Manifold Bases for Join Semidistributive Lattices

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The following are equivalent for a finite lattice.

- $L$ satisfies $SD_{\lor} : u = a \lor b = a \lor c$ implies $u = a \lor (b \land c)$.
- $L$ satisfies $u = \bigvee a_i = \bigvee b_j$ implies $u = \bigvee_{i,j} (a_i \land b_j)$.
- $L$ has canonical joins.

$C$ refines $B$, written $C \ll B$, if $\forall c \exists b \; c \leq b$.

- We consider only the standard representation of $L$ as a closure system on $J(L)$.
- Start with $L$ any finite lattice, later adding JSD or LB.
- The non-binary and binary parts are treated separately.
MJC(\(p\)) is the set of minimal join covers of the JI \(p\).

MMJC(\(p\)) is the set of \(Q \in MJC(\(p\))\) with \(\bigvee Q\) minimal.

MJR(\(x\)) is the set of minimal join representations of \(x\).

\(I_Q\) is the smallest subset of \(L\) such that (1) \(Q \subseteq I_Q\),
(2) \(\bigvee Q \notin I_Q\), and (3) \(s \leq t \in I_Q\) implies \(s \in I_Q\).

\(x\) is join essential if there exists \(X \subseteq L\) such that \(x = \bigvee X\)
and \(\downarrow x \setminus (\{x\} \cup I_X) \neq \emptyset\).

The core \(K(L) = J(L) \cup JE(L)\).
If $Q \in \text{MJR}(x)$, then $x \cup I_Q$ is a sublattice of $L$.

If $x \in \text{JE}(L)$, then there exist $Q \in \text{MJR}(x)$ and $p \in \text{J}(L)$ such that $p \in \downarrow x \setminus (x \cup I_Q)$. Moreover, in this case $Q \in \text{MJC}(p)$.
It is possible to have $I_Q \subseteq I_R$ with both $Q, R \in \text{MJR}(x)$.
In that case, $Q \parallel R$.
If $I_Q \subset I_R$, use $Q$. If $I_Q = I_R$, choose one.
If $L$ is join semidistributive, then $Q$ is unique for $x$. 

Test your $I_Q$
Let the binary part be given.

- $\delta^+$: the dependence or DUB consists of all $Q \to p$ with $p \leq \bigvee Q$ nontrivially and $Q$ minimal w.r.t. set containment.
- $D^+$: the $D$-basis consists of all $Q \to p$ with $Q \in \text{MJC}(p)$.
- $E^+$ (for LB only): the $E$-basis consists of all $Q \to p$ with $Q \in \text{MMJC}(p)$.
- $F^+$ (for LB only): the $F$-basis consists of all $Q \to p$ with $Q \in \text{MMJC}(p)$ and $p$ a maximal JI in $\downarrow (\bigvee Q) \setminus l_Q$.
- $GD^+$: the GD basis consists of all $J(L) \cap l_Q \to p$ with (1) $x = \bigvee Q$ essential, (2) $l_Q$ minimal for $x$, and (3) $p \in \downarrow x \setminus l_Q$.
- $K^+$: the $K$-basis consists of all $Q \to p$ with $Q \in \text{MJC}(p)$, properties (1)–(3), and $p$ a maximal JI in $\downarrow (\bigvee Q) \setminus l_Q$. 

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Join Semidistributive Lattices
The Goldilocks Principle

\[ K^+ \subseteq D^+ \subseteq \delta^+ \]

\[ F^+ \subseteq E^+ \subseteq D^+ \] when it applies.

\[ K^+ \text{ and } F^+ \text{ may be incomparable for a LB lattice, e.g., } N_6. \]
Default is $x \rightarrow y$ whenever $x \geq y$.

It suffices to include only $x \rightarrow y$ whenever $x \succ y$, ordered top-down for an ordered direct basis.

For join semidistributive lattices, the binary part of the $K^-$ basis contains all pairs $c \rightarrow d$ where $d$ is a canonical joinand of $c_\ast$.

Other reductions are sometimes possible! (See next slide.)

These parts can be used separately in different combinations.
Suppose $y \rightarrow Q \rightarrow x \rightarrow Q' \subseteq Q$ with $Q' \neq \emptyset$.
Replace this with $y \rightarrow \{x\} \cup (Q \setminus Q')$ and $Q \rightarrow x \rightarrow Q'$.

Expanding this theme...

The problem of finding an optimal basis for finite bounded lattices is NP-complete.