Bisymmetry and commuting functions

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PART II

- Entropicity
- The role of unary operations
- Commuting lattice polynomial functions
- Rectangular generalized bisymmetry
- Generalized entropic property

Definition

A function $\mathcal{A} : \mathbb{I}^n \to \mathbb{I}$ is bisymmetric (self-commuting, medial, entropic) if, for any $x_{11}, \ldots, x_{nn} \in \mathbb{I}$

$$\mathcal{A}(\mathcal{A}(x_{11}, \ldots, x_{1n}), \ldots, \mathcal{A}(x_{n1}, \ldots, x_{nn})) =$$

$$\mathcal{A}(\mathcal{A}(x_{11}, \ldots, x_{n1}), \ldots, \mathcal{A}(x_{1n}, \ldots, x_{nn})).$$

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In case $n = 2$,

$$\mathcal{A}(\mathcal{A}(x_{11}, x_{12}), \mathcal{A}(x_{21}, x_{22})) = \mathcal{A}(\mathcal{A}(x_{11}, x_{21}), \mathcal{A}(x_{12}, x_{22})).$$

Example

Associative and symmetric (binary) functions are also entropic.

Bisymmetry

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Associative and symmetric (binary) functions are also entropic. Binary entropic functions with some neutral element are associative and symmetric.

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Associative and symmetric (binary) functions are also entropic.
Binary entropic functions with some neutral element are associative and symmetric.
The class of all associative and symmetric (and, therefore entropic) functions with a neutral element consists of all t-norms, t-conorms and uninorms.

Entropicity

Bisymmetry expresses the condition that aggregation of all the elements of any square matrix

$$\begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & & \vdots \\ x_{n1} & \cdots & x_{nn} \end{pmatrix}$$

can be performed first on the rows, then on the columns, or conversely.

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Definition

Let $(\mathcal{A}^{(n)} : \mathbb{I}^n \to \mathbb{R})_{n\in\mathbb{N}}$ be a sequence of functions. For some $k, p \in \mathbb{N}$, two functions $\mathcal{A}^{(k)}$ and $\mathcal{A}^{(p)}$ are said to be entropic (strongly bisymmetric, commuting) if, for all $x_{11}, \ldots, x_{pk} \in \mathbb{I}$,

$$\mathcal{A}^{(p)}(\mathcal{A}^{(k)}(x_{11}, \ldots, x_{1k}), \ldots, \mathcal{A}^{(k)}(x_{p1}, \ldots, x_{pk})) =$$

$$\mathcal{A}^{(k)}(\mathcal{A}^{(p)}(x_{11}, \ldots, x_{p1}), \ldots, \mathcal{A}^{(p)}(x_{1k}, \ldots, x_{pk})).$$

Entropicity

Example

Let M be an R-module. Two module polynomial operations $f(x_1, \ldots, x_m) = \sum_{i=1}^{m} a_i x_i + c$ and $g(x_1, \ldots, x_n) = \sum_{j=1}^{n} b_j x_j + d$ commute if and only if

$$d \sum_{i=1}^{m} a_i + c = c \sum_{j=1}^{n} b_j + d \quad and$$

$$(a_i b_j - b_j a_i)x = 0 \text{ hold for all } x \in M \text{ and } i \leq m,\ j \leq n.$$

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Example

The projection functions: $\mathcal{P}_1(\mathbf{x}) = x_1$ to the first or $\mathcal{P}_n(\mathbf{x}) = x_n$ to the last coordinate is entropic with arbitrary function.

Example

Consider n judges giving a numerical score to each of p candidates.

Entropicity

Example

Consider n judges giving a numerical score to each of p candidates. Put them in a $p \times n$ matrix

$$\begin{array}{c|ccc} & J_1 & \cdots & J_n \\ \hline C_1 & x_{11} & \cdots & x_{1n} \\ & \vdots & & \vdots \\ C_p & x_{p1} & \cdots & x_{pn} \end{array}$$

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First aggregate the scores of each candidate (aggregation over the rows of the matrix), and then aggregate these overall values.

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First aggregate the scores given by each judge (aggregation over the columns of the matrix), and then aggregate these overall values.

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First aggregate the scores of each candidate (aggregation over the rows of the matrix), and then aggregate these overall values.
First aggregate the scores given by each judge (aggregation over the columns of the matrix), and then aggregate these overall values.
The entropicity property means that these two ways to aggregate must lead to the same overall score.

Entropicity

Theorem [J.Aczel, 1948]

A function $\mathcal{A} : \mathbb{I}^n \to \mathbb{R}$ is symmetric, continuous, strictly increasing, idempotent, and entropic if and only if there is a continuous and strictly monotonic function $f : \mathbb{I} \to \mathbb{R}$ such that $\mathcal{A}$ is the quasi-arithmetic mean

$$\mathcal{A}_f(\mathbf{x}) = f^{-1}\left(\frac{1}{n}\sum_{i=1}^{n} f(x_i)\right)$$

generated by f.

Entropicity

Theorem [J.-L.Marichal, 2009]

A function $\mathcal{A} : [a, b]^2 \to [0, 1]$ is symmetric, continuous, nondecreasing, idempotent, and entropic if and only if there exist $\alpha \leq \beta \in [a, b]$ and two symmetric, continuous, nondecreasing, idempotent, and entropic functions $\mathcal{A}_1, \mathcal{A}_2 : [a, b]^2 \to [0, 1]$ fulfilling $\mathcal{A}_1(a, \alpha) = \alpha$, and $\mathcal{A}_2(b, \beta) = \beta$ and a continuous and strictly monotonic function $f : [\alpha, \beta] \to \mathbb{R}$ such that for all $x, y \in [a, b]$

$$\mathcal{A}(x, y) = \begin{cases} \mathcal{A}_1(x, y), & \textit{if} \quad x, y \in [a, \alpha] \\ \mathcal{A}_2(x, y), & \textit{if} \quad x, y \in [\beta, b] \\ f^{-1}\left(\frac{f(Med(\alpha,x,\beta)) + f(Med(\alpha,y,\beta))}{2}\right), & \textit{otherwise}. \end{cases}$$

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A function $\mathcal{A} : \mathbb{I}^n \to \mathbb{R}$ is continuous, strictly increasing, idempotent, and entropic if and only if there exists a continuous and strictly monotonic function $f : \mathbb{I} \to \mathbb{R}$ and real numbers $w_1, \ldots, w_n > 0$ satisfying $\sum_{i=1}^{n} w_i = 1$ such that $\mathcal{A}$ is a quasi-linear mean

$$\mathcal{A}(\mathbf{x}) = f^{-1}(\sum_{i=1}^{n} w_i f(x_i)).$$

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$$\mathcal{A}(\mathbf{x}) = f^{-1}(\sum_{i=1}^{n} p_i f(x_i) + q).$$

Entropicity

Proposition [J.-L.Marichal, 1998]

The Choquet integral $C_\mu : \mathbb{R}^n \to \mathbb{R}$ is entropic if and only if

$$C_\mu \in \{Min_S, Max_S \mid S \subseteq N\} \cup \{\mathcal{WAM}_{\mathbf{w}} \mid \mathbf{w} \in [0,1]^n\}.$$

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Theorem [J.-L.Marichal and P.Mathonet and E.Tousset, 1997]

An aggregation function $\mathcal{A} : \mathbb{I}^n \to \mathbb{I}$ is entropic and satisfies

$$\mathcal{A}(rx_1 + t, \ldots, rx_n + t) = r\mathcal{A}(\mathbf{x}) + t$$

for all $\mathbf{x} \in \mathbb{I}^n$ and all $r > 0$, $t \in \mathbb{R}$ such that $rx_1 + t, \ldots, rx_n + t \in \mathbb{I}$ if and only if

$$\mathcal{A} \in \{Min_S, Max_S \mid S \subseteq N\} \cup \{\mathcal{WAM}_{\mathbf{w}} \mid \mathbf{w} \in [0,1]^n\}.$$

The role of unary operations

$\mathcal{A} : \mathbb{I}^n \to \mathbb{R}$ - an *n*-ary function
$+ : \mathbb{I}^2 \to \mathbb{R}$ - an associative binary operation
(in particular, $+$ is t-norm or t -conorm)

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The generalized Cauchy equation

$$\mathcal{A}(x_1 + x_1', \ldots, x_n + x_n') = \mathcal{A}(x_1, \ldots, x_n) + \mathcal{A}(x_1', \ldots, x_n')$$

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A function which fulfills Cauchy equation we will call *additive*

The role of unary operations

Proposition [J.Aczel, 1966]

A function $\mathcal{A} : \mathbb{I}^n \to \mathbb{R}$ satisfies Cauchy equation if and only if there exist unary additive functions $f_i : \mathbb{R} \to \mathbb{R}$, $i = 1, \ldots, n$, such that, for all $\mathbf{x} \in \mathbb{R}^n$

$$\mathcal{A}(\mathbf{x}) = \sum_{i=1}^{n} f_i(x_i).$$

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$$\mathcal{A}(\mathbf{x}) = \sum_{i=1}^{n} f_i(x_i).$$

Under continuous (or nondecreasing monotonicity), the function $\mathcal{A}$ is of the form

$$\mathcal{A}(\mathbf{x}) = \sum_{i=1}^{n} c_i x_i,$$

where $c_1, \ldots, c_n$ are arbitrary (respectively, nonnegative) real constants.

The role of unary operations

Proposition [D.Dubois and H.Prade, 1990]

A function $\mathcal{A} : [a, b]^n \to \mathbb{R}$ commutes with *Min* function if and only if there exist nondecreasing unary functions $f_i : [a, b], \to \mathbb{R}$, $i = 1, \ldots, n$, such that for all $\mathbf{x} \in \mathbb{R}^n$

$$\mathcal{A}(\mathbf{x}) = Min(f_1(x_1), \ldots, f_n(x_n)).$$

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$$f_{a,i,\mathcal{A}} := \mathcal{A}(a, \ldots, a, x, a, \ldots, a),$$

with x at i-th position.

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Note that if the function $\mathcal{A}$ is nondecreasing then each $f_{a,i,\mathcal{A}}$ is nondecreasing too.

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Note that if the function $\mathcal{A}$ is nondecreasing then each $f_{a,i,\mathcal{A}}$ is nondecreasing too.

Proposition [S.Saminger-Platz, R.Mesiar, D.Dubois, 2007]

For any k-ary aggregation function $\mathcal{A} : [0,1]^k \to [0,1]$ and p-ary aggregation function $\mathcal{B} : [0,1]^p \to [0,1]$ with an idempotent element $a \in [0,1]$, $k, p \in \mathbb{N}$, if $\mathcal{A}$ commutes with $\mathcal{B}$, then functions $f_{a,i,\mathcal{A}}$ and $\mathcal{B}$ also commute.

The role of unary operations

Let

$$\mathcal{F}_{\mathcal{A}} := \{f : [0,1] \to [0,1] \mid f \textit{ is nondecreasing},$$
$$f(\mathcal{A}(x_1, \ldots, x_n)) = \mathcal{A}(f(x_1), \ldots, f(x_n))\}$$

be the set of all nondecreasing unary functions that commute with $\mathcal{A}$.

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The set is non-empty, since it contains at least the identity mapping.

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$$\mathcal{F}_{id} = \{f : [0,1] \to [0,1] \mid f \textit{ is nondecreasing}\}$$

Example

Let $\alpha \in [0,1]$. The function

$$Med_\alpha(x,y) := Med(x,y,\alpha)$$

is an associative and symmetric (therefore entropic) aggregation function.
A nondecreasing unary function $f : [0,1] \to [0,1]$ commutes with Med_α if and only if either

$$\begin{aligned} f(\alpha) &= \alpha \text{ or} \\ f(\alpha) &= f(1) < \alpha \text{ or} \\ f(\alpha) &= f(0) > \alpha. \end{aligned}$$

The role of unary operations

Let $(\mathcal{A}^{(n)} : \mathbb{I}^n \to [0,1])_{n \in \mathbb{N}}$ be a sequence of aggregation functions.

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$$\bigcap_{n\in\mathbb{N}} \mathcal{F}_{\mathcal{A}^{(n)}}$$

is the set of all functions $f \in \mathcal{F}_{id}$ that commute with each operation $\mathcal{A}^{(n)}$, $n \in \mathbb{N}$

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Proposition [S.Saminger-Platz, R.Mesiar, D.Dubois, 2007]

Let $(\mathcal{A}^{(n)} : [0,1]^n \to [0,1])_{n\in\mathbb{N}}$ be a sequence of entropic aggregation functions and for $1 \leq i \leq n$, $f_i \in \bigcap_{n\in\mathbb{N}} \mathcal{F}_{\mathcal{A}^{(n)}}$. Then, for each $n \in \mathbb{N}$, the n-ary function

$$\mathcal{A}^{(n)}(f_1(x_1), \dots, f_n(x_n))$$

commutes with each function $\mathcal{A}^{(m)}$, for any $m \in \mathbb{N}$.

The role of unary operations

Example

Let $f_i(x) = 1$, for all $i \in \{1, \dots, n\}$. Then

$$\mathcal{B}(x_1, \dots, x_n) = \mathcal{A}^{(n)}(f_1(x_1), \dots, f_n(x_n)) = \mathcal{A}^{(n)}(1, \dots, 1) = 1,$$

for arbitrary $x_i \in [0, 1]$. And the boundary condition $\mathcal{B}(0, \dots, 0) = 0$ does not hold.

The role of unary operations

Example

The set

$$\{Min(f_1(x_1), \dots, f_n(x_n)) \mid f_i \in \mathcal{F}_{id},\ f_i(1) = 1 \textit{ for all } i \in \{1, \dots, n\},$$

$$\textit{and } f_i(0) = 0 \textit{ for at least one } i \in \{1, \dots, n\}\}$$

is the class of all aggregation function commuting with the *Min* function.

The role of unary operations

Example

If $(\mathcal{P}_1^{(n)} : [0,1]^n \to [0,1])_{n\in\mathbb{N}}$ is a sequence of the projections to the first coordinate, then

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But

$$\mathcal{P}_1^{(n)}(f_1(x_1), \ldots, f_n(x_n)) = f_1(x_1),$$

hence only functions depending just on the first coordinate can be obtained.

The role of unary operations

Definition - neutral element

An element $e \in [0,1]$ is a neutral of an n-ary function $\mathcal{A} : [0,1]^n \to [0,1]$ if for all $x \in [0,1]$,

$$\mathcal{A}(x, e, \ldots, e) = \mathcal{A}(e, x, e, \ldots, e) = \cdots = \mathcal{A}(e, \ldots, e, x) = x.$$

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An element $e \in [0,1]$ is a neutral of a sequence $(\mathcal{A}^{(n)} : [0,1]^n \to [0,1])_{n \in \mathbb{N}}$ of functions if it is neutral of each function $\mathcal{A}^{(n)}$ for all $n \in \mathbb{N}$.

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If $\mathcal{A}$ and $\mathcal{B}$ are two entropic functions of the same arity with a neutral element e, then $\mathcal{A} = \mathcal{B}$.

The role of unary operations

Proposition [S.Saminger-Platz, R.Mesiar, D.Dubois, 2007]

Let $(\mathcal{A}^{(n)} : [0,1]^n \to [0,1])_{n\in\mathbb{N}}$ be a sequence of entropic aggregation functions with a neutral element. An n-ary function $\mathcal{B}$, $n \in \mathbb{N}$, commutes with the sequence if and only if there exists $f_i \in \bigcap_{n\in\mathbb{N}} \mathcal{F}_{\mathcal{A}^{(n)}}$, $i \in \{1,\ldots,n\}$, such that

$$\mathcal{B} = \mathcal{A}^{(n)}(f_1(x_1),\ldots,f_n(x_n)).$$

The role of unary operations

Recall: Any binary entropic aggregation function with neutral element is either a t-norm, a t-conorm or a uninorm.

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Fact

Let $\mathcal{A}$ and $\mathcal{B}$ be two binary functions with neutral elements $e_{\mathcal{A}}$ and $e_{\mathcal{B}}$, respectively. If $\mathcal{A}$ commutes with $\mathcal{B}$, then $e_{\mathcal{A}} = e_{\mathcal{B}}$.

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Fact

Let $\mathcal{A}$ and $\mathcal{B}$ be two binary functions with neutral elements $e_{\mathcal{A}}$ and $e_{\mathcal{B}}$, respectively. If $\mathcal{A}$ commutes with $\mathcal{B}$, then $e_{\mathcal{A}} = e_{\mathcal{B}}$.

The only binary functions commuting with t-norms, t-conorms, or uninorms are (besides the operation itself) aggregation functions with no neutral elements.

Example

For t-conorm $\mathcal{N} : [0,1]^2 \to [0,1]$, $\mathcal{N}(x,y) := x + y - xy$, we have

$$\mathcal{F}_{\mathcal{N}} = \{\mathbf{0}, \mathbf{0}_{[0,1)}, \mathbf{0}_{\{0\}}, \mathbf{1}\} \cup \{f : [0,1] \to [0,1] \mid$$

$$f(x) = 1 - (1-x)^{\lambda},\ \lambda \in (0,\infty)\},$$

where $\mathbf{0} : [0,1] \to [0,1]$, $x \mapsto 0$, $\mathbf{1} : [0,1] \to [0,1]$, $x \mapsto 1$, and $\mathbf{0}_S : [0,1] \to [0,1]$ is defined by

$$\mathbf{0}_S(x) := \begin{cases} 0, & \textit{if } x \in S \subset [0,1] \\ 1, & \textit{otherwise.} \end{cases}$$

The role of unary operations

OPEN PROBLEM

A full characterization of all entropic aggregation operators with neutral element, (in particular if the neutral element is from the open interval $(0, 1)$), is still unknown.

Commuting lattice polynomial functions

Recall: (Discrete) Sugeno integrals

$$S_\mu(\mathbf{x}) := \bigvee_{T \subseteq N} (\mu(T) \wedge (\bigwedge_{i \in T} x_i))$$

can be seen as idempotent polynomial functions.

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can be seen as idempotent polynomial functions.
Let $(L, \vee, \wedge) = (L, \leq)$ be a distributive lattice, $M := \{1, \ldots, m\}$, and $N := \{1, \ldots, n\}$.

Commuting lattice polynomial functions

Definition

For any lattice polynomial operation $f : L^m \to L$ we will call its disjunctive normal form (DNF)

$$f(x_1, \ldots, x_m) = \bigvee_{S \subseteq M} (a_S \wedge \bigwedge_{i \in S} x_i)$$

maximal if the following conditions hold for the coefficients a_S:

- $a_S \leq a_T$ whenever $S \subseteq T \subseteq M$, and
- $a_\emptyset \neq 1$ if $1 \notin L$ and $a_M \neq 0$ if $0 \notin L$.

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Every polynomial operation of a distributive lattice has a unique maximal DNF.

Commuting lattice polynomial functions

Theorem [M.Behrisch, M.Couceiro, K.A.Kearnes, E.Lehtonen, A.Szendrei, 2010]

$(L, \vee, \wedge)$ - a distributive lattice, f, g - p.o. with maximal DNFs

$$f(x_1, \ldots, x_m) = \bigvee_{S \subseteq M} (a_S \wedge \bigwedge_{i \in S} x_i), \quad g(x_1, \ldots, x_m) = \bigvee_{T \subseteq N} (b_T \wedge \bigwedge_{j \in T} x_j)$$

Then functions f and g commute if and only if the 2-rectangle condition

$$a_\emptyset \vee (a_M \wedge b_\emptyset) \vee (a_{U_1 \cap U_2} \wedge b_{V_1 \cup V_2}) \vee (a_{U_1} \wedge b_{V_1})$$
$$\vee (a_{U_2} \wedge b_{V_2}) \vee (a_{U_1 \cup U_2} \wedge b_{V_1} \wedge b_{V_2}) =$$
$$b_\emptyset \vee (b_N \wedge a_\emptyset) \vee (b_{V_1 \cap V_2} \wedge a_{U_1 \cup U_2}) \vee (b_{V_1} \wedge a_{U_1})$$
$$\vee (b_{V_2} \wedge a_{U_2}) \vee (b_{V_1 \cup V_2} \wedge a_{U_1} \wedge a_{U_2})$$

holds for all $U_1, U_2 \subseteq M$ and $V_1, V_2 \subseteq N$.

Commuting lattice polynomial functions

Corollary

A p.o. f of a distributive lattice $(L, \vee, \wedge)$, with a DNF $f(x_1, \dots, x_m) = \bigvee_{S \subseteq M} (\widetilde{a}_S \wedge \bigwedge_{i \in S} x_i)$ such that the set $\{\widetilde{a}_S \mid S \subseteq M\}$ of coefficients is a chain in $(L, \vee, \wedge, 1, 0)$, is self-commuting if and only if f has a DNF

$$f(x_1, \dots, x_m) = a_\emptyset \vee \bigvee_{i \in M} (a_i \wedge x_i) \vee \bigvee_{l=1}^{r} (a_{S_l} \wedge \bigwedge_{i \in S_l} x_i) \ \ \textit{such that}$$

- $S_1 \subset S_2 \subset \cdots \subset S_r \subseteq M$, $r \geq 0$,
- $\{a_\emptyset\} \cup \{a_i \mid i \in M\}$ is a chain, and

$$a_\emptyset \vee \bigvee_{i \in S_1} a_i = a_\emptyset \vee \bigvee_{i \in M} a_i < a_{S_1} < \cdots < a_{S_r}.$$

Commuting lattice polynomial functions

Corollary

A polynomial operation f of a distributive lattice is symmetric and self-commuting if and only if it has a DNF of the form

$$f(x_1, \ldots, x_m) = a_\emptyset \vee \bigvee_{i \in M} (a_1 \wedge x_i) \vee (a_M \wedge \bigwedge_{i \in M} x_i)$$

for some $a_\emptyset, a_1, a_M \in L \cup \{0, 1\}$ with $a_\emptyset \leq a_1 \leq a_M$.

Commuting lattice polynomial functions

Corollary

Let $(L, \vee, \wedge)$ be a distributive lattice. Two term operations f and g of the lattice commute if and only if they satisfy one of the following conditions:

a) one of f and g is a projection, and the other one is arbitrary,

b) $f = g = \vee$,

c) $f = g = \wedge$.

Rectangular generalized bisymmetry

Definition

Let $2 \leq m, n \in \mathbb{N}$ and $Y_1, \ldots, Y_m, Z_1, \ldots, Z_n, S$ be nonempty sets. We say that "production functions" $F : Z_1 \times \ldots \times Z_n \to S$ and $F_1, \ldots, F_m : S^n \to S$ are consistent with "aggregation functions" $G : Y_1 \times \ldots \times Y_m \to S$ and $G_1, \ldots, G_n : S^m \to S$ if the following rectangular generalized bisymmetry equation holds

$$G(F_1(x_{11}, \ldots, x_{1n}), \ldots, F_m(x_{m1}, \ldots, x_{mn})) =$$

$$F(G_1(x_{11}, \ldots, x_{m1}), \ldots, G_n(x_{1n}, \ldots, x_{mn})),$$

for all $x_{ji} \in S$, $i = 1, \ldots, n$ and $j = 1, \ldots, m$.

Rectangular generalized bisymmetry

Example

Let us assume that we have m producers and each producer $1 \leq j \leq m$ depends upon n kinds of "inputs" $x_{j1}, \ldots, x_{jn}$ (such as goods, services etc.).

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Rectangular generalized bisymmetry

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To "inputs" each producer applies (microeconomic) production function F_j.
PROBLEM: Do there exist aggregator function G such that the aggregated output $G(F_1(x_{11}, \dots, x_{n1}), \dots, F_n(x_{1m}, \dots, x_{nm}))$ will depend only upon the n aggregated values (outputs) $G_i(x_{i1}, \dots, x_{im})$ through a macroeconomic production function F?

Rectangular generalized bisymmetry

If outputs can be measured by their monetary value, then they can be aggregated by addition, what means that we can assume

$$G(x_1, \dots, x_m) = G_1(x_1, \dots, x_m) = \dots = G_n(x_1, \dots, x_m) = x_1 + \dots + x_m.$$

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In such case, $F, F_1, \ldots, F_m$ will be affine (or linear) functions.

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CD (Cobb-Douglas) functions

$$F(x_1, \dots, x_n) = a x_1^{c_1} x_2^{c_2} \cdots x_n^{c_n}, \quad c_1, \dots, c_n, a > 0$$

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CES (Constant Elasticity of Substitution) functions

$$F(x_1,\ldots,x_n) = a(b_1 x_1^c + b_2 x_2^c + \ldots + b_n x_n^c)^{\frac{1}{c}},$$

$b_1,\ldots,b_n, a > 0, c \neq 0$

Rectangular generalized bisymmetry

Theorem [J.Aczel and G.Maksa, 1996]

Functions $F, F_1, \ldots, F_m$ are consistent with functions $G, G_1, \ldots, G_n$ if and only if there exist an abelian group $(T, \star)$, $T \subseteq S$, surjections $f_{ji} : S \to T$, and bijections $g_j : Y_j \to T$, and $h_i : Z_i \to T$ such that

$$\begin{aligned} F(z_1, \ldots, z_n) &= h_1(z_1) \star \cdots \star h_n(z_n), \\ G(y_1, \ldots, y_m) &= g_1(y_1) \star \cdots \star g_m(y_m), \\ G_i(x_{1i}, \ldots, x_{mi}) &= h_i^{-1}(f_{1i}(x_{1i}) \star \cdots \star f_{mi}(x_{mi})), \\ F_j(x_{j1}, \ldots, x_{jn}) &= g_j^{-1}(f_{j1}(x_{j1}) \star \cdots \star f_{jn}(x_{jn})), \end{aligned}$$

for $z_1 \in Z_1, \ldots, z_n \in Z_n$, $y_1 \in Y_1, \ldots, y_m \in Y_m$ and $x_{ji} \in S$ for $i = 1, \ldots, n$, $j = 1, \ldots, m$.

Rectangular generalized bisymmetry

Theorem [J.Aczel and G.Maksa, 1996]

Continuous functions $F, F_1, \ldots, F_m$ are consistent with continuous functions $G, G_1, \ldots, G_n$ if and only if sets $Y_1, \ldots, Y_m, Z_1, \ldots, Z_n, S$ and $T := F(Z_1, \ldots, Z_n) = G(Z_1, \ldots, Z_n)$ are open intervals, and there exist continuous surjections $\beta_{ji} : S \to \mathbb{R}$, and continuous bijections $\alpha_i : Z_i \to \mathbb{R}$, $\gamma_j : Y_j \to \mathbb{R}$ and $\varphi : T \to \mathbb{R}$ such that

$$
\begin{aligned}
F(z_1, \ldots, z_n) &= \varphi^{-1}(\alpha_1(z_1) + \ldots + \alpha_n(z_n)), \\
G(y_1, \ldots, y_m) &= \varphi^{-1}(\gamma_1(y_1) + \ldots + \gamma_m(y_m)), \\
G_i(x_{1i}, \ldots, x_{mi}) &= \alpha_i^{-1}(\beta_{1i}(x_{1i}) + \ldots + \beta_{mi}(x_{mi})), \\
F_j(x_{j1}, \ldots, x_{jn}) &= \gamma_j^{-1}(\beta_{j1}(x_{j1}) + \ldots + \beta_{jn}(x_{jn})),
\end{aligned}
$$

for $z_1 \in Z_1, \ldots, z_n \in Z_n$, $y_1 \in Y_1, \ldots, y_m \in Y_m$ and $x_{ji} \in S$ for $i = 1, \ldots, n$, $j = 1, \ldots, m$.

Rectangular generalized bisymmetry

Theorem [G.Maksa, 1999]

Suppose that

$$G(x_{11} + \ldots + x_{1n}, \ldots, x_{m1} + \cdots + x_{mn}) =$$

$$F(G_1(x_{11}, \ldots, x_{m1}), \ldots, G_n(x_{1n}, \ldots, x_{mn})),$$

holds for all $x_{ji} \in S$, $i = 1, \ldots, n$ and $j = 1, \ldots, m$. Then there exist a real interval $\mathbb{I}$, continuous and strictly monotonic functions $\varphi : \mathbb{I} \to \mathbb{R}$, $\alpha_i : G_i(S^m) \to \mathbb{R}$, $i = 1, \ldots, n$, and $a_1, \ldots, a_m \in \mathbb{R} \setminus \{0\}$ such that

$$\begin{aligned} F(z_1, \ldots, z_n) &= \varphi^{-1}(\alpha_1(z_1) + \ldots + \alpha_n(z_n)) \\ G(y_1, \ldots, y_m) &= \varphi^{-1}(a_1 y_1 + \ldots + a_m y_m), \\ G_i(x_{1i}, \ldots, x_{mi}) &= \alpha_i^{-1}(a_1 x_{1i} + \ldots + a_m x_{mi}), \end{aligned}$$

for $z_i \in G_i(S^m)$, $y_j, x_{ji} \in S$ for $i = 1, \ldots, n$, $j = 1, \ldots, m$.

Rectangular generalized bisymmetry

Theorem [G.Maksa, 1999]

Let $2 \leq m, n \in \mathbb{N}$ and S be real interval. Let $G : S^m \to \mathbb{R}$, and $F : S^n \to S$ be continuous and strictly monotonic functions. Suppose that

$$F(x_{11} + \ldots + x_{1n}, \ldots, x_{m1} + \ldots + x_{mn}) =$$

$$G(x_{11} + \ldots + x_{m1}, \ldots, x_{1n} + \ldots + x_{mn}),$$

holds for all $x_{ji} \in S$, $i = 1, \ldots, n$ and $j = 1, \ldots, m$.
Then there exist continuous and strictly monotonic function $\varphi : S \to \mathbb{R}$ such that

$$\begin{aligned} F(z_1, \ldots, z_n) &= \varphi(z_1 + \ldots + z_n), \quad \textit{and} \\ G(y_1, \ldots, y_m) &= \varphi(y_1 + \ldots + y_m). \end{aligned}$$

Rectangular generalized bisymmetry

Example

If even one production function is addition, then no aggregation function can be CD or CES functions.

Rectangular generalized bisymmetry

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Example

Only CD (CES) functions as production functions are consistent with CD (CES) aggregation functions.

Rectangular generalized bisymmetry

Example

Functions

$$F(z_1, \ldots, z_n) = a z_1^{c_1} z_2^{c_2} \cdots z_n^{c_n} \ \ and$$

$$F_j(x_{j1}, \ldots, x_{jn}) = a_j z_{j1}^{c_{j1}} z_{j2}^{c_{j2}} \cdots z_{jn}^{c_{jn}},$$

are consistent with functions

$$G(y_1, \ldots, y_m) = y_1 y_2 \cdots y_m \ \ and \ \ G_i(x_{1i}, \ldots, x_{mi}) = x_{1i} \cdots x_{mi}$$

if $a = a_1 a_2 \cdots a_m$ and $c_{j1} = \cdots = c_{jn} = c_j$.

Rectangular generalized bisymmetry

Example

Production functions

$$F(z_1, \dots, z_n) = a(\sum_{i=1}^{n} c_i z_i^b)^{\frac{1}{b}} \quad and \quad F_j(x_{j1}, \dots, x_{jn}) = a_j(\sum_{j=1}^{n} c_{ji} x_{ji}^b)^{\frac{1}{b}},$$

with $a^b c_i = a_j^b c_{ji}$, $j = 1, \dots, m$, are consistent with aggregation functions

$$G(y_1, \dots, y_m) = (\sum_{j=1}^{m} y_j^b)^{\frac{1}{b}} \quad and$$

$$G_i(x_{1i}, \dots, x_{mi}) = (\sum_{i=1}^{n} x_{ji}^b)^{\frac{1}{b}}, \quad i = 1, \dots, n.$$

Generalized entropic property

Definition

Let $2 \leq m, n \in \mathbb{N}$ and S be a nonempty set. We say that functions $F : S^n \to S$, $G, G_1, \ldots, G_n : S^m \to S$ satisfy generalized entropic property if the following holds

$$G(F(x_{11}, \ldots, x_{1n}), \ldots, F(x_{m1}, \ldots, x_{mn})) =$$

$$F(G_1(x_{11}, \ldots, x_{m1}), \ldots, G_n(x_{1n}, \ldots, x_{mn})),$$

for all $x_{ji} \in S$, $i = 1, \ldots, n$ and $j = 1, \ldots, m$.

Generalized entropic property

Remark [K. Adaricheva, A. Pilitowska, D. Stanovský, 2008]

Let $e \in S$ and $F : S^n \to S$, and $G, G_1, \ldots, G_n : S^m \to S$ be functions such that e is a neutral element for F and an idempotent element for all functions $G_1, \ldots, G_n$. If functions $F, G, G_1, \ldots, G_n$ satisfy generalized entropic property then

$$G = G_1 = \ldots = G_n.$$

In fact, functions F and G commute.

Remark [K.Adaricheva, A.Pilitowska, D.Stanovský, 2008]

Let $F : S^n \to S$ be an idempotent function, and $G, G_1, \ldots, G_n : S^m \to S$. If functions $F, G, G_1, \ldots, G_n$ satisfy generalized entropic property and $G_1 = \ldots = G_n$ then also

$$G = G_1.$$

In fact, functions F and G commute.

Generalized entropic property

Assume that on a set S is defined an algebraic structure (S, Ω).

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Theorem [K.Adaricheva, A.Pilitowska, D.Stanovský, 2008]

Let $(S, \cdot)$ be an idempotent groupoid, $F = G = \cdot$, $G_1, G_2 : S^2 \to S$ are term operations of the algebra $(S, \cdot)$, and F, G_1, G_2 satisfy generalized entropic property.

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- If at least one of functions G_1 and G_2 derives from a linear term (a term t is linear, if every variable occurs in t at most once) then F is self-commuting.

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- If at least one of functions G_1 and G_2 derives from a linear term (a term t is linear, if every variable occurs in t at most once) then F is self-commuting.
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- If $(S, \cdot)$ is commutative then F is self-commuting.
- If $(S, \cdot)$ is a semigroup then F is self-commuting.

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- If at least one of functions G_1 and G_2 derives from a linear term (a term t is linear, if every variable occurs in t at most once) then F is self-commuting.
- If $(S, \cdot)$ is commutative then F is self-commuting.
- If $(S, \cdot)$ is a semigroup then F is self-commuting.
- If $(S, \cdot)$ is left (or right) cancellative then F is self-commuting

Generalized entropic property

Corollary [K.Adaricheva, A.Pilitowska, D.Stanovský, 2008]

Let (S, Ω) be an algebra and $e \in$ be a neutral and idempotent element for all basic operations in Ω. If $F, G \in \Omega$, $G_1, \ldots, G_n$ are term operations of (S, Ω), and $F, G, G_1, \ldots, G_n$ satisfy generalized entropic property then functions F and G commute.

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Theorem [E.Lehtonen, A.Pilitowska, 2011]

Let $(S, \cdot, ^{-1})$ be an inverse semigroup. If $F = \cdot$, $G =^{-1}$, G_1, G_2 are term operations of $(S, \cdot, ^{-1})$, and F, G, G_1, G_2 satisfy generalized entropic property then the function F is symmetric.

Generalized entropic property

Definition

An algebra (S, Ω) satisfies the generalized entropic property if for every n-ary operation $F \in \Omega$ and m-ary operation $G \in \Omega$, there exist m-ary terms $G_1, \dots, G_n$ such that identities

$$G(F(x_{11}, \dots, x_{1n}), \dots, F(x_{m1}, \dots, x_{mn})) =$$

$$F(G_1(x_{11}, \dots, x_{m1}), \dots, G_n(x_{1n}, \dots, x_{mn})),$$

hold in (S, Ω).

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hold in (S, Ω).
A variety $\mathcal{V}$ of algebras satisfies the generalized entropic property if each algebra in $\mathcal{V}$ satisfies this property.

Generalized entropic property

Definition

An algebra (S, Ω) has the subalgebras property if, for each n-ary operation $F \in \Omega$, the complex product

$$F(S_1, \ldots, S_n) := \{F(a_1, \ldots, a_n) \mid a_1 \in S_1, \ldots, a_n \in S_n\}$$

of its (non-empty) subalgebras $S_1, \ldots, S_n$ is again a subalgebra.

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Theorem [K.Adaricheva, A.Pilitowska, D.Stanovský, 2008]

Each algebra in a variety $\mathcal{V}$ has the subalgebras property if and only if $\mathcal{V}$ satisfies the generalized entropy property. (The result for a variety of groupoids was earlier proved by Evans, 1962.)