The length of chains in algebraic lattices

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Introduction This talk is about the relationship between the length of chains in an algebraic lattice L and the structure of the join-semilattice $K(L)$ of compact elements of L.

The motivation came from posets and in fact from the theory of relations. Let P be an ordered set (poset). An *ideal* of P is any non-empty up-directed initial segment of P. The set $J(P)$ of ideals of P, ordered by inclusion, is an interesting poset associated with P, and it is natural to ask about the relationship between the two posets. For a concrete example, if $P := [\kappa]^{<\omega}$ the set, ordered by inclusion, consisting of finite subsets of a set of size κ, then $J([\kappa]^{<\omega})$ is isomorphic to $\wp(\kappa)$ the power set of κ ordered by inclusion.

Theorem 1. *(I.Chakir, I, A.U 2005) A poset P contains a subset isomorphic to $[\kappa]^{<\omega}$ if and only if $J(P)$ contains a subset isomorphic to $\wp(\kappa)$.*

Maximal chains in $\wp(\kappa)$ are of the form $I(C)$, where $I(C)$ is the chain of initial segments of an arbitrary chain C of size κ. Hence, if $J(P)$ contains a subset isomorphic to $\wp(\kappa)$ it contains a copy of $I(C)$ for every chain C of size κ, whereas chains in P can be small: eg in $P := [\kappa]^{<\omega}$ they are finite or have order type ω. What happens if for a given order type α, particularly a countable one, $J(P)$ contains no chain of type α? A partial answer was given by Zaguia and I, Order 1984 (Theorem 4, pp.62). In order to state the result, I recall that the order type α of a chain C is *indecomposable* if C is embeddable in I or in $C \setminus I$ for every initial segment I of C.

Theorem 2. * *Given an indecomposable countable order type* α, *there is a finite list of ordered sets* $A_1^\alpha, A_2^\alpha, \ldots, A_{n_\alpha}^\alpha$ *such that for every poset* P, *the set* $J(P)$ *of ideals of* P *contains no chain of type* $I(\alpha)$ *if and only if* P *contains no subset isomorphic to one of the* $A_1^\alpha, A_2^\alpha, \ldots, A_{n_\alpha}^\alpha$.

Now, if P is a join-semilattice with a least element, $J(P)$ is an algebraic lattice, and moreover every algebraic lattice is isomorphic to the poset $J(K(L))$ of ideals of the join-semilattice $K(L)$ of the compact elements of L. Due to the importance of algebraic lattices, it was natural to ask whether the two results above have an analog if posets are replaced by join-semilattices and subposets by join-subsemilattices. This

*In Theorem 4, $I(\alpha)$ is replaced by α. This is due to the fact that if α is a countable indecomposable order type and P is a poset, $I(\alpha)$ can be embedded into $J(P)$ if and only if α can be embedded into $J(P)$.

question was the starting point of a research of I.Chakir and I.

We immediately observed that the specialization of Theorem 1 to this case shows no difference. Indeed *a join-semilattice* P *contains a subset isomorphic to* $[\kappa]^{<\omega}$ *if and only if it contains a join-subsemilattice isomorphic to* $[\kappa]^{<\omega}$. Turning to the specialization of Theorem 2, we noticed that it as to be quite different and is far from being immediate. In fact, we do not know yet whether for every countable α there is a finite list as in Theorem 2.

The purpose of this talk is to present the results obtained in that direction. In order to simplify the presentation, I will denote by $\mathbb{J}$ the class of join-semilattices having a least element. If $\mathbb{B} \subseteq \mathbb{J}$, I denote by $Forb(\mathbb{B})$ the class of $P \in \mathbb{J}$ which contain no join-subsemilattice isomorphic to a member of

$\mathbb{B}$. If α denotes an order type, I denote by $\mathbb{J}_\alpha$ the class of members P of $\mathbb{J}$ such that the lattice $J(P)$ of ideals of P contains a chain of order type $I(\alpha)$. Finally I set $\mathbb{J}_{\neg\alpha} := \mathbb{J} \setminus \mathbb{J}_\alpha$.

Our first result expresses that a characterization as Theorem 2 is possible.

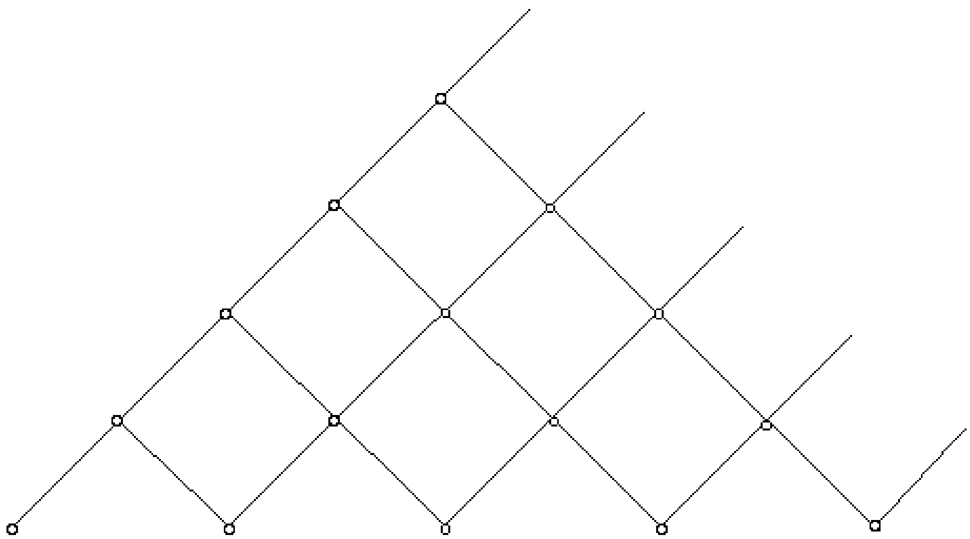
Theorem 3. *For every order type α there is a subset $\mathbb{B}$ of $\mathbb{J}$ of size at most $2^{|\alpha|}$ such that $\mathbb{J}_{\neg\alpha} = Forb(\mathbb{B})$.*

This is very weak. Indeed, we cannot answer the following question.

Question 1. *If α is countable, is there a finite $\mathbb{B}$?*

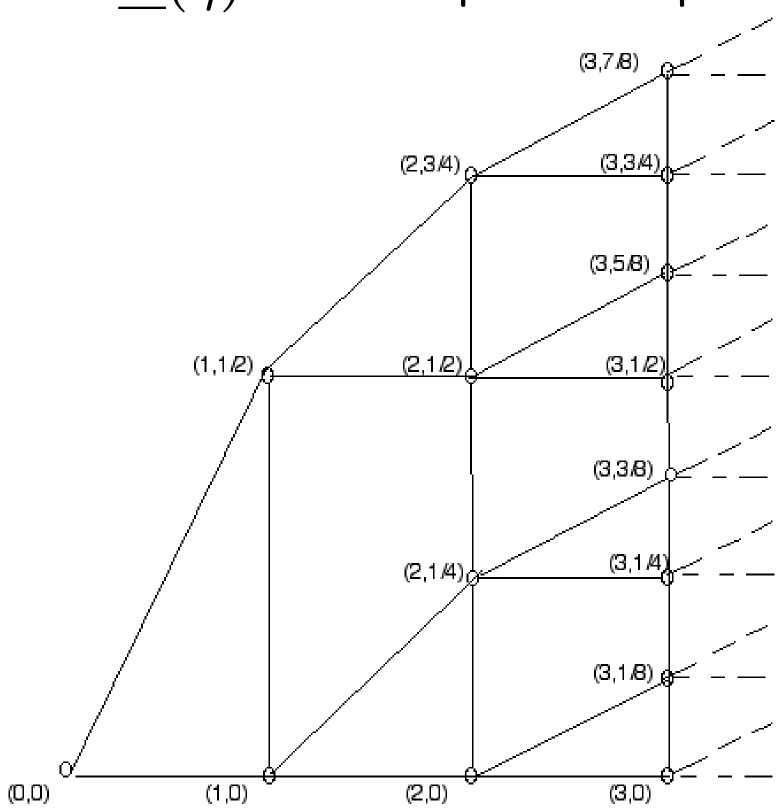
Examples are in order. Beyond finite chains, there are three unavoidable countable chains: ω, the order type of the chain $\mathbb{N}$ of non negative integers equipped with the natural order, ω^* the order type of $\mathbb{N}$ equipped with the reverse order and η the order type of the chain $\mathbb{Q}$ of rational numbers. It is immediate to see that if α is finite or ω, then the list in Theorem 2 has just one member, namely $A_1^\alpha = \{\alpha'\}$ (where α' is such that $\alpha = 1 + \alpha'$). In this case, the specialization to join-semilattice yields the same result. What happens if $\alpha = \omega^*$ or η?

Let $\underline{\Omega}(\omega^*)$ be the join-semilattice obtained by adding a least element to the set $[\omega]^2$ of two-element subsets of ω, identified to pairs (i, j), $i < j < \omega$, ordered so that $(i, j) \leq (i', j')$ if and only if $i' \leq i$ and $j \leq j'$.



$\underline{\Omega}(\omega^*)$

Let $\underline{\Omega}(\eta)$ be the poset represented below:



$\underline{\Omega}(\eta)$

With Zaguia, I proved that the list has

two members: α and $\underline{\Omega}(\alpha)$. In this last case the specialization to join-semilattice is certainly different: the analogous list has at least three members, namely α, $[\omega]^{<\omega}$ and $\underline{\Omega}(\alpha)$. If $\alpha = \omega^*$, these three members suffice. If $\alpha = \eta$, we do not know. These very specific cases take into account important classes of posets.

Let us say that a poset P is *well-founded*, resp. *scattered*, if it contains no chain of type ω^*, resp. η.

Theorem 4. *(Chakir and I, 2005) An algebraic lattice L is well-founded if and only if $K(L)$ is well-founded and contains no join-subsemilattice isomorphic to $\underline{\Omega}(\omega^*)$ or to $[\omega]^{<\omega}$.*

Question 2. *Is it true that an algebraic lattice L is scattered if and only if $K(L)$ is scattered and contains no join-subsemilattice isomorphic to $\underline{\Omega}(\eta)$ or to $[\omega]^{<\omega}$?*

The join-semilattice $[\omega]^{<\omega}$ never appeared in the list mentionned in Theorem 2 but, as Theorem 4 illustrates, it might appear in the specialization to join-semilattices. This raises two questions. For which α it appears? If it does not appears, are they particular join-semilattices of $[\omega]^{<\omega}$ which appear? Here are the answers:

Theorem 5. *Let α be a countable order type.*

(i) The join-semilattice $[\omega]^{<\omega}$ belongs to every list $\mathbb{B}$ characterizing the class $\mathbb{J}_{\neg\alpha}$ of join-semilattices P such that $J(P)$ contains no chain of type $I(\alpha)$ if and only if α is not an ordinal.

(ii) If α is an ordinal, then among the join-subsemilattices P of $[\omega]^{<\omega}$ which does not belong to $\mathbb{J}_{\neg\alpha}$ there is one, say Q_α, which embeds as a join-semilattice in all the others.

The poset Q_α is of the form $I_{<\omega}(S_\alpha)$ where S_α is some sierpinskisation of α and ω. We recall that a *sierpinskisation* of a countable order type α and ω, or simply of α, is any poset $(S,\leq)$ such that the order on S is the intersection of two linear orders on S, one of type α, the other of type ω. Such a sierpinskisation can be obtained from a bijective map $\varphi : \omega \to \alpha$, setting $S := \mathbb{N}$ and $x \leq y$ if $x \leq y$ w.r.t. the natural order on $\mathbb{N}$ and $\varphi(x) \leq \varphi(y)$ w.r.t. the order of type α. Let $\omega.\alpha$ be the ordinal sum of α copies of the chain ω; a sierpinskization of $\omega.\alpha$ and ω is *monotonic* if it obtained from a bijective map $\varphi : \omega \to \omega\alpha$ such that φ^{-1} is order-preserving on each subset of the form $\omega \times \{\beta\}$ where $\beta \in \alpha$. With that in hand, if α is an ordinal, we set $S_\alpha := \alpha$ if $\alpha < \omega$. If $\alpha = \omega\alpha' + n$ with $\alpha' \neq 0$ and $n < \omega$, let $S_\alpha := \Omega(\alpha') \oplus n$ be the direct sum of $\Omega(\alpha')$ and the chain n, where $\Omega(\alpha')$ is a monotonic sierpinskisation of $\omega\alpha'$ and ω. We note that for countably infinite α's, S_α is a sierpinskisation of α and ω.

Among the monotonic sierpinskisations of $\omega\alpha$ and ω there are some which are join-subsemilattices of the direct product $\omega\times\alpha$ that we call *lattice sierpinskisations*. Indeed, to every countable order type α, we may associate a join-subsemilattice $\Omega_L(\alpha)$ of the direct product $\omega\times\alpha$ obtained via a monotonic sierpinskisation of $\omega\alpha$ and ω. We add a least element, if there is none, and we denote by $\underline{\Omega}_L(\alpha)$ the resulting poset. Posets $\underline{\Omega}(\omega^*)$ and $\underline{\Omega}(\eta)$ represented Figure 1 and Figure 2 fall in this category. We also associate the join-semilattice P_α defined as follows:

If $1+\alpha \nleq \alpha$, in which case $\alpha = n+\alpha'$ with $n<\omega$ and α' without a first element, we set $P_\alpha := n+\underline{\Omega}_L(\alpha')$. If not, and if α is equimorphic to $\omega+\alpha'$ we set $P_\alpha := \underline{\Omega}_L(1+\alpha')$, otherwise, we set $P_\alpha = \underline{\Omega}_L(\alpha)$.

The importance of this kind of lattice sierpinskization steems from the following result:

Theorem 6. *If α is countably infinite, P_α belongs to every list $\mathbb{B}$ characterizing $\mathbb{J}_{\neg\alpha}$.*

This work leaves open the following questions.

Questions 3. *1. If α is a countably infinite ordinal, does the minimal obstructions are α, P_α, Q_α and some lexicographical sums of obstructions corresponding to smaller ordinal?*

2. If α is a scattered order type which is not an ordinal, does the minimal obstructions are α, P_α, $[\omega]^{<\omega}$ and some lexicographical sums of obstructions corresponding to smaller scattered order types?

We only have some examples of ordinals for which the answer to the first question is positive. We conjecture that the answer is always positive.

Sierpinskisations and a proof of Theorem 5 Proof of item (i) of Theorem 5 We start with the following lemma

Lemma 1. *If α is a countably infinite order type and S is a sierpinskisation of α and ω then the join-semilattice $I_{<\omega}(S)$, made of finitely generated initial segments of S, is isomorphic to a join-subsemilattice of $[\omega]^{<\omega}$ and belongs to $\mathbb{J}_{\alpha}$.*

Proof. By definition, the order on a sierpinskisation S of α and ω has a linear extension such that the resulting chain $\overline{S}$ has order type α. Hence, from a result of [0], the chain $I(\overline{S})$ is a maximal chain of $I(S)$ of type $I(\alpha)$. The lattices $I(S)$ and $J(I_{<\omega}(S))$ are isomorphic, thus $I_{<\omega}(S) \in \mathbb{J}_{\alpha}$. The order on S has a linear extension of type ω, thus every principal initial segment of S is finite and more generally every finitely generated initial segment of S is finite. This tells us that $I_{<\omega}(S)$ is a join-subsemilattice of $[S]^{<\omega}$. Since S is countable, $I_{<\omega}(S)$ identifies to a join-subsemilattice of $[\omega]^{<\omega}$. $\square$

The proof of the "only if" part goes as follows. Let S be a sierpinskization of α and ω. According to Lemma 1, the join-semilattice $I_{<\omega}(S)$, made of finitely generated initial segments of S, is isomorphic to a join-subsemilattice of $[\omega]^{<\omega}$ and belongs to $\mathbb{J}_{\alpha}$. If $[\omega]^{<\omega}$ is a minimal member of $\mathbb{J}_{\alpha}$, then $[\omega]^{<\omega}$ is embeddedable as a join-subsemilattice in $I_{<\omega}(S)$. To conclude that α cannot be an ordinal, it suffices to prove:

Lemma 2. *If α is an ordinal and S is a sierpinskisation of α and ω, then $[\omega]^{<\omega}$ is not embeddable in $I_{<\omega}(S)$.*

This simple fact relies on the important notion of well-quasi-ordering introduced by Higman. We recall that a poset P is *well-quasi-ordered* (briefly w.q.o.) if every non-empty subset A of P has at least a minimal element and the number of these minimal elements is finite. As shown by Higman, this is equivalent to the fact that $I(P)$ is well-founded.

Well-ordered set are trivially w.q.o. and, as it is well known, the direct product of finitely many w.q.o. is w.q.o. Lemma 2 follows immediately from this. Indeed, if S is a sierpinskisation of α and ω, it embeds in the direct product $\omega \times \alpha$. Thus S is w.q.o. and consequently $I(S)$ is well-founded. This implies that $[\omega]^{<\omega}$ is not embeddable in $I_{<\omega}(S)$. Otherwise $J([\omega]^{<\omega})$ would be embeddable in $J(I_{<\omega}(S))$, that is $\mathfrak{P}(\omega)$ would be embeddable in $I(S)$. Since $\mathfrak{P}(\omega)$ is not well-founded, this would contradict the well-foundedness of $I(S)$.

The "if" part is based on our earlier work on well-founded algebraic lattices, and essentially on the following corollary of Theorem 4.

Theorem 7. *A join-subsemilattice P of $[\omega]^{<\omega}$ contains either $[\omega]^{<\omega}$ as a join-semilattice or is well-quasi-ordered. In the latter case, $J(P)$ is well-founded.*

With this result, the proof of the "if" part of Theorem 5 is immediate. Indeed, suppose that α is not an ordinal. Let $P \in \mathbb{J}_{\alpha}$. The lattice $J(P)$ contains a chain isomorphic to $I(\alpha)$. Since α is not an ordinal, $\omega^* \leq \alpha$. Hence, $J(P)$ is not well-founded. If P is embeddable in $[\omega]^{<\omega}$ as a join-semilattice then, from Theorem 7, P contains a join-subsemilattice isomorphic to $[\omega]^{<\omega}$. Thus $[\omega]^{<\omega}$ is minimal in $\mathbb{J}_{\alpha}$.

A sierpinskisation S of a countable order type α and ω is embeddable into $[\omega]^{<\omega}$ as a poset. A consequence of Theorem 7 is the following

Corollary 1. *If S can be embedded in $[\omega]^{<\omega}$ as a join-semilattice, α must be an ordinal.*

Proof. Otherwise, S contains an infinite antichain and by Theorem 7 it contains a copy of $[\omega]^{<\omega}$. But this poset cannot be embedded in a sierpinskisation. Indeed, a sierpinskisation is embeddable into a product of two chains, whereas $[\omega]^{<\omega}$ cannot be embedded in a product of finitely many chains (for every integer n, it contains the power set $\mathfrak{P}(\{0, \dots, n-1\})$ which cannot be embedded into a product of less than n chains; its dimension, in the sense of Dushnik-Miller's notion of dimension, is infinite, see Trotter). □

Proof of item (ii) of Theorem 5

We prove first that there is a sierpinskisation S of α and ω such that $Q := I_{<\omega}(S) \in \mathbb{J}_\alpha$

is embeddable in P by a map preserving finite joins.

Theorem 8. *Let α be a countable ordinal and $P \in \mathbb{J}_\alpha$. If P is embeddable in $[\omega]^{<\omega}$ by a map preserving finite joins there is a sierpinskisation S of α and ω such that $I_{<\omega}(S) \in \mathbb{J}_\alpha$ and $I_{<\omega}(S)$ is embeddable in P by a map preserving finite joins.*

Proof. We construct first R such that $I_{<\omega}(R) \in \mathbb{J}_\alpha$ and $I_{<\omega}(R)$ is embeddable in P by a map preserving finite joins.

We may suppose that P is a subset of $[\omega]^{<\omega}$ closed under finite unions. Thus $J(P)$ identifies with the set of arbitrary unions of members of P. Let $(I_\beta)_{\beta<\alpha+1}$ be a strictly increasing sequence of ideals of P. For each $\beta < \alpha$ pick $x_\beta \in I_{\beta+1} \setminus I_\beta$ and $F_\beta \in P$ such that $x_\beta \in F_\beta \subseteq I_{\beta+1}$. Set $X := \{x_\beta : \beta < \alpha\}$, $\rho := \{(x_{\beta'}, x_{\beta''}) : \beta' < \beta'' < \alpha$ and $x_{\beta'} \in F_{\beta''}\}$. Let $\widehat{\rho}$ be the reflexive transitive closure of ρ. Since $\theta :=$

$\{(x_{\beta'}, x_{\beta''}) : \beta' < \beta'' < \alpha\}$ is a linear order containing ρ, $\widehat{\rho}$ is an order on X. Let $R := (X, \widehat{\rho})$ be the resulting poset.

Claim 1. $I_{<\omega}(R) \in \mathbb{J}_\alpha$.

Proof of claim 1. The linear order θ extends the order $\widehat{\rho}$ and has type α, thus $I(R)$ has a maximal chain of type $I(\alpha)$. Since $J(I_{<\omega}(R))$ is isomorphic to $I(R)$, $I_{<\omega}(R)$ belongs to $\mathbb{J}_\alpha$ as claimed. □

Claim 2. *For each $x \in X$, the initial segment $\downarrow x$ in R is finite.*

Proof of claim 2. Suppose not. Let β be minimum such that for $x := x_\beta$, $\downarrow x$ is infinite. For each $y \in X$ with $y < x$ in R select a finite sequence $(z_i(y))_{i \leq n_y}$ such that:

1. $z_0(y) = x$ and $z_{n_y} = y$.

2. $(z_{i+1(y)}, z_i(y)) \in \rho$ for all $i < n_y$.

According to item 2, $z_1(y) \in F_\beta$. Since F_β is finite, it contains some $x' := x_{\beta'}$ such that $z_1(y) = x'$ for infinitely many y. These elements belong to $\downarrow x'$. The fact that $\beta' < \beta$ contradicts the choice of x. □

Claim 3. *Let ϕ be defined by $\phi(I) := \bigcup\{F_\beta : x_\beta \in I\}$ for each $I \subseteq X$. Then:*

ϕ induces an embedding of $I(R)$ in $J(P)$ and an embedding of $I_{<\omega}(R)$ in P.

Proof of claim 3. We prove the first part of the claim. Clearly, $\phi(I) \in J(P)$ for each $I \subseteq X$. And trivially, ϕ preserves arbitrary unions. In particular, ϕ is order preserving. Its remains to show that ϕ is one-to-one. For that, let $I, J \in I(R)$ such that $\phi(I) = \phi(J)$. Suppose $J \nsubseteq I$. Let $x_\beta \in J \setminus I$, Since $x_\beta \in J$, $x_\beta \in F_\beta \subseteq \phi(J)$. Since $\phi(J) = \phi(I)$, $x_\beta \in \phi(I)$. Hence $x_\beta \in F_{\beta'}$ for some $\beta' \in I$. If $\beta' < \beta$ then since $F_{\beta'} \subseteq I_{\beta'+1} \subseteq I_\beta$ and $x_\beta \notin I_\beta$, $x_\beta \notin F_{\beta'}$. A contradiction. On the other hand, if $\beta < \beta'$ then, since $x_\beta \in F_{\beta'}$, $(x_\beta, x_{\beta'}) \in \rho$. Since I is an initial segment of R, $x_\beta \in I$. A contradiction too. Consequently $J \subseteq I$. Exchanging the roles of I and J, yields $I \subseteq J$. The equality $I = J$ follows. For the second part of the claim, it suffices to show that $\phi(I) \in P$ for every $I \in I_{<\omega}(R)$. This fact is a straightforward consequence of Claim 2. Indeed, from this claim I is finite. Hence $\phi(I)$ is finite and thus belongs to P.

Claim 4. *The order $\widehat{\rho}$ has a linear extension of type ω.*

Proof of claim 4. Clearly, $[\omega]^{<\omega}$ has a linear extension of type ω. Since R embeds in $[\omega]^{<\omega}$, via an embedding in P, the induced linear extension on R has order type ω. □

Let ρ' be the intersection of such a linear extension with the order θ and let $S := (X, \rho')$.

Claim 5. *For every* $I \in I(S)$*, resp.* $I \in I_{<\omega}(S)$ *we have* $I \in I(R)$*, resp.* $I \in I_{<\omega}(R)$*.*

Proof of claim 5. The first part of the proof follows directly from the fact that ρ' is a linear extension of $\widehat{\rho}$. The second part follows from the fact that each $I \in I_{<\omega}(S)$ is finite. □

It is then easy to check that the poset S satisfies the properties stated in the theorem. □

In order to conclude, it suffices to prove that one can replace Q by $Q_\alpha := I_{<\omega}(S_\alpha)$, where S_α is the sierpinskization defined in the introduction.

This fact follows directly from Lemma 5 below. It relies on properties of monotonic sierpinskizations, some already in Pouzet-Zaguia.

We recall that for a countable order type α', two monotonic sierpinskisations of $\omega\alpha'$ and ω are embeddable in each other and denoted by the same symbol $\Omega(\alpha')$ and we recall the following result (cf. Pouzet-ZaguiaProposition 3.4.6. pp. 168).

Lemma 3. *Let α' be a countable order type. Then $\Omega(\alpha')$ is embeddable in every sierpinskisation S' of $\omega\alpha'$ and ω.*

Lemma 4. *Let α be a countably infinite order type and S be a sierpinskisation of α and ω. Assume that $\alpha = \omega\alpha' + n$ where $n < \omega$. Then there is a subset of S which is the direct sum $S' \oplus F$ of a sierpinskisation S' of $\omega\alpha'$ and ω with an n-element poset F.*

Proof. Assume that S is given by a bijective map φ from $\mathbb{N}$ onto a chain C having order type α. Let A' be the set of the n last elements of C, $A := \varphi^{-1}(A')$ and a be the largest element of A in $\mathbb{N}$. The image of $]a \to)$ has order type $\omega\alpha'$, thus S induces on $]a \to)$ a sierpinskisation S' of $\omega\alpha'$ and ω. Let F be the poset induced by S on A. Since every element of S' is incomparable to every element of F these two posets form a direct sum. □

Let α be a countably infinite order type such that $\alpha = \omega\alpha' + n$ where $n < \omega$. We set $S_\alpha := \Omega(\alpha') \oplus n$ and $Q_\alpha := I_{<\omega}(S_\alpha)$.

Lemma 5. *$Q_\alpha \in \mathbb{J}_\alpha$ and for every sierpinskisation S of α and ω, Q_α is embeddable in $I_{<\omega}(S)$ by a map preserving finite joins.*

Proof. For the the first part, apply Lemma 1.

Case 1. $n = 0$. By Lemma 3, $\Omega(\alpha')$ is embeddable in S. Thus Q_α is embeddable in $I_{<\omega}(S)$ by a map preserving finite joins.

Case 2. $n \neq 0$. Apply Lemma 4. According to Case 1, $I_{<\omega}(\Omega(\alpha'))$ is embeddable in $I_{<\omega}(S')$. On an other hand $n +$

1 is embeddable in $I_{<\omega}(F) = I(F)$. Thus Q_α which is isomorphic to the product $I_{<\omega}(\Omega(\alpha')) \times (n+1)$ is embeddable in the product $I_{<\omega}(S') \times I_{<\omega}(F)$. This product is itself isomorphic to $I_{<\omega}(S' \oplus F)$. Since $S' \oplus F$ is embeddable in S, $I_{<\omega}(S' \oplus F)$ is embeddable in $I_{<\omega}(S)$ by a map preserving finite joins. It follows that Q_α is embeddable in $I_{<\omega}(S)$ by a map preserving finite joins. $\square$

Partial answers to Question 2 There is no need for sierpinskizations of η if our lattices are modular:

Theorem 9. *(Chakir and I 2007) An algebraic modular lattice is order-scattered iff the semilattice of compact elements is order-scattered and does not contain as a subsemilattice the semilattice $[\omega]^{<\omega}$ of finite subsets of a countable set.*

What happen for lattices of convex geometries? This is a joint work with Kira Adaricheva.

An example: Relatively convex sets Let V be a real vector space and $X \subseteq V$. Let $Co(V, X)$ be the collection of sets $C \cap X$, where C is a convex subset of V. Ordered by inclusion, $Co(V, X)$ is an algebraic convex geometry.

We prove the following analogue of Theorem 9.

$Co(V, X)$ is order scattered iff the semilattice S of compact elements of $Co(V, X)$ is order scattered and does not have $[\omega]^{<\omega}$ as a subsemilattice.

First, we start from the analysis of independent subsets of $Co(V) = Co(V, V)$. As before a subset $Y \subseteq V$ is called *independent*, if $y \notin Co(Y \setminus y)$, for every $y \in Y$. Here $Co(Z)$ denotes the closure of a subset $Z \subseteq X$ in $Co(V)$ and we call it the *convex hull* of Z.

Lemma 6. *Let $X \subseteq V$. Then either X is contained in a finite union of lines, or X contains an infinite independent subset (with respect to the closure in $Co(V)$).*

Proof.

The proof uses the same arguments as in the proof of Erdos-Szekeres theorem (see Morris and Soltan) We suppose first that $V = \mathbb{R}^2$.

If X is not contained in the finite union of lines, then one can find a a countable subset $X_1 \subseteq X$ such that no three points from X_1 are on a line. Indeed, pick two points x_1, x_2 from X randomly, and if $x_1, \dots, x_k$ are already picked, choose $x_{k+1} \in X$ so that it does not belong to any line that goes through any two points from $x_1, \dots, x_k$.

Now form F, the set of 4-element subsets of X_1, and colour elements of F red, if one point of four is in the convex hull of the others, and colour it blue otherwise. According to the infinite form of Ramsey's theorem, there exists an infinite subset $X_2 \subseteq X_1$ such that all four-element subsets of X_2 are coloured in one colour. But it cannot be red colour, because, even for a 5-element subset of points from X_1, at least one 4-element subset would be coloured blue, an easy fact. Hence, X_2 has all 4-element subsets coloured blue. It follows that X_2 is in infinite independent subset of X. Indeed, if any point $x \in X_2$ was in the closure of some finite subset $X' \subseteq X_2 \setminus \{x\}$, then, due to Carathéodory property of the plane, x would be in the closure of 3 points from X', which contradicts the choice of X_2.

Now, we show how to reduce the general case to the case above. For this purpose, let $Af(V, X)$ be the set $A \cap X$, where A is an affine subset of V. Ordered by inclusion, $Af(V, X)$ is an algebraic geometric lattice, that is an algebraic lattice and, as a closure system, it satisfies the exchange property. Every subset Y of X contains an affinely independent subset Y' with the same affine span S as Y; moreover, the size of Y' is equal to $dim_{af}(S) + 1$ where $dim_{af}S$, the affine dimension of S, is the ordinary dimension of the translate of S containing $\{0\}$.

Suppose that X is not contained in a finite union of lines. Let λ be the least cardinal such that X contains a subset X' such that X' is not contained in a finite union of lines and the affine dimension of its affine span is λ. Necessarily, $\lambda \geq 2$. If λ is infinite then X contains an infinite convexely independent subset. Indeed, X' contains an affinely independent subset of size $\lambda+1$ and every affinely independent set is convexely independent. Suppose that λ is finite. We proceed by induction on λ. We may assume with no loss of generality that $X' \subseteq \mathbb{R}^{\lambda}$. If $\lambda = 2$, the first case applies.

Suppose $\lambda > 2$. Let X'' be a projection of X' on an hyperplane V'. If X'' is not contained in a finite union of lines, then induction yields an infinite convexely independent subset of X''. For each element a' in this subset, select some element a

in X' whose projection is a'. The resulting set is convexely independent. If X'' is contained in a finite union of lines, then there is some line such that its inverse image in X' cannot be covered by finitely many lines. This inverse image being a plane, the first case applies. □

Corollary 2. *If X contains an independent sets of arbitrary large finite size, then it contains an infinite independent set.*

Let $Co^{<\omega}(\mathbb{N})$ be the (semi)lattice of finite intervals of the chain of natural numbers $\mathbb{N}$, ordered by inclusion. With a least element added, this semilattice is order isomorphic to $\underline{\Omega}(\omega^*)$.

Corollary 3. *If X is infinite, then the semilattice of compact elements of $L = Co(V, X)$ contains either $Co^{<\omega}(\mathbb{N})$ or $[\omega]^{<\omega}$ as a join semilattice.*

Proof. If X contains an infinite independent subset, then the semilattice of compact elements of $L = Co(\mathbb{R}^2, X)$ will have a semilattice isomorphic to $[\omega]^{<\omega}$. Otherwise, X must be covered by finitely many lines. If X is infinite, then one of the lines will have infinitely many points from X. Choose a coordinate system on that line. Then one can find either increasing or decreasing infinite countable sequence of elements of X on that line. Hence, the semilattice of compact elements of L has $Co^{<\omega}(\mathbb{N})$ as a subsemilattice. □

This leads to ask if in the case of a convex geometry one can get rid of sierpinskizations of η.

Surely not: $\Omega(\eta)$ is a typical subsemilattice of compact elements in some special convex geometries, coming from *multichains*. Indeed, let E be a set with a set $(\mathcal{L}_i : i \in I)$ of linear orders on E. For each $i \in I$, let $C_i := I(E, \mathcal{L}_i)$, be the set of initial segments of $(E, \mathcal{L}_i)$. Let $C = \bigvee C_i$ be the closure system on E with closed sets $X = \bigcap X_i$, where X_i is closed in C_i for each i. Then C is a convex geometry, which is algebraic if I is finite.

Suppose I finite and that each L_i has a least element. In order to get a convenient presentation of the semilattice of compact elements of C let $L = \Pi(E, \mathcal{L}_i)$ and $\delta : E \to L$ be *the diagonal mapping*, i.e., $\delta(x) = \langle x, \ldots, x \rangle$, and let $\Delta(L)$ be the semilattice in L generated by $\delta(E) \subseteq L$.

Lemma 7. *$\Delta(L)$ is $\vee$-isomorphic to the semilattice of compact elements of C.*

Lemma 8. *If C is the convex geometry associated with a chain of type ω and a chain containing η then $\Omega(\eta)$ and $\Delta(L)$ embeds in each other as subsemilattices.*

We say that a semilattice P with 0 has $\vee$-dimension $dim_{\vee}(P) = \kappa$, if κ is the smallest cardinal for which there exist κ chains C_i,

$i < \kappa$, with minimal element 0_i, and injective map $f : P \to \Pi C_i$ such that $f(a \vee b) = f(a) \vee f(b)$ and $f(0) = (0_i, i < \kappa)$.

Note that the presence of join-preserving map is essential in this definition. In particular, $dim_\vee(P)$ differs from the dimension of P treated as a partially ordered set. We recall that *the order-dimension* of a poset P is defined as a minimal cardinal λ for which there exist chains C_i, $i < \lambda$, such that $P \leq \Pi C_i$, where $\leq$ is a poset embedding.

Theorem 10. *Let P be the semilattice of compact elements of algebraic convex geometry $C = J(P)$. If $dim_\vee P = n < \omega$, then C is order scattered iff P is order scattered and $\Omega(\eta)$ is not a subsemilattice of P.*

The proof relies on the famous unpublished theorem by F. Galvin $\eta \to [\eta]^2_2$.

An easier form we use is this:

Theorem 11. *Suppose the pairs of rationals are divided into finitely many classes* $A_1, \ldots, A_n$. *Fix the ordering on the rationals with order type* Ω. *Then there is a subset* X *of rationals of order type* η *and indices* i, j *(with possibly* $i = j$*) such that all pairs of* X *on which two orders coinside belong to* A_i*, and all pairs of* X *on which the two orders disagree belong to* A_j.

The proof of Galvin's Theorem can be found in Fraïssé.

Topological scatteredness On an algebraic lattice, there is a topology which make it compact (indeed, a closure system on a set X is algebraic if and only if it is a closed subset of the power set $\wp(X)$ equipped with the product topology).

Theorem 12. $Co(V, X)$ *is topologically scattered iff it is order-scattered.*

This is reminiscent of the beautiful result of Mislove:

Theorem 13. *A distributive algebraic lattice is topologically scattered iff it is order-scattered.*

Theorem 14. *Let P be a semilattice with $dim_{\vee}(P) = n < \infty$. Then the following properties are equivalent:*

(1) $P \leq_{\vee} \Pi C_i, i \leq n$, for some order scattered chains;

(2) $J(P)$ is topologically scattered;

(3) $J(P)$ is order scattered;

Note that the similar statement does not hold with the ordinary dimension.

The infinite binary tree T_2 with a top element added has dimension 2, $J(T_2)$ is order scattered but not topologically scattered.

With Eric Milner, we proved long time ago that if P is a join-semilattice, $J(P)$ is topologically scattered iff it is order scattered and does not embed T_2 as a subposet.

Problem 1. *Find a list S of join-semilattices such that the lattice of ideals of a semilattice P is topologically scattered iff P contains no member of the list as semilattice.*

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