Manifold Bases for Join Semidistributive Lattices

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Join semidistributive lattices

The following are equivalent for a finite lattice.

- L satisfies $\mathrm{SD}_\vee$: $u = a \vee b = a \vee c$ implies $u = a \vee (b \wedge c)$.
- L satisfies $u = \bigvee a_i = \bigvee b_j$ implies $u = \bigvee_{i,j}(a_i \wedge b_j)$.
- L has canonical joins.

C **refines** B, written $C \ll B$, if $\forall c \, \exists b \; c \leq b$.

- We consider only the standard representation of L as a closure system on $\mathrm{J}(L)$.
- Start with L any finite lattice, later adding JSD or LB.
- The non-binary and binary parts are treated separately.

Notation and terminology

- MJC(p) is the set of minimal join covers of the JI p.
- MMJC(p) is the set of $Q \in$ MJC(p) with $\bigvee Q$ minimal.
- MJR(x) is the set of minimal join representations of x.
- I_Q is the smallest subset of L such that (1) $Q \subseteq I_Q$, (2) $\bigvee Q \notin I_Q$, and (3) $s \leq t \in I_Q$ implies $s \in I_Q$.
- x is **join essential** if there exists $X \subseteq L$ such that $x = \bigvee X$ and $\downarrow x \setminus (\{x\} \cup I_X) \neq \emptyset$.
- The **core** $K(L) = J(L) \cup JE(L)$.

Duquenne's Lemmate

If $Q \in \mathrm{MJR}(x)$, then $x \cup I_Q$ is a sublattice of L.

If $x \in \mathrm{JE}(L)$, then there exist $Q \in \mathrm{MJR}(x)$ and $p \in \mathrm{J}(L)$ such that $p \in \downarrow x \setminus (x \cup I_Q)$. Moreover, in this case $Q \in \mathrm{MJC}(p)$.

Test your I_Q

- It is possible to have $I_Q \subseteq I_R$ with both $Q, R \in \mathrm{MJR}(x)$.
- In that case, $Q \parallel R$.
- If $I_Q \subset I_R$, use Q. If $I_Q = I_R$, choose one.
- If L is join semidistributive, then Q is unique for x.

Manifold bases, non-binary part

Let the binary part be given.

- δ^+: the dependence or DUB consists of all $Q \to p$ with $p \leq \bigvee Q$ nontrivially and Q minimal w.r.t. set containment.
- D^+: the D-basis consists of all $Q \to p$ with $Q \in \mathrm{MJC}(p)$.
- E^+ (for LB only): the E-basis consists of all $Q \to p$ with $Q \in \mathrm{MMJC}(p)$.
- F^+ (for LB only): the F-basis consists of all $Q \to p$ with $Q \in \mathrm{MMJC}(p)$ and p a maximal JI in $\downarrow (\bigvee Q) \setminus I_Q$.
- GD^+: the GD basis consists of all $\mathrm{J}(L) \cap I_Q \to p$ with (1) $x = \bigvee Q$ essential, (2) I_Q minimal for x, and (3) $p \in \downarrow x \setminus I_Q$.
- K^+: the K-basis consists of all $Q \to p$ with $Q \in \mathrm{MJC}(p)$, properties (1)–(3), and p a maximal JI in $\downarrow (\bigvee Q) \setminus I_Q$.

The Goldilocks Principle

$K^+ \subseteq D^+ \subseteq \delta^+$

$F^+ \subseteq E^+ \subseteq D^+$ when it applies.

K^+ and F^+ may be incomparable for a LB lattice, e.g., N_6.

Manifold bases, binary part

- Default is $x \rightarrow y$ whenever $x \geq y$.
- It suffices to include only $x \rightarrow y$ whenever $x \succ y$, ordered top-down for an ordered direct basis.
- For join semidistributive lattices, the binary part of the K^{-} basis contains all pairs $c \rightarrow d$ where d is a canonical joinand of c_{*}.
- Other reductions are sometimes possible! (See next slide.)
- These parts can be used separately in different combinations.

Wilder reductions

Suppose $y \rightarrow Q \rightarrow x \rightarrow Q' \subseteq Q$ with $Q' \neq \emptyset$.
Replace this with $y \rightarrow \{x\} \cup (Q \setminus Q')$ and $Q \rightarrow x \rightarrow Q'$.

Expanding this theme...

The problem of finding an optimal basis for finite bounded lattices is NP-complete.