# What is a Finite Lattice?

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# ... Trojae qui primus ab oris ...

## Original

A **lattice** is an ordered set in which every pair of elements has a l.u.b. and g.l.b.

## Extra crispy

A **lattice** is an algebra $\mathbf{L} = \langle L, \vee, \wedge \rangle$ satisfying

$$\begin{aligned} x \vee x &\approx x & x \wedge x &\approx x \\ x \vee y &\approx y \vee x & x \wedge y &\approx y \wedge x \\ x \vee (y \vee z) &\approx (x \vee y) \vee z & x \wedge (y \wedge z) &\approx (x \wedge y) \wedge z \\ x \vee (x \wedge y) &\approx x & x \wedge (x \vee y) &\approx x \end{aligned}$$

# Half price

- A finite **lattice** is a join semilattice with 0, or dually,
- a meet semilattice with 1.

A finite lattice can be represented as a closure system/Moore family on any set $S$ with $J(L) \subseteq S \subseteq L$.

A finite lattice can be represented as Sub $\mathbf{A}$ for some finite algebra $A$.

# ... ist ein Weg.

**FBA or Formal Concept Analysis**

A finite lattice can be represented by the closure lattice of the Galois connection LE $\subseteq S \times T$ whenever $J(L) \subseteq S \subseteq L$ and $M(L) \subseteq T \subseteq L$.

# Slow boat to China

## Moore families

- Each finite lattice is isomorphic to a subsemilattice of $\langle \mathbf{2}^S, \cdot, \mathbf{1} \rangle$ with $J(L) \subseteq S \subseteq L$.
- The lattice of closure systems on $S$ is isomorphic to $\mathrm{Sub}(\mathbf{2}^S, \cdot, \mathbf{1})$ ordered by reverse inclusion.

A finite lattice can be obtained from $\mathbf{2}^S$ by a sequence of operations removing a meet irreducible element.

More generally, remove $\uparrow a - \uparrow b$ for some pair $a < b$.

# Fast boat to nowhere

## Closure operators

- Each finite lattice is isomorphic to a homomorphic image of the free 0-semilattice on $S$.
- The lattice of closure operators on $S$ is isomorphic to $\mathrm{Con}(\mathbf{F}(S), \vee, 0)$.
- In particular, $\mathrm{Clop}(S)$ is upper bounded and satisfies $\mathrm{SD}_\wedge$.

## Join irreducibles

- $\mathrm{J}(\mathrm{Clop}(S))$ consists of all nontrivial implications of the form $X \to y$.
- $(X \to z) \leq (Y \to z)$ iff $X \supseteq Y$.
- $(X \to z) \leq \bigvee (Y_i \to t_i)$ iff this follows from the order relations and a sequence of applications of the cut rule $(X \cup Y \to b) \leq (X \to a) \vee (\{a\} \cup Y \to b)$.

# Bases

A finite lattice is determined by a join representation $\Sigma = \bigvee T_i$ of the corresponding closure system (semilattice congruence) in Clop($S$):

- direct unit basis
- D-basis
- G-D canonical basis
- K-basis

# Meet representation

- Clop($S$) has canonical meets - the subdirect decomposition of $L$ as a join semilattice
- Coatoms of Clop($S$) are $\psi_B$ where $S = B \dot{\cup} T$ with $B \neq S$, given by

$$b \to b' \qquad t \to t' \qquad t \to b$$

  for all $b, b' \in B$ and $t, t' \in T$.
- $\psi_B \geq \bigwedge \psi_{C_j}$ minimally if $B = \bigcap C_j$.

# Meet representation cont.

- Given a finite lattice $L$ and $S = J(L)$, let

$$\mathcal{B}(L) = \{\psi_B : B = \downarrow p \cap J(L) \text{ for some } p \in M(L)\}$$

- $L = F(S)/\theta$ where $\theta = \bigwedge\{\psi_B : B \in \mathcal{B}(L)\}$.
- The number of meetands is $|M(L)|$, out of a total of $2^{|J(L)|} - 1$ coatoms.
- Removing one meet irreducible at a time is adding one relation $b \rightarrow t$ that is below $\theta$ but not below exactly one $\psi_D$ with $\psi_D \not\geq \theta$.

# Subdirectly irreducible lattices

- Likewise, every lattice is a subdirect product of subdirectly irreducible lattices.
- Let $U$ be a filter on a lattice $L$. Then there is a unique largest congruence $\hat{\psi}_U$ on $L$ separating $U$.
- Let $L$ be a finite subdirectly irreducible lattice, and let $u \in J(L)$ be a critical element. Let $X = J(L)$. Then there is an endomorphism $\sigma$ of $\mathrm{FL}(X)$ such that (1) for all $x \in X$, $\sigma(x) \leq x$ and $\sigma(x) \in X^{\wedge\vee\wedge}$ and (2) $L \cong \mathrm{FL}(X)/\hat{\psi}_U$ for the filter generated by $\{\sigma^k(u) : k \in \omega\}$.
- $U$ is lower bounded iff $U$ is principal.
- Convesely, let $\sigma$ be an endomorphism of $\mathrm{FL}(X)$, with $X$ finite, satisfying (1). If $u \in X$ and $U$ is defined as in (2), then $\mathrm{FL}(X)/\hat{\psi}_U$ is finite.

# Congruence lattices of finite lattices

## Tischendorf extension

- Con($L$) $\cong^d$ Id($J(L), D$) where $I$ is an ideal if $x \in I$ and $x\, D\, y$ implies $y \in I$.
- So the congruence lattice of $L$ is determined by the non-binary part of its D-basis.
- Removing the binary part of the D-basis produces an atomistic lattice $K$ with Con($K$) $\cong$ Con($L$).
- OR you could add new binary relations $x \;\rightarrow\; y$ to obtain a smaller lattice $M$ with Con($M$) $\cong$ Con($L$), so long as the D-relation remains intact.

## Compare

- Belief revision
- Belief propogation

# Congruence representations

- (Whitman, Pudlák, Tuma) Every finite lattice can be represented by equivalence relations on a finite set.
- Every finite lattice can be embedded into the lattice of subgroups of a finite group.
- Can we uniformly bound the exponent of the group?
- Every finite lattice is isomorphic to a congruence lattice Con **A** for some algebra **A**.
- (Finite representation problem) Can we take **A** to be finite?
- Candidates for a counterexample: $\mathbf{2} \times \mathbf{3}$ with a wing, or nondesarguean projective planes.