate elements of AS with elements of AP.

EXAMPLE: Distributive lattice $(L, +, \cdot)$, with absorption $\leq_+ = \leq_\cdot$; more generally **disemilattice** with potentially distinct $\leq_+$ and $\leq_\cdot$.

EXAMPLE: For semilattice $(S, \cdot)$, **stammered semilattice** $(S, \cdot, \cdot)$.

PROPERTIES OF MODALS

PROPERTIES OF MODALS

Function $f : (A, \Omega) \to (D, +, \Omega)$ from mode to modal is (Ω)-**convex**

PROPERTIES OF MODALS

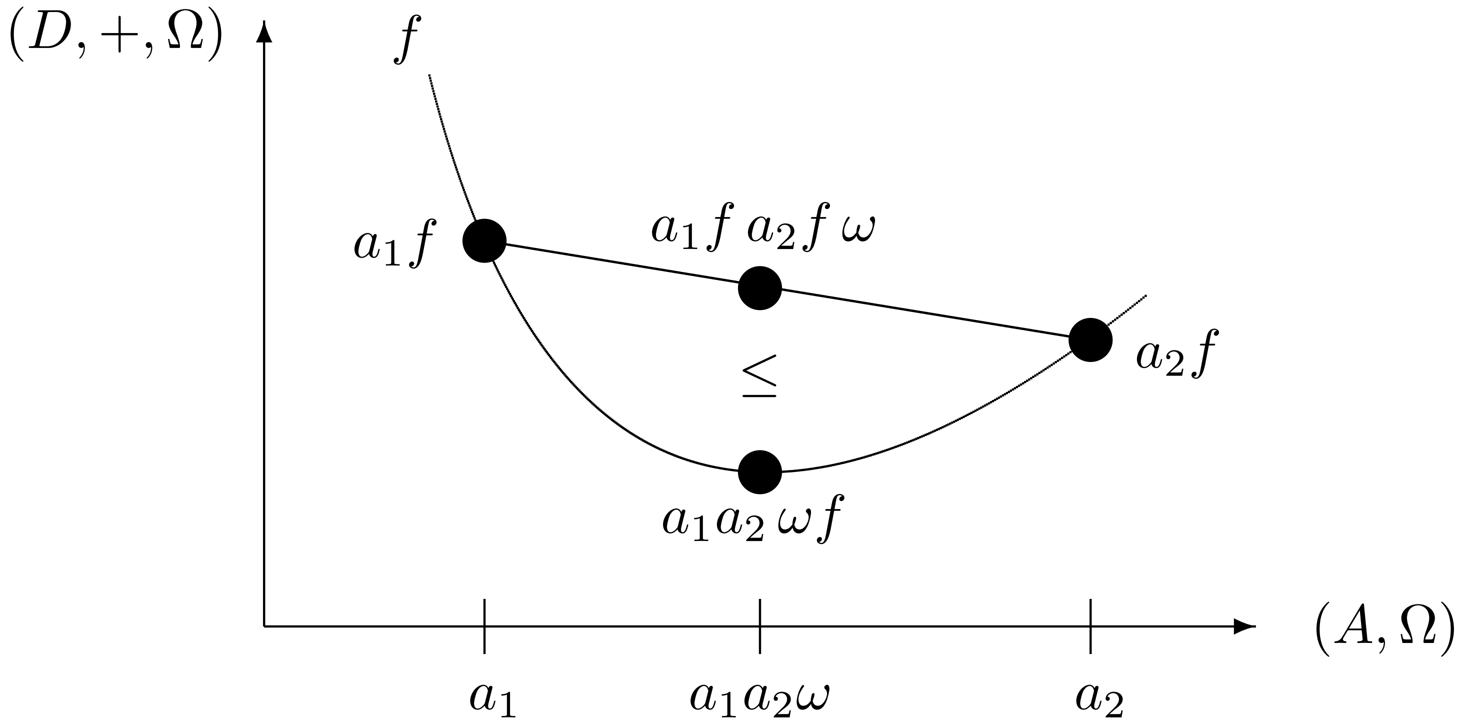
Function $f : (A, \Omega) \to (D, +, \Omega)$ from mode to modal is (Ω)-**convex** if

$\forall\, \omega \in \Omega\,,\ \forall\, a_1, \ldots, a_{\omega\tau} \in A\,,\ a_1 \ldots a_{\omega\tau}\omega f \leq a_1 f \ldots a_{\omega\tau} f\omega$.

PROPERTIES OF MODALS

Function $f : (A, \Omega) \to (D, +, \Omega)$ from mode to modal is (Ω)-**convex** if

$\forall\, \omega \in \Omega\,,\ \forall\, a_1, \ldots, a_{\omega\tau} \in A\,,\ a_1 \ldots a_{\omega\tau}\omega f \leq a_1 f \ldots a_{\omega\tau} f\omega$.



PROPERTIES OF MODALS

Function $f : (A, \Omega) \to (D, +, \Omega)$ from mode to modal is (Ω)-**convex** if

$\forall\, \omega \in \Omega\,,\ \forall\, a_1, \ldots, a_{\omega\tau} \in A\,,\ a_1 \ldots a_{\omega\tau}\omega f \leq a_1 f \ldots a_{\omega\tau} f\omega$.

PROPERTIES OF MODALS

Function $f : (A, \Omega) \to (D, +, \Omega)$ from mode to modal is (Ω)-**convex** if

$\forall\, \omega \in \Omega\,,\ \forall\, a_1, \ldots, a_{\omega\tau} \in A\,,\ a_1 \ldots a_{\omega\tau}\omega f \leq a_1 f \ldots a_{\omega\tau} f\omega$.

Consider modal $(D, +, \Omega)$.

PROPERTIES OF MODALS

Function $f : (A, \Omega) \to (D, +, \Omega)$ from mode to modal is (Ω)-**convex** if

$\forall\, \omega \in \Omega\,,\ \forall\, a_1, \ldots, a_{\omega\tau} \in A\,,\ a_1 \ldots a_{\omega\tau}\omega f \leq a_1 f \ldots a_{\omega\tau} f \omega$.

Consider modal $(D, +, \Omega)$.

Monotonicity Lemma: Each operation $\omega : D^{\omega\tau} \to D$ is monotone.

PROPERTIES OF MODALS

Function $f : (A, \Omega) \to (D, +, \Omega)$ from mode to modal is (Ω)-**convex** if

$\forall\, \omega \in \Omega\,,\ \forall\, a_1, \ldots, a_{\omega\tau} \in A\,,\ a_1 \ldots a_{\omega\tau}\omega f \leq a_1 f \ldots a_{\omega\tau} f\omega$.

Consider modal $(D, +, \Omega)$.

Monotonicity Lemma: Each operation $\omega : D^{\omega\tau} \to D$ is monotone.

Convexity Lemma: For each positive integer r,

$\sum_r : D^r \to D; (x_1, \ldots, x_r) \mapsto x_1 + \cdots + x_r$ is convex.

PROPERTIES OF MODALS

Function $f : (A, \Omega) \to (D, +, \Omega)$ from mode to modal is (Ω)-**convex** if

$\forall\, \omega \in \Omega\,,\ \forall\, a_1, \ldots, a_{\omega\tau} \in A\,,\ a_1 \ldots a_{\omega\tau}\omega f \leq a_1 f \ldots a_{\omega\tau} f\omega$.

Consider modal $(D, +, \Omega)$.

Monotonicity Lemma: Each operation $\omega : D^{\omega\tau} \to D$ is monotone.

Convexity Lemma: For each positive integer r,

$\sum_r : D^r \to D; (x_1, \ldots, x_r) \mapsto x_1 + \cdots + x_r$ is convex.

Sum-Superiority Lemma: $\forall\, \omega \in \Omega\,,\ \omega \leq \sum_{\omega\tau}$.

BARYCENTRIC ALGEBRAS

BARYCENTRIC ALGEBRAS

On a field F, define . . .

BARYCENTRIC ALGEBRAS

On a field F, define . . .

. . . **complementation** $p' = 1 - p$,

BARYCENTRIC ALGEBRAS

On a field F, define . . .

. . . **complementation** $p' = 1 - p$,

. . . **dual multiplication** $p \circ q = (p'q')'$,

BARYCENTRIC ALGEBRAS

On a field F, define . . .

. . . **complementation** $p' = 1 - p$,

. . . **dual multiplication** $p \circ q = (p'q')'$, and

. . . **implication** $p \to q =$ **if** $p = 0$ **then** 1 **else** q/p.

BARYCENTRIC ALGEBRAS

On a field F, define ...

... **complementation** $p' = 1 - p$,

... **dual multiplication** $p \circ q = (p'q')'$, and

... **implication** $p \to q =$ **if** $p = 0$ **then** 1 **else** q/p.

A **barycentric algebra** (A, I°) has binary operation $\underline{p}$

for each p in open real interval $I^\circ = \{p \in \mathbb{R} \mid 0 < p < 1\}$,

BARYCENTRIC ALGEBRAS

On a field F, define ...

... **complementation** $p' = 1 - p$,

... **dual multiplication** $p \circ q = (p'q')'$, and

... **implication** $p \to q =$ **if** $p = 0$ **then** 1 **else** q/p.

A **barycentric algebra** (A, I°) has binary operation $\underline{p}$

for each p in open real interval $I^\circ = \{p \in \mathbb{R} \mid 0 < p < 1\}$, satisfying ...

... **idempotence** $xx\underline{p} = x$;

BARYCENTRIC ALGEBRAS

On a field F, define . . .

. . . **complementation** $p' = 1 - p$,

. . . **dual multiplication** $p \circ q = (p'q')'$,   and

. . . **implication** $p \to q =$ **if** $p = 0$ **then** 1 **else** q/p.

A **barycentric algebra** (A, I°) has binary operation $\underline{p}$

for each p in open real interval $I^\circ = \{p \in \mathbb{R} \mid 0 < p < 1\}$, satisfying . . .

. . . **idempotence** $xx\underline{p} = x$;

. . . **skew-commutativity** $xy\underline{p} = yx\underline{p'}$,

BARYCENTRIC ALGEBRAS

On a field F, define ...

... **complementation** $p' = 1 - p$,

... **dual multiplication** $p \circ q = (p'q')'$, and

... **implication** $p \to q =$ **if** $p = 0$ **then** 1 **else** q/p.

A **barycentric algebra** (A, I°) has binary operation $\underline{p}$

for each p in open real interval $I^\circ = \{p \in \mathbb{R} \mid 0 < p < 1\}$, satisfying ...

... **idempotence** $xx\underline{p} = x$;

... **skew-commutativity** $xy\underline{p} = yx\underline{p'}$, and

... **skew-associativity** $xy\underline{p}\, z\underline{q} = x\, yz\underline{(p \circ q \to q)}\, \underline{p \circ q}$.

BARYCENTRIC ALGEBRAS

On a field F, define ...

... **complementation** $p' = 1 - p$,

... **dual multiplication** $p \circ q = (p'q')'$, and

... **implication** $p \to q =$ **if** $p = 0$ **then** 1 **else** q/p.

A **barycentric algebra** (A, I°) has binary operation $\underline{p}$

for each p in open real interval $I^\circ = \{p \in \mathbb{R} \mid 0 < p < 1\}$, satisfying ...

... **idempotence** $xx\underline{p} = x$;

... **skew-commutativity** $xy\underline{p} = yx\underline{p'}$, and

... **skew-associativity** $xy\underline{p}\, z\underline{q} = x\, yz\underline{(p \circ q \to q)}\, \underline{p \circ q}$.

Barycentric algebras are modes.

EXAMPLES OF BARYCENTRIC ALGEBRAS

EXAMPLES OF BARYCENTRIC ALGEBRAS

Semilattice with $xy\,\underline{p} = x \cdot y$ for each p in I°.

EXAMPLES OF BARYCENTRIC ALGEBRAS

Semilattice with $xy\,\underline{p} = x \cdot y$ for each p in I°. ("Skew-" disappears.)

EXAMPLES OF BARYCENTRIC ALGEBRAS

Semilattice with $xy\,\underline{p} = x \cdot y$ for each p in I°.   (“Skew-” disappears.)

Real affine space with $xy\,\underline{p} = x(1-p) + yp$ for each p in I°.

EXAMPLES OF BARYCENTRIC ALGEBRAS

Semilattice with $xy\,\underline{p} = x \cdot y$ for each p in I°. (“Skew-” disappears.)

Real affine space with $xy\,\underline{p} = x(1-p) + yp$ for each p in I°.

Extended real line $\mathbb{R}^\infty = \mathbb{R} \cup \{\infty\}$ with $\{\infty\}$ as a sink.

EXAMPLES OF BARYCENTRIC ALGEBRAS

Semilattice with $xy\,\underline{p} = x \cdot y$ for each p in I°. ("Skew-" disappears.)

Real affine space with $xy\,\underline{p} = x(1-p) + yp$ for each p in I°.

Extended real line $\mathbb{R}^\infty = \mathbb{R} \cup \{\infty\}$ with $\{\infty\}$ as a sink.

The dual $A^* = \mathrm{Hom}\,(A, \mathbb{R}^\infty)$ for a barycentric algebra A.

EXAMPLES OF BARYCENTRIC ALGEBRAS

Semilattice with $xy\,\underline{p} = x \cdot y$ for each p in I°.   ("Skew-" disappears.)

Real affine space with $xy\,\underline{p} = x(1-p) + yp$ for each p in I°.

Extended real line $\mathbb{R}^\infty = \mathbb{R} \cup \{\infty\}$ with $\{\infty\}$ as a sink.

The dual $A^* = \mathrm{Hom}\,(A, \mathbb{R}^\infty)$ for a barycentric algebra A.   Note $A \leq A^{**}$.

EXAMPLES OF BARYCENTRIC ALGEBRAS

Semilattice with $xy\,\underline{p} = x \cdot y$ for each p in I°.   ("Skew-" disappears.)

Real affine space with $xy\,\underline{p} = x(1-p) + yp$ for each p in I°.

Extended real line $\mathbb{R}^\infty = \mathbb{R} \cup \{\infty\}$ with $\{\infty\}$ as a sink.

The dual $A^* = \mathrm{Hom}\,(A, \mathbb{R}^\infty)$ for a barycentric algebra A.   Note $A \leq A^{**}$.

Convex set with $xy\,\underline{p} = x(1-p) + yp$ for each p in I°.

EXAMPLES OF BARYCENTRIC ALGEBRAS

Semilattice with $xy\,\underline{p} = x \cdot y$ for each p in I°. ("Skew-" disappears.)

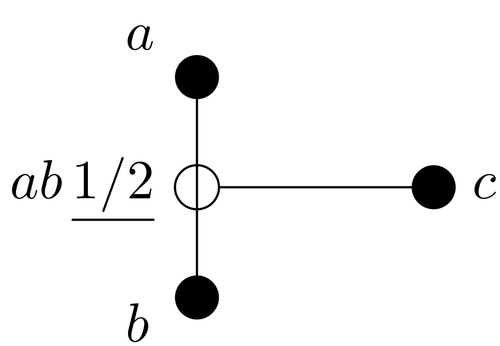
Real affine space with $xy\,\underline{p} = x(1-p) + yp$ for each p in I°.

Extended real line $\mathbb{R}^\infty = \mathbb{R} \cup \{\infty\}$ with $\{\infty\}$ as a sink.

The dual $A^* = \mathrm{Hom}\,(A, \mathbb{R}^\infty)$ for a barycentric algebra A. Note $A \leq A^{**}$.

Convex set with $xy\,\underline{p} = x(1-p) + yp$ for each p in I°.

The "T" with $xy\,\underline{p} = \left(ab\,\underline{1/2}\right)(1-p) + yp$ for $x \in \langle a, b\rangle$ and $y \notin \langle a, b\rangle$.



FREE BARYCENTRIC ALGEBRAS

FREE BARYCENTRIC ALGEBRAS

Variety $\underline{\underline{B}}$ of barycentric algebras.

FREE BARYCENTRIC ALGEBRAS

Variety $\underline{\underline{B}}$ of barycentric algebras.

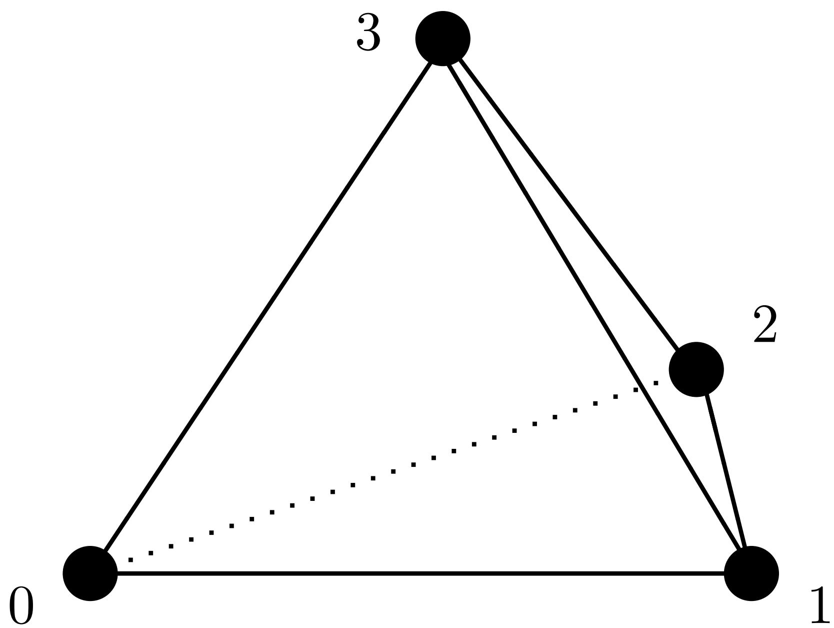
Quasivariety $\underline{\underline{C}}$ of convex sets $\left(xy\underline{p} = xz\underline{p} \Rightarrow y = z\right)$.

FREE BARYCENTRIC ALGEBRAS

Variety $\underline{\underline{B}}$ of barycentric algebras.

Quasivariety $\underline{\underline{C}}$ of convex sets $\big(xy\underline{p} = xz\underline{p} \Rightarrow y = z\big)$.

Free barycentric algebra XB or convex set XC over set X is simplex



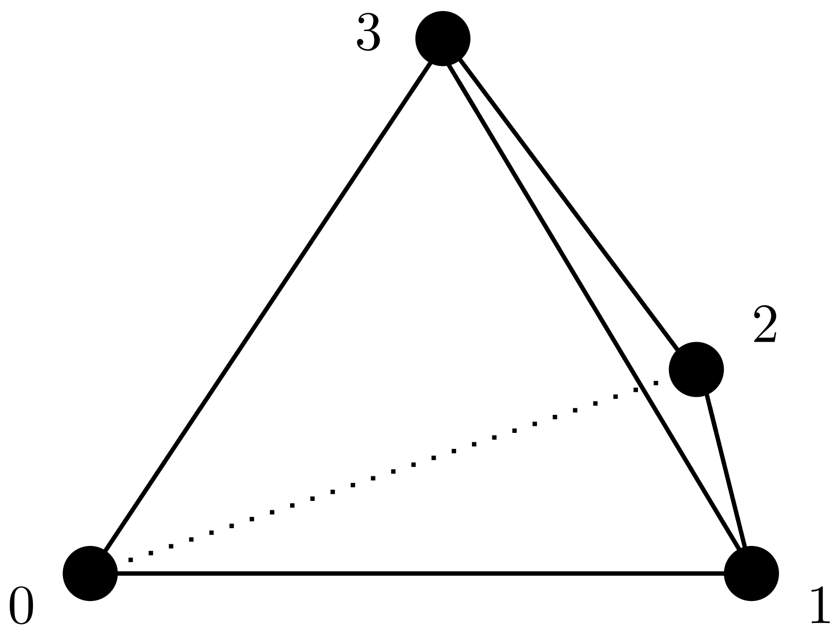
$\{0, 1, 2, 3\}B$

FREE BARYCENTRIC ALGEBRAS

Variety $\underline{\underline{B}}$ of barycentric algebras.

Quasivariety $\underline{\underline{C}}$ of convex sets $\big(xy\underline{p} = xz\underline{p} \Rightarrow y = z\big)$.

Free barycentric algebra XB or convex set XC over set X is simplex



$\{0, 1, 2, 3\}B$

or set of finite probability distributions on X.

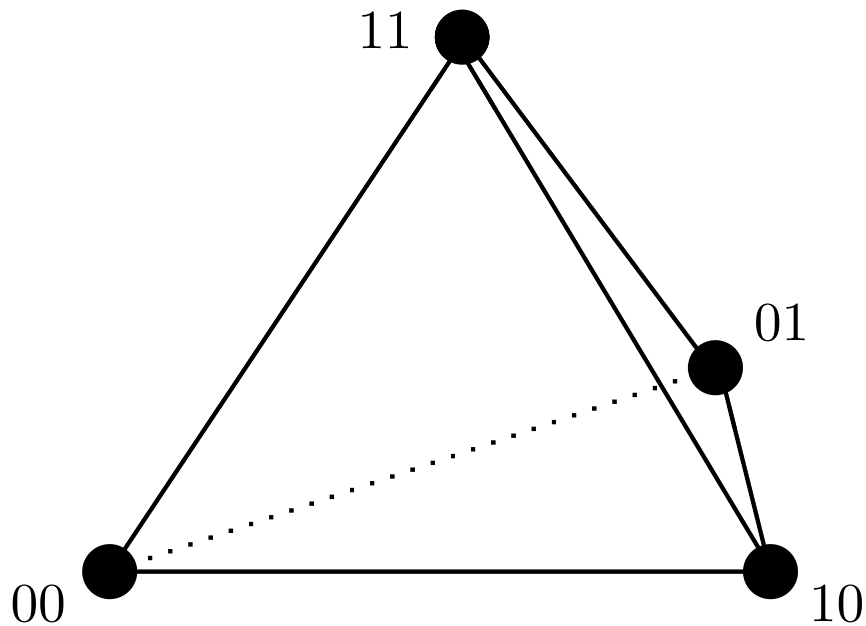
TENSOR PRODUCTS AND DIRECT PRODUCTS

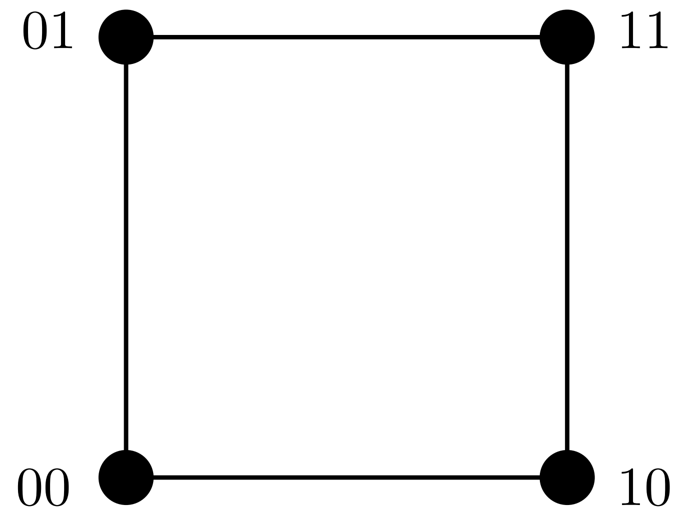
TENSOR PRODUCTS AND DIRECT PRODUCTS

Barycentric algebra $A = \{0, 1\}B$ (one bit). Compare $A \otimes A$ with $A \times A$.

TENSOR PRODUCTS AND DIRECT PRODUCTS

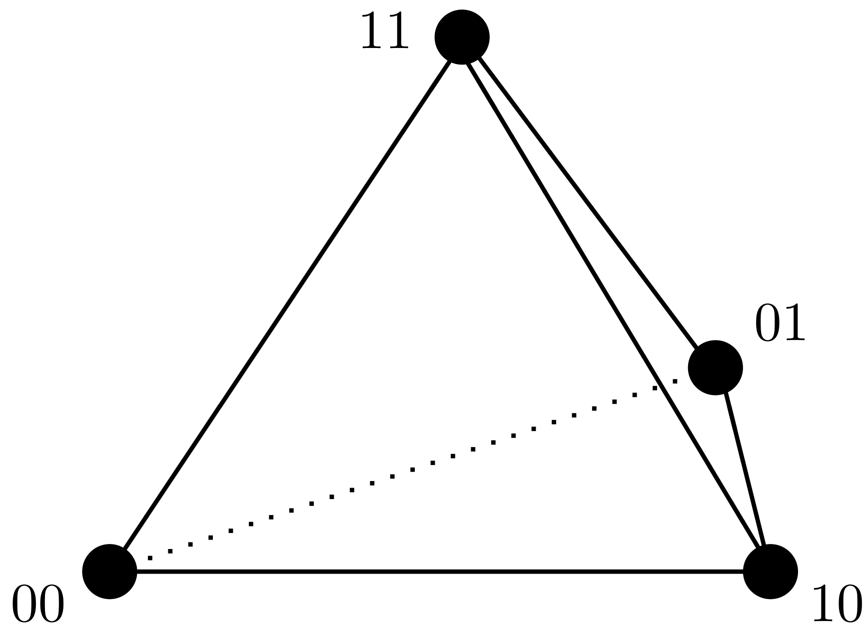
Barycentric algebra $A = \{0, 1\}B$ (one bit). Compare $A \otimes A$ with $A \times A$.



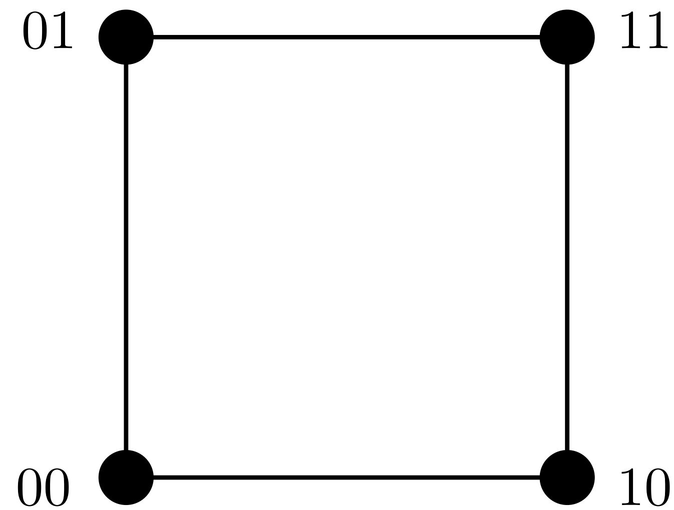


TENSOR PRODUCTS AND DIRECT PRODUCTS

Barycentric algebra $A = \{0, 1\}B$ (one bit). Compare $A \otimes A$ with $A \times A$.



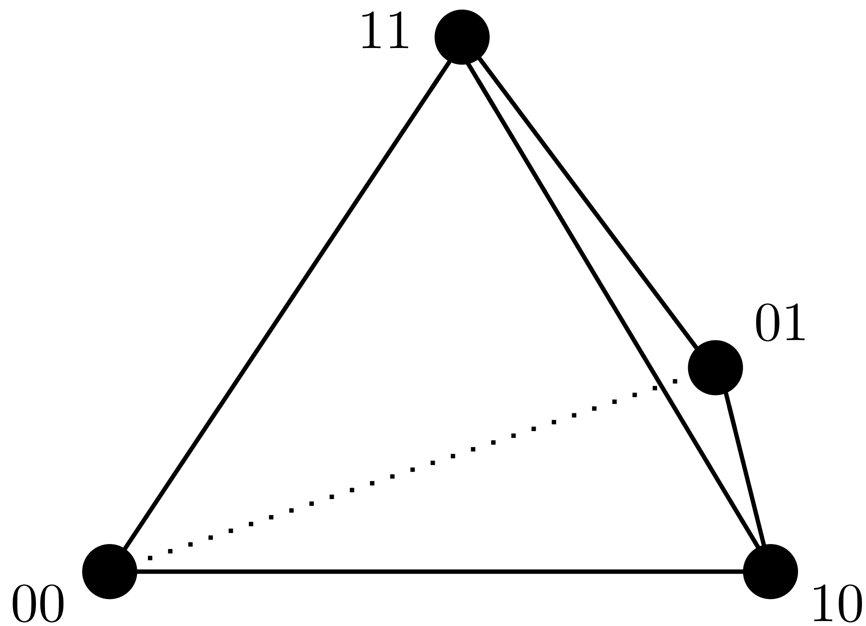
$A \otimes A$, entangled bits



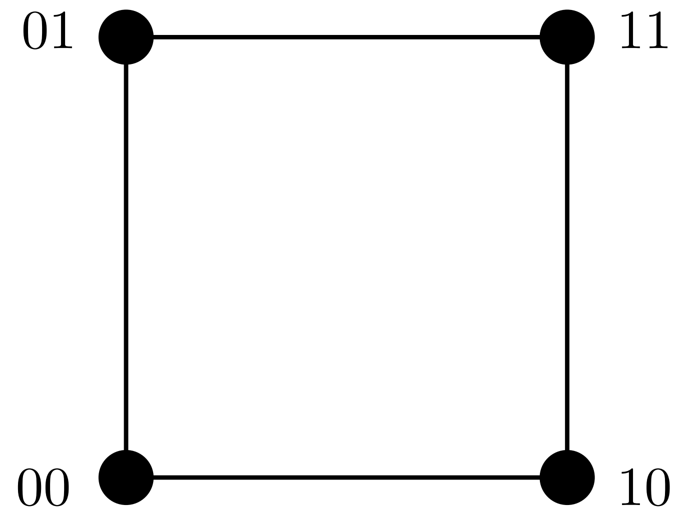
$A \times A$, independent bits

TENSOR PRODUCTS AND DIRECT PRODUCTS

Barycentric algebra $A = \{0, 1\}B$ (one bit). Compare $A \otimes A$ with $A \times A$.



$A \otimes A$, entangled bits

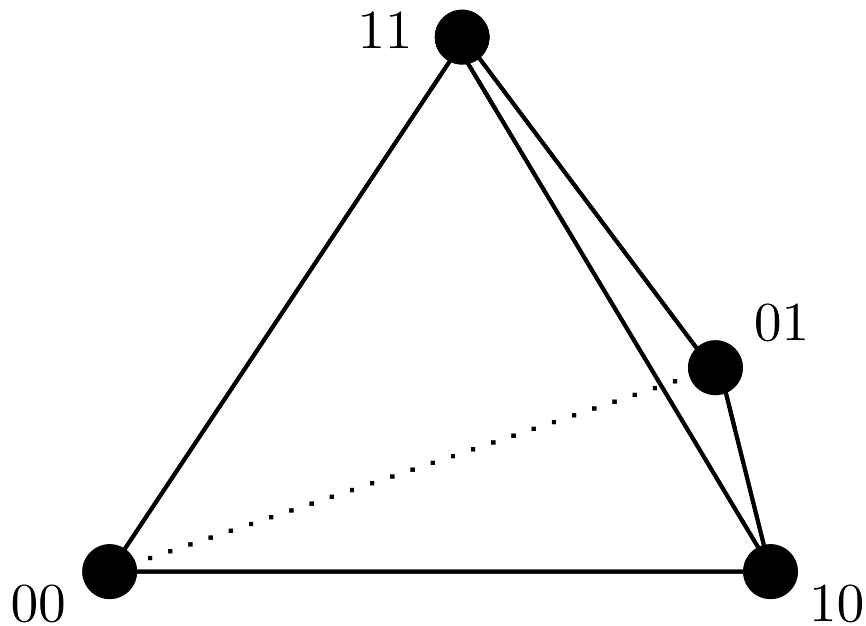


$A \times A$, independent bits

classical

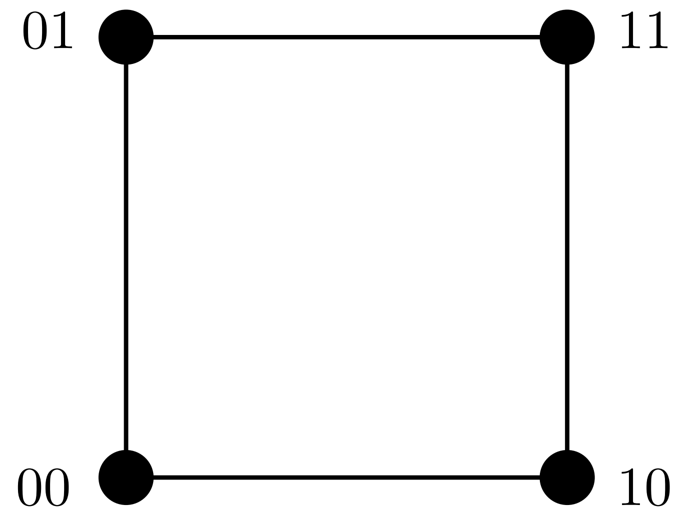
TENSOR PRODUCTS AND DIRECT PRODUCTS

Barycentric algebra $A = \{0, 1\}B$ (one bit). Compare $A \otimes A$ with $A \times A$.



$A \otimes A$, entangled bits

quantum?



$A \times A$, independent bits

classical

HIERARCHICAL STATISTICAL MECHANICS

HIERARCHICAL STATISTICAL MECHANICS

State space: Finite set X.

HIERARCHICAL STATISTICAL MECHANICS

State space: Finite set X.

Valuation function: $f : X \to A$ with barycentric algebra A generated by Xf

HIERARCHICAL STATISTICAL MECHANICS

State space: Finite set X.

Valuation function: $f : X \to A$ with barycentric algebra A generated by Xf

e.g. energy $E : X \to [\min_{x \in X} E(x), \max_{x \in X} E(x)]$

(say in joules) of some physical system X.

HIERARCHICAL STATISTICAL MECHANICS

State space: Finite set X.

Valuation function: $f : X \to A$ with barycentric algebra A generated by Xf

e.g. energy $E : X \to [\min_{x \in X} E(x), \max_{x \in X} E(x)]$

(say in joules) of some physical system X.

Entropy: $H : A \to [0, \log |X|]$;

$$\alpha \mapsto \sup\{-\textstyle\sum_{x \in X} p_x \log p_x \mid \alpha = \textstyle\sum_{x \in X} x f p_x\}$$

HIERARCHICAL STATISTICAL MECHANICS

State space: Finite set X.

Valuation function: $f : X \to A$ with barycentric algebra A generated by Xf

e.g. energy $E : X \to [\min_{x\in X} E(x), \max_{x\in X} E(x)]$

(say in joules) of some physical system X.

Entropy: $H : A \to [0, \log|X|]$;

$$\alpha \mapsto \sup\{-\sum_{x\in X} p_x \log p_x \mid \alpha = \sum_{x\in X} xfp_x\}$$

cf. thermodynamic entropy $S = kH$ with Boltzmann's constant k.

HIERARCHICAL STATISTICAL MECHANICS

State space: Finite set X.

Valuation function: $f : X \to A$ with barycentric algebra A generated by Xf

e.g. energy $E : X \to [\min_{x\in X} E(x), \max_{x\in X} E(x)]$

(say in joules) of some physical system X.

Entropy: $H : A \to [0, \log|X|]$;

$$\alpha \mapsto \sup\{-\textstyle\sum_{x\in X} p_x \log p_x \mid \alpha = \sum_{x\in X} xfp_x\}$$

cf. thermodynamic entropy $S = kH$ with Boltzmann's constant k.

Potential: $\beta \in A^* = \mathrm{Hom}(A, \mathbb{R}^\infty)$

HIERARCHICAL STATISTICAL MECHANICS

State space: Finite set X.

Valuation function: $f : X \to A$ with barycentric algebra A generated by Xf

e.g. energy $E : X \to [\min_{x \in X} E(x), \max_{x \in X} E(x)]$

(say in joules) of some physical system X.

Entropy: $H : A \to [0, \log |X|]$;

$$\alpha \mapsto \sup\{-\sum_{x \in X} p_x \log p_x \mid \alpha = \sum_{x \in X} xfp_x\}$$

cf. thermodynamic entropy $S = kH$ with Boltzmann's constant k.

Potential: $\beta \in A^* = \mathrm{Hom}(A, \mathbb{R}^\infty)$

Partition function: $Z : A^* \to [0, \infty]; \beta \mapsto \sum_{x \in X} \exp(-xf\beta)$

HIERARCHICAL STATISTICAL MECHANICS

State space: Finite set X.

Valuation function: $f : X \to A$ with barycentric algebra A generated by Xf

e.g. energy $E : X \to [\min_{x \in X} E(x), \max_{x \in X} E(x)]$

(say in joules) of some physical system X.

Entropy: $H : A \to [0, \log |X|]$;

$$\alpha \mapsto \sup\{-\sum_{x \in X} p_x \log p_x \mid \alpha = \sum_{x \in X} xfp_x\}$$

cf. thermodynamic entropy $S = kH$ with Boltzmann's constant k.

Potential: $\beta \in A^* = \mathrm{Hom}(A, \mathbb{R}^\infty)$

Partition function: $Z : A^* \to [0, \infty]; \beta \mapsto \sum_{x \in X} \exp(-xf\beta)$

cf. free energy $F = -kT \log Z(\beta)$ at temperature $T^\circ K$.

RIEMANN ZETA FUNCTION

RIEMANN ZETA FUNCTION

State space $\{1, 2, \ldots, N\}$.

RIEMANN ZETA FUNCTION

State space $\{1, 2, \ldots, N\}$.

Valuation function $\log : \{1, 2, \ldots, N\} \to [0, \log N]$.

RIEMANN ZETA FUNCTION

State space $\{1, 2, \ldots, N\}$.

Valuation function $\log : \{1, 2, \ldots, N\} \to [0, \log N]$.

For a real number s, consider the **potential**

$$s : [0, \log N] \to \mathbb{R}^*; r \mapsto rs.$$

RIEMANN ZETA FUNCTION

State space $\{1, 2, \ldots, N\}$.

Valuation function $\log : \{1, 2, \ldots, N\} \to [0, \log N]$.

For a real number s, consider the **potential**

$$s : [0, \log N] \to \mathbb{R}^*; r \mapsto rs.$$

Then the **partition function** is the partial sum

$$Z(s) = \sum_{n=1}^{N} n^{-s}$$

of the **Riemann zeta function** $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$.

LEGENDRE TRANSFORMS

LEGENDRE TRANSFORMS

Modal $(\mathbb{R}^{\infty}, \max, I^{\circ})$.

LEGENDRE TRANSFORMS

Modal $(\mathbb{R}^\infty, \max, I^\circ)$.

Legendre Transform Theorem. Consider $f : X \to \langle X f \rangle = A$.

LEGENDRE TRANSFORMS

Modal $(\mathbb{R}^\infty, \max, I^\circ)$.

Legendre Transform Theorem. Consider $f : X \to \langle X f \rangle = A$.

(a) $\forall\, \alpha \in A\,,\, \forall\, \beta \in A^*\,,\ \alpha\beta \geq H(\alpha) - \log Z(\beta)$.

LEGENDRE TRANSFORMS

Modal $(\mathbb{R}^\infty, \max, I^\circ)$.

Legendre Transform Theorem. Consider $f : X \to \langle X f \rangle = A$.

(a) $\forall\, \alpha \in A\,,\ \forall\, \beta \in A^*\,,\ \alpha\beta \geq H(\alpha) - \log Z(\beta)$.

(b) $-\log Z : A^* \to \mathbb{R}^\infty$ is concave,

LEGENDRE TRANSFORMS

Modal $(\mathbb{R}^\infty, \max, I^\circ)$.

Legendre Transform Theorem. Consider $f : X \to \langle X f \rangle = A$.

(a) $\forall\, \alpha \in A\,,\ \forall\, \beta \in A^*\,,\ \alpha\beta \geq H(\alpha) - \log Z(\beta)$.

(b) $-\log Z : A^* \to \mathbb{R}^\infty$ is concave, with

$$\forall\, \beta \in A^*\,,\ -\log Z(\beta) = \inf\{\alpha\beta - H(\alpha) \mid \alpha \in A\}.$$

LEGENDRE TRANSFORMS

Modal $(\mathbb{R}^\infty, \max, I^\circ)$.

Legendre Transform Theorem. Consider $f : X \to \langle X f \rangle = A$.

(a) $\forall\, \alpha \in A\,,\ \forall\, \beta \in A^*\,,\ \alpha\beta \geq H(\alpha) - \log Z(\beta)$.

(b) $-\log Z : A^* \to \mathbb{R}^\infty$ is concave, with

$$\forall\, \beta \in A^*\,,\ -\log Z(\beta) = \inf\{\alpha\beta - H(\alpha) \mid \alpha \in A\}.$$

(c) $H : A \to [0, |X|]$ is concave,

LEGENDRE TRANSFORMS

Modal $(\mathbb{R}^\infty, \max, I^\circ)$.

Legendre Transform Theorem. Consider $f : X \to \langle X f \rangle = A$.

(a) $\forall\, \alpha \in A\,,\ \forall\, \beta \in A^*\,,\ \alpha\beta \geq H(\alpha) - \log Z(\beta)$.

(b) $-\log Z : A^* \to \mathbb{R}^\infty$ is concave, with

$$\forall\, \beta \in A^*\,,\ -\log Z(\beta) = \inf\{\alpha\beta - H(\alpha) \mid \alpha \in A\}.$$

(c) $H : A \to [0, |X|]$ is concave, with

$$\forall\, \alpha \in A\,,\ H(\alpha) = \inf\{\alpha\beta + \log Z(\beta) \mid \beta \in A^*\}.$$

SEMILATTICE CHARACTERS

SEMILATTICE CHARACTERS

Valuation function $\mathrm{id}_A : A \to A$ on finite (join) semilattice A.

SEMILATTICE CHARACTERS

Valuation function $\mathrm{id}_A : A \to A$ on finite (join) semilattice A.

Consider 2-element semilattice $S = \{0, \infty\} \leq \mathbb{R}^\infty$.

SEMILATTICE CHARACTERS

Valuation function $\mathrm{id}_A : A \to A$ on finite (join) semilattice A.

Consider 2-element semilattice $S = \{0, \infty\} \leq \mathbb{R}^\infty$.

Character of A is a semilattice homomorphism $\chi : A \to S$.

SEMILATTICE CHARACTERS

Valuation function $\mathrm{id}_A : A \to A$ on finite (join) semilattice A.

Consider 2-element semilattice $S = \{0, \infty\} \leq \mathbb{R}^\infty$.

Character of A is a semilattice homomorphism $\chi : A \to S$.

For $a \in A$, define **principal wall** $[a] = \{x \in A \mid x \leq a\}$.

SEMILATTICE CHARACTERS

Valuation function $\mathrm{id}_A : A \to A$ on finite (join) semilattice A.

Consider 2-element semilattice $S = \{0, \infty\} \leq \mathbb{R}^\infty$.

Character of A is a semilattice homomorphism $\chi : A \to S$.

For $a \in A$, define **principal wall** $[a] = \{x \in A \mid x \leq a\}$.

Define character $\widehat{a} : A \to S; x \mapsto$ **if** $x \in [a]$ **then** 0 **else** ∞.

SEMILATTICE CHARACTERS

Valuation function $\mathrm{id}_A : A \to A$ on finite (join) semilattice A.

Consider 2-element semilattice $S = \{0, \infty\} \leq \mathbb{R}^{\infty}$.

Character of A is a semilattice homomorphism $\chi : A \to S$.

For $a \in A$, define **principal wall** $[a] = \{x \in A \mid x \leq a\}$.

Define character $\widehat{a} : A \to S; x \mapsto$ **if** $x \in [a]$ **then** 0 **else** ∞.

Concavity of $-\log Z : A^* \to \mathbb{R}^{\infty}$ and $H : A \to [0, \log |A|]$ give

$$\big|[a] \cap [b]\big| \leq \big|[a]\big|^{1-p}\big|[b]\big|^{p} \leq \big|[a] \cup [b]\big|$$

for $p \in I^{\circ}$ and $a, b \in A$.

A TOY MODEL FOR COMPLEX SYSTEMS

A TOY MODEL FOR COMPLEX SYSTEMS

Ecology: Competition between species A and B.

A TOY MODEL FOR COMPLEX SYSTEMS

Ecology: Competition between species A and B.

Demography: Species A has stages $A1$ and $A2$ (e.g. larva and adult).

A TOY MODEL FOR COMPLEX SYSTEMS

Ecology: Competition between species A and B.

Demography: Species A has stages $A1$ and $A2$ (e.g. larva and adult).

Hierarchical statistical mechanics with state space $\{A1, A2, B\}$,

A TOY MODEL FOR COMPLEX SYSTEMS

Ecology: Competition between species A and B.

Demography: Species A has stages $A1$ and $A2$ (e.g. larva and adult).

Hierarchical statistical mechanics with state space $\{A1, A2, B\}$,

values of energy function in barycentric algebra T:

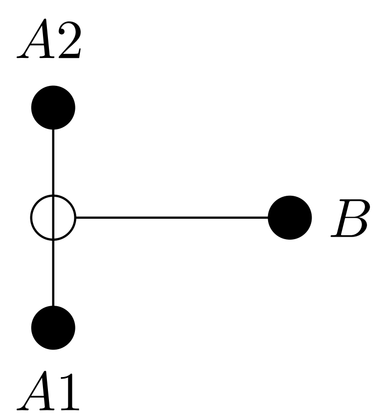
A TOY MODEL FOR COMPLEX SYSTEMS

Ecology: Competition between species A and B.

Demography: Species A has stages $A1$ and $A2$ (e.g. larva and adult).

Hierarchical statistical mechanics with state space $\{A1, A2, B\}$,

values of energy function in barycentric algebra T:



OTHER APPLICATIONS, DEVELOPMENTS, PROBLEMS

OTHER APPLICATIONS, DEVELOPMENTS, PROBLEMS

1. Support functions for arbitrary convex sets.

OTHER APPLICATIONS, DEVELOPMENTS, PROBLEMS

1. Support functions for arbitrary convex sets.

2. Which functions are differences of support functions?

OTHER APPLICATIONS, DEVELOPMENTS, PROBLEMS

1. Support functions for arbitrary convex sets.
2. Which functions are differences of support functions?
3. Combinatorial differentiation (**differential groupoids**).

OTHER APPLICATIONS, DEVELOPMENTS, PROBLEMS

1. Support functions for arbitrary convex sets.
2. Which functions are differences of support functions?
3. Combinatorial differentiation (**differential groupoids**).
4. Duality for barycentric algebras.

OTHER APPLICATIONS, DEVELOPMENTS, PROBLEMS

1. Support functions for arbitrary convex sets.
2. Which functions are differences of support functions?
3. Combinatorial differentiation (**differential groupoids**).
4. Duality for barycentric algebras.
5. Embedding modes into (semi)modules.

OTHER APPLICATIONS, DEVELOPMENTS, PROBLEMS

1. Support functions for arbitrary convex sets.
2. Which functions are differences of support functions?
3. Combinatorial differentiation (**differential groupoids**).
4. Duality for barycentric algebras.
5. Embedding modes into (semi)modules.
6. Embedding posets into linear orders (order-constrained probability).

OTHER APPLICATIONS, DEVELOPMENTS, PROBLEMS

1. Support functions for arbitrary convex sets.
2. Which functions are differences of support functions?
3. Combinatorial differentiation (**differential groupoids**).
4. Duality for barycentric algebras.
5. Embedding modes into (semi)modules.
6. Embedding posets into linear orders (order-constrained probability).
7. Polytopes over dyadic rationals.

OTHER APPLICATIONS, DEVELOPMENTS, PROBLEMS

1. Support functions for arbitrary convex sets.
2. Which functions are differences of support functions?
3. Combinatorial differentiation (**differential groupoids**).
4. Duality for barycentric algebras.
5. Embedding modes into (semi)modules.
6. Embedding posets into linear orders (order-constrained probability).
7. Polytopes over dyadic rationals.
8. “Fuzzy logic” of gene expression in cells.