Survey on aggregation theory

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PART I - INTRODUCTION

- Definition, basic examples, elementary property
- Some classes of aggregation functions
  - Means
  - Associative aggregation functions
  - t-norms, t-conorms, uninorms
  - Fuzzy integrals

Aggregation

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The process of aggregation

A single output value $\mathcal{A}(\mathbf{x})$

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An arbitrarily long vector of inputs $\mathbf{x} = (x_1, \ldots, x_n)$

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In order to reach a consensus on these judgements, aggregation functions may be applied.

Aggregation functions

Aggregation functions are used in:

- pure mathematics (theory of means and averages, measure and integration theory),
- applied mathematics (probability, statistics),
- computer and engineering sciences (artificial intelligence, information theory, automated reasoning),
- economics and finance (game theory, voting theory, decision making),
- social sciences,
- physics and natural sciences

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Requirements:

- boundary conditions
- nondecreasing monotonicity

Aggregation functions

Definition

Let $\mathbb{I} \subseteq \mathbb{R}$, $n \in \mathbb{N}$ and let $\mathbf{x} := (x_1, \ldots, x_n) \in \mathbb{I}^n$. *An aggregation function* with n variables is a function $\mathcal{A} : \mathbb{I}^n \to \mathbb{I}$ such that

- $\mathcal{A}$ is nondecreasing: for any $\mathbf{x}, \mathbf{x}^* \in \mathbb{I}^n$

$$\mathbf{x} \leq \mathbf{x}^* \quad \Rightarrow \quad \mathcal{A}(\mathbf{x}) \leq \mathcal{A}(\mathbf{x}^*)$$

- $\mathcal{A}$ fulfills the boundary conditions

$$\inf_{\mathbf{x} \in \mathbb{I}^n} \mathcal{A}(\mathbf{x}) = \inf \mathbb{I} \quad \textit{and} \quad \sup_{\mathbf{x} \in \mathbb{I}^n} \mathcal{A}(\mathbf{x}) = \sup \mathbb{I}.$$

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If $\mathbb{I} = [0, 1]$, the boundary conditions say:

$$\mathcal{A}(0, \ldots, 0) = 0 \;\; \textit{and} \;\; \mathcal{A}(1, \ldots, 1) = 1.$$

Aggregation functions

An extended aggregation function

The mapping

$$\mathcal{A} : \bigcup_{n\in\mathbb{N}} \mathbb{I}^n \to \mathbb{I}$$

such that

$$\mathcal{A}^{(n)} := \mathcal{A}_{|\mathbb{I}^n}$$

is an n-ary aggregation function for any $n \in \mathbb{N}$.

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A sequence of functions
$$(\mathcal{A}^{(n)} : \mathbb{I}^n \to \mathbb{I})_{n\in\mathbb{N}},$$
whose nth element is an n-ary function $\mathcal{A}^{(n)} : \mathbb{I}^n \to \mathbb{I}$.

Aggregation functions - Basic examples

Let $\mathbf{x} := (x_1, \ldots, x_n) \in \mathbb{I}^n$.

- The arithmetic mean function

$$\mathcal{AM}(\mathbf{x}) := \frac{1}{n}\sum_{i=1}^{n} x_i$$

- The geometric mean function (for $\mathbb{I} \subseteq [0, \infty]$)

$$\mathcal{GM}(\mathbf{x}) := (\prod_{i=1}^{n} x_i)^{\frac{1}{n}}$$

- The k-projection function, for any $k \in \{1, \ldots, n\}$

$$\mathcal{P}_k(\mathbf{x}) := x_k$$

Aggregation functions - Basic examples

Let π be a permutation on the set $\{1, \ldots, n\}$ such that $x_{\pi(1)} \leq \ldots \leq x_{\pi(n)}$.

- The k-order statistic function, for any $k \in \{1, \ldots, n\}$

$$\mathcal{OS}_k(\mathbf{x}) := x_{\pi(k)}$$

  The extreme order statistic functions

$$\mathcal{OS}_1(\mathbf{x}) = \min\{x_1, \ldots, x_n\} = Min(\mathbf{x})$$

$$\mathcal{OS}_n(\mathbf{x}) = \max\{x_1, \ldots, x_n\} = Max(\mathbf{x})$$

- The median of values $x_1, \ldots, x_n$

$$Med(x_1, \ldots, x_n) = \begin{cases} x_{\pi(k)} & \textit{if } n = 2k - 1 \\ \frac{x_{\pi(k)} + x_{\pi(k+1)}}{2} & \textit{if } n = 2k. \end{cases}$$

Aggregation functions - Basic examples

Let $\mathbf{w} := (w_1, \ldots, w_n) \in \mathbb{I}^n$ be any (weight) vector such that $\sum_{i=1}^{n} w_i = 1$.

- The weighted arithmetic mean function

$$\mathcal{WAM}_{\mathbf{w}}(\mathbf{x}) := \sum_{i=1}^{n} w_i x_i$$

- The ordered weighted averaging function

$$\mathcal{OWA}_{\mathbf{w}}(\mathbf{x}) := \sum_{i=1}^{n} w_i x_{\pi(i)}$$

Aggregation functions - Basic examples

Let S be a nonempty subset of $\{1, \ldots, n\}$.

- The partial minimum function (associated with S)

$$Min_S(\mathbf{x}) := \min_{i \in S} x_i$$

- The partial maximum function (associated with S)

$$Max_S(\mathbf{x}) := \max_{i \in S} x_i$$

- The sum $\Sigma : (0, \infty)^n \to (0, \infty)$

$$\Sigma(\mathbf{x}) := \sum_{i=1}^{n} x_i$$

- The product $\Pi : (0, \infty)^n \to (0, \infty)$

$$\Pi(\mathbf{x}) := \prod_{i=1}^{n} x_i$$

Aggregation functions - Examples

Assume $\mathbb{I} = [a, b]$ is a closed interval.

The smallest aggregation function

$$\mathcal{A}_{\perp}(\mathbf{x}) := \begin{cases} b, & \text{if } x_i = b \text{ for all } i \in \{1, \ldots, n\} \\ a, & \text{otherwise} \end{cases}$$

The greatest aggregation function

$$\mathcal{A}^{\top}(\mathbf{x}) := \begin{cases} a, & \text{if } x_i = a \text{ for all } i \in \{1, \ldots, n\} \\ b, & \text{otherwise} \end{cases}$$

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Non aggregation function

The constant function $K_c : \mathbb{I}^n \to \mathbb{I}$, given by

$$K_c(\mathbf{x}) := c,$$

where $c \in \mathbb{I}$ is a fixed constant (unless $\mathbb{I} = \{c\}$).

Elementary properties

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Example

If we consider the aggregation of opinions in voting procedures, and as usual, the voters are anonymous, the aggregation function must be symmetric.

Continuity

Example

The function $\mathcal{A} : (0, \infty)^n \to (0, \infty)$

$$\mathcal{A}(\mathbf{x}) := \frac{(\sum_{i=1}^{n} x_i)^2 - \sum_{i=1}^{n} x_i^2}{\sum_{i=1}^{n} x_i}$$

is continuous aggregation function.

Idempotency

Definition

$\mathcal{A} : \mathbb{I}^n \to \mathbb{R}$ is an idempotent function if, for all $x \in \mathbb{I}$,

$$\mathcal{A}(x, \ldots, x) = x.$$

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$\mathcal{AM}$, $\mathcal{WAM}_{\mathbf{w}}$, $\mathcal{OWA}_{\mathbf{w}}$, *Min*, *Max* and *Med* are idempotent aggregation functions.

Idempotency

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Let $c \in (0, 1)$ be an arbitrarily chosen element. The aggregation function $\mathcal{A}_c : [0, 1]^n \to [0, 1]$

$$\mathcal{A}_c(\mathbf{x}) := Med(0, c + \sum_{i=1}^{n}(x_i - c), 1)$$

is not idempotent but has a nonextreme idempotent element c.

Idempotency

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Every aggregation function on $\mathbb{I} = [a, b]$ is weakly idempotent.

Symmetry

Definition

$\mathcal{A} : \mathbb{I}^n \to \mathbb{R}$ is symmetric function if, for all $\mathbf{x} \in \mathbb{I}^n$ and any permutation π on the set $\{1, \ldots, n\}$

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Functions: $\mathcal{AM}$, $\mathcal{GM}$ and $\mathcal{OWA}_{\mathbf{w}}$ are symmetric.
Nonsymmetric aggregation function is $\mathcal{WAM}_{\mathbf{w}}$.

Symmetry

Proposition [J.J.Rotman, 1995]

$\mathcal{A} : \mathbb{I}^n \to \mathbb{R}$ is symmetric function if and only if, for all $\mathbf{x} \in \mathbb{I}^n$ we have

- $\mathcal{A}(x_2, x_1, x_3, \ldots, x_n) = \mathcal{A}(x_1, x_2, x_3, \ldots, x_n)$
- $\mathcal{A}(x_2, x_3, \ldots, x_n, x_1) = \mathcal{A}(x_1, x_2, x_3, \ldots, x_n)$

Conjunction

Definition

A function $\mathcal{A} : \mathbb{I}^n \to \mathbb{R}$ is conjunctive if, for all $\mathbf{x} \in \mathbb{I}^n$

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Min is the greatest conjunctive aggregation function.
Min is also the only idempotent conjunctive aggregation function.

Disjunction

Definition

A function $\mathcal{A} : \mathbb{I}^n \to \mathbb{R}$ is disjunctive if, for all $\mathbf{x} \in \mathbb{I}^n$

$$Max(\mathbf{x}) \leq \mathcal{A}(\mathbf{x}) \leq \sup \mathbb{I}.$$

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Internality

Definition

A function $\mathcal{A} : \mathbb{I}^n \to \mathbb{R}$ is internal if, for all $\mathbf{x} \in \mathbb{I}^n$

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Proposition [B.de Finetti, 1931]

If $\mathcal{A} : \mathbb{I}^n \to \mathbb{R}$ is internal, then it is idempotent.

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If $\mathcal{A} : \mathbb{I}^n \to \mathbb{R}$ is internal, then it is idempotent.
If $\mathcal{A} : \mathbb{I}^n \to \mathbb{R}$ is nondecreasing and idempotent then it is internal.

Associativity

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Extended associativity

A sequence of functions $(\mathcal{A}^{(n)} : \mathbb{I}^n \to \mathbb{R})_{n \in \mathbb{N}}$ is associative if, for any $x \in \mathbb{I}$, $\mathbf{x}_1 \in \mathbb{I}^p$ and $\mathbf{x}_2 \in \mathbb{I}^k$ with $n = p + k$,

$$\mathcal{A}^{(1)}(x) = x, \; \textit{and}$$

$$\mathcal{A}^{(n)}(\mathbf{x}_1, \mathbf{x}_2) = \mathcal{A}^{(2)}(\mathcal{A}^{(p)}(\mathbf{x}_1), \mathcal{A}^{(k)}(\mathbf{x}_2)).$$

Associativity

Min and *Max* are associative functions.
Functions $\mathcal{AM}$ and $\mathcal{GM}$ are not associative.

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The aggregation procedure can be decomposed into partial aggregation and we can start with the aggregation procedure before knowing all inputs to be aggregated.

Means

Definition

An n-ary internal aggregation function $\mathcal{A} : \mathbb{I}^n \to \mathbb{I}$ is called a mean.

Quasi-arithmetic mean

Definition

Let $f : \mathbb{I} \to \mathbb{R}$ be a continuous and strictly monotonic function. The n-ary quasi-arithmetic mean (generated by f) is the function $\mathcal{A}_f : I^n \to \mathbb{I}$ defined as

$$\mathcal{A}_f(\mathbf{x}) := f^{-1}(\frac{1}{n}\sum_{i=1}^{n} f(x_i)).$$

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Remark

Quasi-arithmetic means are symmetric functions.

Quasi-arithmetic means - Examples

- Arithmetic mean

$$\mathcal{AM}(\mathbf{x}) := \frac{1}{n}\sum_{i=1}^{n} x_i, \quad \textit{with} \;\; f(x) = x$$

- Quadratic mean

$$\mathcal{QM}(\mathbf{x}) := (\frac{1}{n}\sum_{i=1}^{n} x_i^2)^{\frac{1}{2}}, \quad \textit{with} \;\; f(x) = x^2$$

- Geometric mean

$$\mathcal{GM}(\mathbf{x}) := (\prod_{i=1}^{n} x_i)^{\frac{1}{n}}, \quad \textit{with} \;\; f(x) = \log x$$

Quasi-arithmetic means - Examples

- Harmonic mean

$$\mathcal{HM}(\mathbf{x}) := \frac{1}{\frac{1}{n}\sum_{i=1}^{n}\frac{1}{x_i}}, \quad with \quad f(x) = x^{-1}$$

- Root mean power

$$\mathcal{M}_{id^\alpha}(\mathbf{x}) := (\frac{1}{n}\sum_{i=1}^{n} x_i^\alpha)^{\frac{1}{\alpha}}, \quad with \quad f(x) = x^\alpha,\ 0 \neq \alpha \in \mathbb{R}$$

- Exponential mean

$$\mathcal{EM}_\alpha(\mathbf{x}) := \frac{1}{\alpha}\ln(\frac{1}{n}\sum_{i=1}^{n} e^{\alpha x_i}), \quad with \quad f(x) = e^{\alpha x},\ 0 \neq \alpha \in \mathbb{R}$$

Quasi-arithmetic means - Examples

$\mathcal{A} : (0,1)^n \to (0,1)$

$$\mathcal{A}(\mathbf{x}) := \frac{\mathcal{GM}(\mathbf{x})}{\mathcal{GM}(\mathbf{x}) + \mathcal{GM}(\mathbf{1} - \mathbf{x})},$$

where $\mathbf{1} := (1, \ldots, 1)$.
The function $\mathcal{A}$ is a quasi-arithmetic mean generated by the increasing function $f(x) = \log \frac{x}{1-x}$, whose inverse function is given by $f^{-1}(x) = \frac{e^x}{1+e^x}$.

Quasi-linear mean (weighted quasi-arithmetic mean)

Definition

Let $f : \mathbb{I} \to \mathbb{R}$ be a continuous, strictly monotonic function and let $w_1, \ldots, w_n > 0$ be real numbers fulfilling $\sum_{i=1}^{n} w_i = 1$. The n-ary quasi-linear mean is the function $\mathcal{A}_{f_{\mathbf{w}}} : I^n \to \mathbb{I}$ defined as

$$\mathcal{A}_{f_{\mathbf{w}}}(\mathbf{x}) := f^{-1}\Big(\sum_{i=1}^{n} w_i f(x_i)\Big).$$

Quasi-linear means - Examples

- Weighted arithmetic mean

$$\mathcal{WAM}_{\mathbf{w}}(\mathbf{x}) := \sum_{i=1}^{n} w_i x_i, \quad with \quad f(x) = x$$

- Weighted quadratic mean

$$\mathcal{WQM}_{\mathbf{w}}(\mathbf{x}) := (\sum_{i=1}^{n} w_i x_i^2)^{\frac{1}{2}}, \quad with \quad f(x) = x^2$$

Quasi-linear means - Examples

- Weighted geometric mean

$$\mathcal{WGM}_{\mathbf{w}}(\mathbf{x}) := \prod_{i=1}^{n} x_i^{w_i}, \quad \textit{with} \quad f(x) = \log x$$

- Weighted root mean power

$$\mathcal{WM}_{id^\alpha,\mathbf{w}}(\mathbf{x}) := (\sum_{i=1}^{n} w_i x_i^\alpha)^{\frac{1}{\alpha}}, \quad \textit{with} \quad f(x) = x^\alpha, \ 0 \neq \alpha \in \mathbb{R}$$

Associative aggregation functions

Theorem, [J.Aczel, 1948]

Let $\mathbb{I}$ be a real interval, which is open on one side. $\mathcal{A} : \mathbb{I}^2 \to \mathbb{I}$ is continuous, strictly increasing, and associative if and only if there exists a continuous and strictly monotonic function $f : \mathbb{I} \to \mathbb{R}$ such that

$$\mathcal{A}(x, y) = f^{-1}(f(x) + f(y)).$$

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Corollary

Every continuous, strictly increasing, and associative function is symmetric.

Associative aggregation functions

$$\mathcal{A}(x, y) = f^{-1}(f(x) + f(y))$$

Example

The sum $\Sigma : (0, \infty)^2 \to (0, \infty)$

$$\Sigma(x, y) = x + y, \quad \textit{with} \;\; f(x) = x$$

and the product $\Pi : (0, \infty)^2 \to (0, \infty)$

$$\Pi(x, y) = x \cdot y = 10^{\log(x)+\log(y)}, \quad \textit{with} \;\; f(x) = \log(x)$$

are continuous, strictly increasing, and associative functions.

Associative aggregation functions

$$\mathcal{A}(x, y) = f^{-1}(f(x) + f(y))$$

If $x \in \mathbb{I}$ then

$$\mathcal{A}(x, x) = f^{-1}(f(x) + f(x)) = f^{-1}(2f(x)) = x \quad \Leftrightarrow \quad f(x) = 0.$$

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Corollary

There is no continuous, strictly increasing, idempotent, and associative function.

Theorem [C.-H. Ling, 1965]

Let $\mathbb{I} = [a, b]$ be a real interval. $\mathcal{A} : \mathbb{I}^2 \to \mathbb{I}$ is continuous, nondecreasing, associative, and such that

$$\mathcal{A}(b, x) = x, \quad \textit{for } x \in \mathbb{I} \ (\textit{b is an identity for } \mathcal{A})$$

$$\mathcal{A}(x, x) < x, \quad \textit{for } x \in (a, b) \ (\textit{there is no idempotents for } \mathcal{A} \textit{ in } (a, b))$$

if and only if there exists a continuous and strictly decreasing function $f : \mathbb{I} \to [0, \infty]$, with $f(b) = 0$, such that

$$\mathcal{A}(x, y) = f^{-1}(Min(f(x) + f(y), f(a))).$$

Associative aggregation functions

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If $\mathcal{A} : \mathbb{I}^2 \to \mathbb{I}$ is an associative function then $(\mathbb{I}, \mathcal{A})$ is a semigroup.

Definition

Let $\{(\mathbb{I}_k, \mathcal{A}_k) \mid k \in K\}$ be a collection of disjoint semigroups indexed by a set K. The *sum* of $\{(\mathbb{I}_k, \mathcal{A}_k) \mid k \in K\}$ is the semigroup $(\bigcup_{k\in K} \mathbb{I}_k, \mathcal{A})$ defined on the set-theoretic union $\bigcup_{k\in K} \mathbb{I}_k$ under the following binary operation:

$$\mathcal{A}(x,y) := \begin{cases} \mathcal{A}_k(x,y), & \textit{if exists } k \in K \textit{ such that } x,y \in \mathbb{I}_k \\ Min(x,y), & \textit{if exist } k_1 \neq k_2 \in K \textit{ with } x \in \mathbb{I}_{k_1},\ y \in \mathbb{I}_{k_2} \end{cases}$$

Definition

A semigroup $(\mathbb{I} = [a, b], \mathcal{A})$ is called *Archimedean*, if the function $\mathcal{A} : \mathbb{I}^2 \to \mathbb{I}$ is continuous, nondecreasing, $\mathcal{A}(b, x) = x$, for $x \in \mathbb{I}$, and $\mathcal{A}(x, x) < x$, for $x \in (a, b)$ or $\mathcal{A}(a, x) = x$, for $x \in \mathbb{I}$, and $\mathcal{A}(x, x) < x$, for $x \in (a, b)$ (one endpoint of $\mathbb{I}$ is an identity for $\mathcal{A}$, and there are no idempotents for $\mathcal{A}$ in (a, b)).

Associative aggregation functions

Theorem [P.S.Mostert and A.L.Shields, 1957]

A function $\mathcal{A} : [a, b]^2 \to [a, b]$ is continuous, associative, and such that $\mathcal{A}(a, x) = \mathcal{A}(x, a) = a$ (a is a zero element) and $\mathcal{A}(b, x) = \mathcal{A}(x, b) = x$ (b is an identity) if and only if

- either $\mathcal{A}(x, y) = Min(x, y)$,
- or $([a, b], \mathcal{A})$ is a conjunctive Archimedean semigroup,
- or $([a, b], \mathcal{A})$ is the sum of conjunctive Archimedean semigroups and one-point semigroups.

Associative aggregation functions

Theorem [J.C.Fodor, 1991]

A function $\mathcal{A} : \mathbb{I}^2 \to I$ is continuous, nondecreasing, idempotent, and associative if and only if there exist $\alpha, \beta \in \mathbb{I}$ such that for all $x, y \in \mathbb{I}$

$$\mathcal{A}(x, y) = Max(Min(\alpha, x), Min(\beta, x), Min(x, y)).$$

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Theorem [L.W.Fung and K.S.Fu, 1975]

A function $\mathcal{A} : \mathbb{I}^2 \to I$ is symmetric, continuous, nondecreasing, idempotent, and associative if and only if there exists $\alpha \in \mathbb{I}$ such that for all $x, y \in \mathbb{I}$

$$\mathcal{A}(x, y) = Med(x, y, \alpha).$$

Associative aggregation functions

Theorem [E.Czogala and J.Drewniak, 1984]

If a function $\mathcal{A} : \mathbb{I}^2 \rightarrow I$ is nondecreasing, idempotent, associative, and has an identity element $e \in \mathbb{I}$, then there exists a decreasing function $g : \mathbb{I} \rightarrow \mathbb{I}$, with $g(e) = e$, such that for all $x, y \in \mathbb{I}$

$$\mathcal{A}(x,y) := \begin{cases} Min(x,y), & if \quad y < g(x), \\ Max(x,y), & if \quad y > g(x), \\ Max(x,y) \ or \ Min(x,y), & if \ y = g(x). \end{cases}$$

Associative aggregation functions

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If a function $\mathcal{A} : \mathbb{I}^2 \to I$ is continuous, nondecreasing, idempotent, associative, and has an identity, then

$$\mathcal{A}(x,y) = Min(x,y) \; or \; \mathcal{A}(x,y) = Max(x,y).$$

t-norms, t-conorms, uninorms

Definition

A t -norm is a symmetric, nondecreasing, and associative function $\mathcal{A} : [0,1]^2 \to [0,1]$ having 1 as the identity.

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t-norms are the most important class of conjunctive aggregation functions. ($\inf \mathbb{I} \leq \mathcal{A}(\mathbf{x}) \leq Min(\mathbf{x})$)

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t-norms are the most important class of conjunctive aggregation functions. $(\inf \mathbb{I} \leq \mathcal{A}(\mathbf{x}) \leq Min(\mathbf{x}))$
t-conorms are the most important class of disjunctive aggregation functions. $(Max(\mathbf{x}) \leq \mathcal{A}(\mathbf{x}) \leq \sup \mathbb{I})$

t-norms, t-conorms, uninorms - Examples

t-norms

$$\mathcal{N} : [0,1]^2 \to [0,1], \;\; \mathcal{N}(x,y) := xy,$$

$$\mathcal{N} : [0,1]^2 \to [0,1], \;\; \mathcal{N}(x,y) := Max(x+y-1,0).$$

t-conorms

$$\mathcal{N} : [0,1]^2 \to [0,1], \;\; \mathcal{N}(x,y) := x+y-xy,$$

$$\mathcal{N} : [0,1]^2 \to [0,1], \;\; \mathcal{N}(x,y) := Min(x+y,1).$$

Uninorms with neutral element $e = \frac{1}{2}$

$$\mathcal{N} : [0,1]^2 \to [0,1],\ \ \mathcal{N}(x,y) := \begin{cases} 2xy, & if\ (x,y) \in [0, \frac{1}{2}]^2 \\ Max(x,y), & otherwise \end{cases}$$

$$\mathcal{N} : [0,1]^2 \to [0,1],\ \ \mathcal{N}(x,y) := \begin{cases} Max(x,y),\ if\ (x,y) \in [\frac{1}{2}, 1]^2 \\ 4xy,\ if\ (x,y) \in [0, \frac{1}{4}]^2 \\ \frac{1}{4}(4x-1)(4y-1) + \frac{1}{4},\ if \\ (x,y) \in [\frac{1}{4}, \frac{1}{2}]^2 \\ Min(x,y),\ otherwise. \end{cases}$$

Theorem [J.-L.Marichal, 2009]

A function $\mathcal{A} : [0, 1]^2 \to [0, 1]$ is continuous, nondecreasing, weakly idempotent, associative and has an identity in $[0, 1]$ if and only if $\mathcal{A}$ is a continuous t-norm or a continuous t-conorm.

Fuzzy integrals

Definition

A (discrete) fuzzy measure on $N := \{1, \dots, n\}$ is a set function

$$\mu : 2^N \to [0,1]$$

that is

- monotonic ($S \subseteq T \;\Rightarrow\; \mu(S) \leq \mu(T)$),
- fulfills the boundary conditions

$$\mu(\emptyset) = 0 \textit{ and } \mu(N) = 1.$$

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- fulfills the boundary conditions

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$\mathcal{F}_N :=$ the set of all fuzzy measures on N

The Choquet integrals

Definition

Let $\mu \in \mathcal{F}_N$. The (discrete) Choquet integral of $\mathbf{x} \in \mathbb{R}^n$ with respect to μ is defined by

$$C_\mu(\mathbf{x}) := \sum_{i=1}^{n} x_{\pi(i)}(\mu(A_{\pi(i)}) - \mu(A_{\pi(i+1)})),$$

where $A_{\pi(i)} := \{\pi(i), \ldots, \pi(n)\}$ and $A_{\pi(n+1)} := \emptyset$.
$(x_{\pi(1)} \leq \ldots \leq x_{\pi(n)})$

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$(x_{\pi(1)} \leq \ldots \leq x_{\pi(n)})$

Example

If $x_3 \leq x_1 \leq x_2$, we have $C_\mu(x_1, x_2, x_3) =$

$$x_3(\mu(\{3,1,2\}) - \mu(\{1,2\})) + x_1(\mu(\{1,2\}) - \mu(\{2\})) + x_2\mu(\{2\}).$$

The Choquet integral - properties

- continuous
- nondecreasing
- idempotent
- internal
- comonotonic additive:

$$f(x_1 + x'_1, \dots, x_n + x'_n) = f(x_1, \dots, x_n) + f(x'_1, \dots, x'_n),$$

  for all vectors such that there is no $i, j \in N$ such that $x_i > x_j$ and $x'_i < x'_j$

- linear with respect to the fuzzy measure:

$$C_{\lambda_1\mu_1+\lambda_2\mu_2} = \lambda_1 C_{\mu_1} + \lambda_2 C_{\mu_2},$$

  for any $\mu_1, \mu_2 \in \mathcal{F}_N$ and all $\lambda_1, \lambda_2 \in \mathbb{R}$ such that $\lambda_1\mu_1 + \lambda_2\mu_2 \in \mathcal{F}_N$.

The Choquet integrals

Proposition [D.Schmeidler, 1986]

A function $f : \mathbb{R}^n \to \mathbb{R}$ is nondecreasing, comonotonic additive, and fulfills $f(\mathbf{1}) = 1$ if and only if there exists $\mu \in \mathfrak{F}_N$ such that $f = C_\mu$.

Proposition [T.Murofushi and M.Sugeno, 1993]

The Choquet integral $C_\mu : \mathbb{R}^n \to \mathbb{R}$ is additive $(f(x_1 + x'_1, \ldots, x_n + x'_n) = f(x_1, \ldots, x_n) + f(x'_1, \ldots, x'_n)$, for all vectors $\mathbf{x}, \mathbf{x}' \in \mathbb{R}^n)$ if and only if there exists $\mathbf{w} \in [0,1]^n$ with $\sum_{i=1}^{n} w_i = 1$ such that $C_\mu = \mathcal{WAM}_\mathbf{w} = \sum_{i=1}^{n} w_i x_i$.

Proposition [M.Grabisch, 1995]

The Choquet integral $C_\mu : \mathbb{R}^n \to \mathbb{R}$ is a symmetric function if and only if there exists a weight vector $\mathbf{w} \in [0,1]^n$ with $\sum_{i=1}^{n} w_i = 1$ such that

$$C_\mu = \mathcal{OWA}_\mathbf{w} = \sum_{i=1}^{n} w_i x_{\pi(i)}.$$

The Sugeno integrals

The unit interval $[0, 1]$ has naturally defined a structure of a distributive lattice (chain) with binary operations:

$$x \wedge y := \min\{x, y\} \ \ \textit{and} \ \ x \vee y := \max\{x, y\}.$$

The Sugeno integrals

Definition

Let $\mu \in \mathcal{F}_N$. The (discrete) Sugeno integral of $\mathbf{x} \in [0,1]^n$ with respect to μ is defined by

$$S_\mu(\mathbf{x}) := \bigvee_{i=1}^{n} (x_{\pi(i)} \wedge \mu(A_{\pi(i)})),$$

where $A_{\pi(i)} := \{\pi(i), \ldots, \pi(n)\}$ and $A_{\pi(n+i)} := \emptyset$.
$(x_{\pi(1)} \leq \ldots \leq x_{\pi(n)})$

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$(x_{\pi(1)} \leq \ldots \leq x_{\pi(n)})$

Example

If $x_3 \leq x_1 \leq x_2$, then we have $S_\mu(x_1, x_2, x_3) =$

$$(x_3 \wedge \mu(\{3,1,2\})) \vee (x_1 \wedge \mu(\{1,2\})) \vee (x_2 \wedge \mu(\{2\})).$$

The Sugeno integrals

$$S_\mu(\mathbf{x}) := \bigvee_{T \subseteq N} (\mu(T) \wedge (\bigwedge_{i \in T} x_i))$$

The Sugeno integrals - properties

- continuous
- nondecreasing
- idempotent
- internal
- comonotonic minitivity:

  $$f(x_1 \wedge x_1', \ldots, x_n \wedge x_n') = f(x_1, \ldots, x_n) \wedge f(x_1', \ldots, x_n'),$$

  for all vectors such that there is no $i, j \in N$ such that $x_i > x_j$ and $x_i' < x_j'$
- comonotonic maxitivity:

  $$f(x_1 \vee x_1', \ldots, x_n \vee x_n') = f(x_1, \ldots, x_n) \vee f(x_1', \ldots, x_n'),$$

  for all vectors such that there is no $i, j \in N$ such that $x_i > x_j$ and $x_i' < x_j'$

Proposition [J.-L.Marichal, 1998]

A function $f : [0,1]^n \to [0,1]$ is nondecreasing, idempotent, comonotonic minitive and maxitive if and only if there exists $\mu \in \mathcal{F}_N$ such that $f = S_\mu$.

Theorem [J.-L.Marichal, 2001]

The set of Sugeno integrals coincides with the set of idempotent lattice polynomial functions.

The Sugeno integrals

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The set of Sugeno integrals coincides with the set of idempotent lattice polynomial functions.

Lattice term functions are Sugeno integrals defined from fuzzy measures $\mu : 2^N \to \{0,1\}$ having their value in the set $\{0,1\}$.

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