Consensus theory, median rule and median graphs

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Outline

1 Consensus and multiconsensus rules

2 Median procedure

3 Median graphs

4 Convex expansions in graphs

5 Median graphs and median semilattices

6 Hierarchies

Literature

William H.E.Day(Nova Scotia) and F.R.McMorris(IIT, Chicago)
'"Axiomatic consensus theory in group choice and biomathematics"

Consensus rule

Let X be a generic set of objects to be aggregated, X^k be the set of all k-tuples of X.
Element $(x_1, \ldots, x_k) \in X^k$, $k \geq 2$ is called *a profile*.
Consensus rule is $C : X^k \to X$.
Multiconsensus rule: the domain of C is $\bigcup_{k \leq \omega} X^k$ and the range is $2^X \setminus \emptyset$.

Example with quasi-orders

- S is the set of objects that has to be ranked
- k is the number of individuals (experts) that establish a quasi-order on S
- X is the set of all quasi-orders on S
- $(R_1, \ldots, R_k) \in X^k$ is a profile of quasi-orders established by k experts
- the consensus rule will choose a quasi-order $R = C(R_1, \ldots, R_k)$ that *aggregates* k quasi-orders $R_1, \ldots, R_k$
- example of such rule: $C = Maj$, $(s_1, s_2) \in Maj(R_1, \ldots, R_k))$ iff $|i : (s_1, s_2) \in R_i| \geq |i : (s_2, s_1) \in R_i|$.
- not every profile is *admissible* for Maj: $R_1 = (s_1 > s_2 > s_3), R_2 = (s_2 > s_3 > s_1), R_3 = (s_3 > s_1 > s_2)$

Consensus theory: studies the consensus and multi-consensus functions that could be applied to various models.
Typical result: describing the class of particular functions by the set of their properties.

Example of properties for quasi-orders

Examples of desirable properties:

- *Free triples*: for all $Z = (x, y, z) \in S^3$, for all $R \in X^k$ there exists an admissible $R^* \in X^k$ such that $R|_Z = R^*|_Z$.
- *Independence*: for all $S' \subseteq S$ and all $R, R^* \in X^k$, if $R|_{S'} = R^*|_{S'}$, then $C(R)|_{S'} = C(R^*)|_{S'}$
- *Pareto Optimality*: for all $s_1, s_2 \in S$, and $R \in X^k$, if $(s_1, s_2) \in R_i$, for all $i \leq k$, then $(s_1, s_2) \in C(R)$

Example of undesirable property:

- *Dictatorship*
  there exists $i \leq k$ such that for all $s_1, s_2 \in S$ and all $R \in X^k$, if $(s_1, s_2) \in R_i$, then $(s_1, s_2) \in C(R)$.

Theorem (K. Arrow, Journal of political economy, 1950)
If $C : X^k \to X$ satisfies Free Triples, Independence and Pareto Optimality, then it satisfies the Dictatorship.

Median procedure

Let (X, d) be a finite metric space.

- $\pi = (x_1, \ldots, x_n) \in X^k$ is a profile.
- An element $x \in X$ is called *a median* for π, if $\Sigma_{i \leq k} d(x, x_i)$ is minimum.
- $M(\pi)$ is the set of medians for profile π
- The *median procedure* on (X, d) is a function $M : \bigcup_{k<\omega} X^k \to 2^X \setminus \{\emptyset\}$ such that $M(\pi)$ is the set of medians for $\pi \in X^k$
- Median procedures have a wide applications in studies of consensus and location.

Results on median procedure

- *X* is a distributive semilattice, and *d* is geodesic distance in covering graph of *X*:
  B.Monjardet, *Theorie et application de la mediane dans les trellis distributifs finis*, Ann. Disc. Math. 9 (1980), 87-91.
- *X* is a finite graph and *d* is a geodesic distance:
  F.R.McMorris, Henry M. Mulder, Fred S. Roberts, *The median procedure on median graphs*, Discrete Applied Mathematics 4 (1998), 165-181.

Other examples of consensus rules on metric spaces

- The center function:
  $Cen(\pi) = \{x \in X : e(x, \pi) \text{ is minimum}\}$,
  where $e(x, \pi) = max\{d(x, x_1), \ldots, d(x, x_k)\}$.
- The mean function:
  $Mean(\pi) = \{x \in X : \Sigma d^2(x, x_i) \text{ is minimum}\}$.

Definition of median graph

- $X = (V, E)$ is a connected graph
- $d(x, y) =$ shortest path between $x, y \in V$, also called *geodesics*
- $I(x, y) = \{w \in V : d(x, w) + d(w, y) = d(x, y)\}$ is *the interval* between vertices x and y
- X is a *median* graph, if $|I(x, y) \cap I(y, z) \cap I(z, x)| = 1$, for all $x, y, z \in V$
- If $\pi = \{x, y, z\}$ is a profile, then $M(\pi) = w$, where w is a unique vertex in $I(x, y) \cap I(y, z) \cap I(z, x)$.

Examples

- First introduced/studied by P.Avann (1960), Nebesky (1971), Mulder and Schrijver (1979).
- Trees
- n-cube Q_n: vertices are presented by n-sequences of 0s and 1s, two vertices are adjacent, if they differ at a single component; $M(x, y, z)$ is a majority rule
- grid graphs
- covering graphs of distributive lattices

Bipartite graphs and median graphs

- Every median graph is bipartite: if $x_0, x_1, \ldots x_k, x_{k+1}, \ldots, x_{2k}, x_0$ is a shortest cycle of odd length, then x_k, x_{k+1} would be two different medians for (x_0, x_k, x_{k+1})
- Not every bipartite graph is median: $K_{2,3}$ is a graph where independent subset (x, y, z) has two medians

Convex subgraphs

- $G = (V, E)$ is a graph
- $W \subseteq V$ is called a *convex* subgraph, if $I(x, y) \subseteq W$, for every $x, y \in W$.
- H.M.Mulder, 1980: If G is a median graph, then $I(x, y)$ is convex.
- If $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are two graphs, then $G_1 \cup G_2$ is defined as $G = (V, E)$ with $V = V_1 \cup V_2$ and $E = E_1 \cup E_2$.
- A *proper cover* of G is two convex subgraphs G_1, G_2 such that $G = G_1 \cup G_2$ and $G_1 \cap G_2 \neq \emptyset$ (meaning $V_1 \cap V_2 \neq \emptyset$).
- $G_1 = G_2 = G$ gives an example of a *trivial* proper cover.

Convex expansion

- Suppose G_1, G_2 is a proper cover of G, and $G_0 = G_1 \cap G_2$.
- Graph $G[G_0]$ is called a *convex expansion* of G with respect to this cover, if it is obtained via doubling vertices $u \in G_0$ into pair u_1, u_2, and placing a new edge:$(u_1, u_2) \in E$. Besides, every $(x, u) \in E_1$ is replaced by $(x, u_1) \in E$, and every $(y, u) \in E_2$ is replaced by $(y, u_2) \in E$.

Convex Expansion Theorem

Theorem (H.M.Mulder, 1980). A graph G is a median graph if and only if G can be obtained from one vertex graph K_1 by successive convex expansions.

Characterization of finite bounded lattices

A lattice L is called *bounded*, if every homomorphism $F : FL(n) \to L$ is bounded. Inner characterization of finite bounded lattices: lattices that do not have D-cycles and dual D-cycles. N_5 is an example of bounded lattices, while M_3 has a D-cycle.

Theorem. (A. Day, 1979) A finite lattice L is bounded if and only if L can be obtained by successive doubling of intervals from one-element lattice.

Median semilattice

A finite $\wedge$ semilattice S is called *median*, if

- it is distributive, i.e. every interval $[0, x]$ is a distributive lattice, $x \in S$;
- it satisfies *coronation property*: if join exists for every pair of triple x, y, z, then $x \vee y \vee z$ also exists.

Theorem.(H.M.Mulder) A graph G is a median graph iff G is a covering graph of a median semilattice. Moreover, there exists one-to-one correspondence between the median semilattices defined on some finite set V, and pairs $(G, 0)$, where G is a median graph with set of vertices V, and $0 \in V$.

Questions

- Characterize median graphs for which the corresponding median semilattice is a lattice.
- Verify whether the Day's doubling construction that leads to some finite distributive lattice is equivalent to convex expansions of its covering graph.

Median procedure on median graphs

Suppose $L : V^* \to 2^V \setminus \{\emptyset\}$ is some consensus rule on V, and we have graph $G = (V, E)$ with regular geodesic distance d.

Theorem. (McMorris, Mulder and Roberts, 1998)

If G is a median graph, then L is the median rule iff the following properties hold:

- (S) Symmetry: $L(\sigma(\overline{x})) = L(\overline{x})$
- (C) Consistency: if $L(\pi) \cap L(\pi') \neq \emptyset$ for some profiles π, π', then $L(\pi) \cap L(\pi') = L(\pi^\smile \pi')$, where $\pi^\smile \pi'$ is concatenation of π and π'.
- (B) Betweenness: $L((x, y)) = I(x, y)$.
- (Cv) Convexity: For profiles π of length $k \geq 2$, if $\bigcap_{i \leq k} L(\pi \setminus x_i) = \emptyset$, then $L(\pi) = Con(\bigcup_{i \leq k} L(\pi \setminus x_i))$.

Properties

- (S) Symmetry: $L(\sigma(\overline{x})) = L(\overline{x})$;
- (C) Consistency: if $L(\pi) \cap L(\pi') \neq \emptyset$ for some profiles π, π', then $L(\pi) \cap L(\pi') = L(\pi^\smile \pi')$, where $\pi^\smile \pi'$ is concatenation of π and π';
- (B) Betweenness: $L((x, y)) = I(x, y)$.

hold for median rule on arbitrary metric space.

A median graph G is called *cube-free*, if it does not contain Q_3 as a subgraph (Q_3 is isomorphic to the graph formed by vertices and edges of 3-dimensional cube)
Theorem. (McMorris, Mulder and Roberts, 1998) If G is a median cube-free graph, then L is the median rule iff L satisfies properties (S),(C) and (B).

Phylogenetics

- Systematics biology: relationships between groups of organisms.
- Phylogenetics: analysis of trees and networks to represent the evolution of species, populations and individuals.
- Phylogenetic tree shows the evolutionary relationships among various species believed to have a common ancestor
- The same data set may be analyzed with several methods, which bring to different evolutionary trees.
- There exists a need to generate a consensus tree: a summary of agreement between evolutionary trees.

Polygenetic tree of crocodilian outgroups

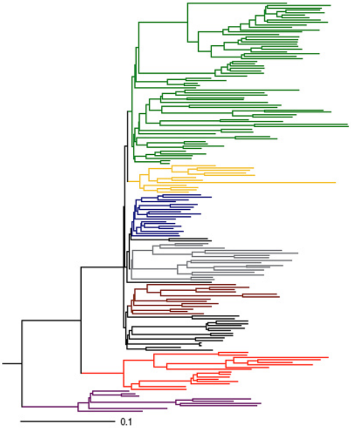

Figure: Polygenetic tree

Trees and hierarchies

Most typical evolutionary tree T:

- has a root of degree at least two
- the external nodes represent the most recent biological entities
- internal nodes (of degree at least three) represent common ancestors of their descendant-nodes

Suppose X is the set of external nodes. For each internal node y, let $X_y \subseteq X$ be a set of external nodes such that y belongs to the path from root r to $x \in X_y$.
Each X_y is called a cluster, and the tree can be represented by a system of clusters. For a tree T, let $H(T)$ be the system of clusters. It is called a hierarchy on X.

Example

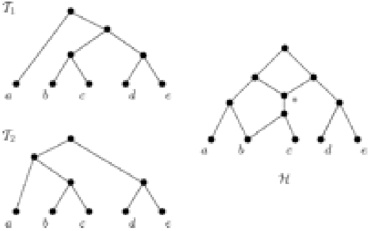

Figure: Examples of poly-trees

If $X = \{a, b, c, d, e\}$ is the set of external nodes of tree T_2, then hierarchy $H(T_2) =$
$\{\{a\}, \{b\}, \{c\}, \{d\}, \{e\}, \{b, c\}, \{a, b, c\}, \{d, e\}, \{a, b, c, d, e\}\}$.

Characterization

A hierarchy H on X is $H \subseteq 2^X$:

- $\emptyset \notin H$
- $\{x\} \in H$, for all $x \in X$
- if $Y, Z \in H$, then $Y \cap Z \in \{Y, Z, \emptyset\}$

Theorem. (Semple and Steel, 2003) H is a hierarchy on X iff $H = H(T)$ for some rooted tree T with external set of vertices X.

Poset of hierarchies

- All hierarchies on X can be ordered by inclusion, forming a poset $PH(X)$.
- Smallest element in $PH(X)$ is a trivial hierarchy H_0 comprising one-element subsets of X and X itself.
- (W.H. Day and F.R. McMorris, 2003) $PH(X)$ is a median semilattice.
- Characterize median semilattices that can be represented as $PH(X)$.
- Is every median semilattice a sub-semilattice of $PH(X)$, for some X?

Last slide: future program

Several directions in connecting this topic with convex geometries:

- $PH(X)$ represents a lower set of some convex geometry. What are the minimal and largest extensions of this lower set to the whole convex geometry?
- Use convex geometry of point configurations as the domain for establishing consensus/location rules with some list of desired properties.