Hereditary classes of relational structures and profile

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Algebra Across the Borders
Stern college, NY
August 9, 2011

This talk is about a counting function: the profile.

I will present some old and some new results and some conjectures.

I will divide this talk into three parts:

PAST
PRESENT
and
FUTURE

THE PAST

Definitions

A *relational structure* is a realization of a language whose non-logical symbols are predicates.

This is a pair $R := (E, (\rho_i)_{i \in I})$ made of a set E and of a family of m_i-ary relations ρ_i on E. The set E is the *domain* or *base* of R. The family $\mu := (m_i)_{i \in I}$ is the *signature* of R.

The *profile* of R is the function φ_R which counts for every integer n the number $\varphi_R(n)$ of substructures of R induced on the n-element subsets, isomorphic substructures being identified.

Clearly, this function only depends upon the set $\mathcal{A}(R)$ of finite induced substructures of R considered up to an isomorphism, a set introduced by Roland Fraïssé under the name of *age* of R .

If the signature μ is finite (in the sense that I is finite), there are only finitely many relational structures with signature μ on an n-element domain, hence $\varphi_R(n)$ is necessarily an integer for each integer n.
In order to capture examples coming from algebra and group theory, we cannot preclude I to be infinite. But then, $\varphi_R(n)$ could be an infinite cardinal. I will exclude this case. I make the assumption that φ_R is integer valued, no matter how large I is. With this assumption, *profiles of relational structures with bounded signature are profiles of relational structures with finite signature*, structures that R. Fraïssé call *multirelations*.

Several counting functions are profiles. Here is some simple minded examples.

1. *The binomial coefficient* $\binom{n+k}{k}$. Let $R := (\mathbb{Q}, \leq, u_1, \ldots, u_k)$ where $\leq$ is the natural order on the set $\mathbb{Q}$ of rational numbers, $u_1, \ldots, u_k$ are k unary relations which divide $\mathbb{Q}$ into $k+1$ intervals. Then $\varphi_R(n) = \binom{n+k}{k}$.

2. *The exponential* $n \hookrightarrow k^n$. Let $R := (\mathbb{Q}, \leq, u_1, \ldots, u_k)$, where again $u_1, \ldots, u_k$ are k unary relations, but which divide $\mathbb{Q}$ into k "colors" in such a way that between two rational numbers all colors appear. Then $\varphi_R(n) = k^n$.

3. *The factorial* $n \hookrightarrow n!$. Let $R := (\mathbb{Q}, \leq, \leq')$, where $\leq'$ is an other linear order on $\mathbb{Q}$ such a way that the finite restrictions induce all

possible pairs of two linear orders on a finite set (eg take for $\leq'$ an order with the same type as the natural order on the set $\mathbb{N}$ of non-negative integers). Then $\varphi_R(n) = n!$

4. *The partition function* which counts the number (n) of partitions of the integer n. Let $R := (\mathbb{N}, \rho)$ be the infinite path on the integers whose edges are pairs $\{x, y\}$ such that $y = x + 1$. Then $\varphi_R(n) = p(n)$. The determination of its asymptotic growth is a famous achievement, the difficulties encountered to prove that $\mathfrak{p}(n) \simeq \frac{1}{4\sqrt{3n}} e^{\pi\sqrt{\frac{2n}{3}}}$ (Hardy and Ramanujan, 1918) suggest some difficulties in the general study of profiles.

Orbital profiles An important class of functions comes from permutation groups. The *orbital profile* of a permutation group G acting on a set E is the function θ_G which counts for each integer n the number, possibly infinite, of orbits of the n-element subsets of E. As it is easy to see, θ_G is the profile of some relational structure $R := (E, (\rho_i)_{i \in I})$ on E. In fact, as it is easy to see:

Lemma 1. *For every permutation group G acting on a set E there is a relational structure R on E such that:*

1. *Every isomorphism f from a finite restriction of R onto an other extends to an automorphism of R.*

2. *$Aut(R) = \overline{G}$ where $\overline{G}$ is the topological adherence of G into the symmetric group $\mathfrak{S}(E)$, equipped with the topology induced by the product topology on E^E, E being equipped with the discrete topology.*

Structures satisfying condition 1) are called *homogeneous* (or *ultrahomogeneous*). They are now considered as one of the basic objects of model theory. Ages of such structures are called *Fraïssé classes* after their characterization by R.Fraïssé in 1954. In many cases, I is infinite, even if $\theta_G(n)$ is finite. Groups for which $\theta_G(n)$ is always finite are said *oligomorphic* by P.J.Cameron. The study of their profile is whole subject by itself. Their relevance to model theory steems from the following result of Ryll-Nardzewski.

Theorem 2. *Let G acting on a denumerable set E and R be a relational structure such that $AutR = \overline{G}$. Then G is oligomorphic if and only if the complete theory of R is $\aleph_0$-categorical.*

A Sample of Results

The profile grows
Theorem 3. *If R is a relational structure on an infinite set then φ_R is non-decreasing.*

This result was conjectured by R.Fraïssé and I. I proved it in 1971; the proof - for a single relation- appeared in 1971 in R.Fraïssé's course in logic, Exercise 8 p. 113. The proof relies on Ramsey theorem. In 1976, I gave a proof using linear algebra (a straigthforward application of a nice result on incidence matrices discovered independently by Gottlieb (1966) and Kantor (1972)).

There are jumps in the Growth of the Profile Beyond bounded profiles, and provided that the relational structures satisfy some mild conditions, there are jumps in the behavior of the profiles: eg. no profile grows as $log\ n$ or $nlog\ n$.

Let $\varphi : \mathbb{N} \to \mathbb{N}$ and $\psi : \mathbb{N} \to \mathbb{N}$. Recall that $\varphi = O(\psi)$ and ψ *grows as fast as* φ if $\varphi(n) \leq a\psi(n)$ for some positive real number a and n large enough. We say that φ and ψ have the *same growth* if φ grows as fast as ψ and ψ grows as fast as φ. The growth of φ is *polynomial of degree* k if φ has the same growth as $n \hookrightarrow n^k$; in other words there are positive real numbers a and b such that $an^k \leq \varphi \leq bn^k$ for n large enough. Note that the growth of φ is as fast as every polynomial if and only if $lim_{n \to +\infty} \frac{\varphi(n)}{n^k} = +\infty$ for every non negative integer k.

Theorem 4. *Let $R := (E, (\rho_i)_{i \in I})$ be a relational structure. The growth of φ_R is either polynomial or as fast as every polynomial provided that either the signature $\mu := (n_i)_{i \in I}$ is bounded or the kernel $K(R)$ of R is finite.*

The *kernel* of R is the set $K(R)$ of $x \in E$ such that $\mathcal{A}(R_{\restriction E \setminus \{x\}}) \neq \mathcal{A}(R)$. Relational structures with empty kernel are those for which their age has the *disjoint embedding property*, meaning that two arbitrary members of the age can be embedded into a third in such a way that their domains are disjoint. In Fraïssé's terminology, ages with the *disjoint embedding property* are said *inexhaustible* and relational structures whose age is inexhaustible are said *age-inexhaustible*. We will say that relational structures with finite kernel are *almost age-inexhaustible*.

At this point, enough to know that the kernel of any relational structure which encodes an oligomorphic permutation group is finite (this fact immediate: if R encodes a permutation group G acting on a set E then $K(R)$ is the set union of the orbits of the 1-element subsets of E which are finite. Since the number of these orbits is at most $\theta_G(1)$, if G is oligomorphic then $K(R)$ is finite).

Corollary 1. *The orbital profile of an oligomorphic group is either polynomial or faster than every polynomial.*

Some hypotheses on R are needed in Theorem 4, indeed

Theorem 5. *For every non-decreasing and unbounded map $\varphi : \mathbb{N} \to \mathbb{N}$, there is a relational structure R such that φ_R is unbounded and eventually bounded above by φ.*

The hypothesis about the kernel is not ad hoc. As it turns out, *if the growth of the profile of a relational structure with a bounded signature is bounded by a polynomial then its kernel is finite*.

Theorems 4 and 5 were obtained in 1978. Theorem 5 and a part of Theorem 4 appeared in 1981, with a detailed proof showing that the growth of unbounded profiles of relational structures with bounded signature is at least linear. The notion of kernel is in several papers.

Polynomial Growth

It is natural to ask:

Problem 1. *If the profile of a relational structure R with finite kernel has polynomial growth, is $\varphi_R(n) \simeq cn^{k'}$ for some positive real c and some non-negative integer k'?*

The problem was raised by P.J.Cameron for the special case of orbital profiles. Up to now, it is unsolved, even in this special case.

An example, pointed out by P.J.Cameron, suggests that a stronger property holds.

Let G' be the wreath product $G' := G \wr \mathfrak{S}_\omega$ of a permutation group G acting on $\{1, \ldots, k\}$ and of $\mathfrak{S}_\omega$, the symmetric group on ω. Looking at G' as a permutation group acting on $E' := \{1, \ldots, k\} \times \omega$, then - as observed by Cameron- $\theta_{G'}$ is the *Hilbert function* $h_{Inv(G)}$

of the subalgebra $Inv(G)$ of $\mathbb{C}[x_1, \ldots, x_k]$ consisting of polynomials in the k indeterminates $x_1, \ldots, x_k$ which are invariant under the action of G. The value of $h_{Inv(G)}(n)$ is, by definition, the dimension $dim(Inv_n(G))$ of the subspace of homogeneous polynomials of degree n. As it is well known, the *Hilbert series* of $Inv(G)$,

$$\mathcal{H}(Inv(G), x) := \sum_{n=0}^{\infty} h_{Inv(G)}(n)x^n$$

is a rational fraction of the form

$$\frac{P(x)}{(1-x)\cdots(1-x^k)} \tag{1}$$

with $P(0) = 1$, $P(1) > 0$, and all coefficients of P being non negative integers.

Let us associate to a relational structure R whose profile takes only finite values its *generating series*

$$\mathcal{H}_{\varphi_R} := \sum_{n=0}^{\infty} \varphi_R(n)x^n$$

Problem 2. *If R has a finite kernel and φ_R is bounded above by some polynomial, is the series $\mathcal{H}_{\varphi_R}$ a rational fraction of the form*

$$\frac{P(x)}{(1-x)(1-x^2)\cdots(1-x^k)}$$

with $P \in \mathbb{Z}[x]$?

Under the hypothesis above we do not know if $\mathcal{H}_{\varphi_R}$ is a rational fraction.

It is well known that if a generating function is of the form $\frac{P(x)}{(1-x)(1-x^2)\cdots(1-x^k)}$ then for n large enough, a_n is a *quasi-polynomial* of degree k', with $k' \leq k-1$, that is a polynomial $a_{k'}(n)n^{k'} + \cdots + a_0(n)$ whose coefficients $a_{k'}(n), \ldots, a_0(n)$ are periodic functions. Hence, a subproblem is:

Problem 3. *If R has a finite kernel and φ_R is bounded above by some polynomial, is $\varphi_R(n)$ a* quasi-polynomial *for n large enough?*

Remark 1. *Since the profile is non-decreasing, if $\varphi_R(n)$ is a quasi-polynomial for n large enough then $a_{k'}(n)$ is eventually constant. Hence the profile has polynomial growth in the sense that $\varphi_R(n) \sim cn^{k'}$ for some positive real c and $k' \in \mathbb{N}$. Thus, in this case, Problem 1 has a positive solution.*

Partial answers to Problem 2. True for relational structures which have a finite monomorphic decomposition, Thiéry and I, (2005). In the case of tournaments, these are finite lexicographic sums of acyclic tournaments (such decompositions have been studied by Culus and Jouve (2005) and by Boudabbous and I, (2010). True for graphs (Balogh, Bollobas, Saks, Sos (2009)).

Relationship with language theory

In the theory of languages, one of the basic results is that the generating series of a regular language is a rational fraction. This result is not far away from our considerations. Indeed, if A is a finite alphabet, with say k elements, and A^* is the set of words over A, then each word can be viewed as a finite chain coloured by k colors and A^* can be viewed as the age of the relational structure made of the chain $\mathbb{Q}$ of rational numbers divided into k colors in such a way that, between two distinct rational numbers, all colors appear.

Problem 4. *Does the members of the age of a relational structure with polynomial growth can be coded by words forming a regular language?*

Problem 5. *Extend the properties of regular languages to subsets of* Ω_μ.

PRESENT During the last ten years, they have been many results on enumeration of classes of concrete structures like ordered graphs, graphs, tournaments and also on permutations in connection with the famous Stanley-Wilf conjecture, solved by Marcus and Tardös in 2004.

These results are about hereditary classes of relational structures.

A class $\mathcal{C}$ of finite relational structures, all with the same arity μ, is *hereditary* if for every $R \in \mathcal{C}$, every S which is isomorphic to an induced substructure of R belongs to $\mathcal{C}$. Let us call *profile* the function $\varphi_{\mathcal{C}}(n)$ which counts for each integer n the number of non isomorphic n-element structures which belong to $\mathcal{C}$.

Essentially, the results asserts that there are jumps in the growth rate of the profile of hereditary classes of ordered graphs, graphs, tournaments and permutations.

Why permutations fall in the category of relational structures?

Let $\mathfrak{S}_n$ be the set of permutations σ on $\{1,\dots,n\}$ and $\mathfrak{S} := \bigcup_n \mathfrak{S}_n$.

Let $\sigma \in \mathfrak{S}_n$, set $B_\sigma := (\{1,\dots,n\},(\leq,\leq_\sigma))$, where $\leq$ is the natural order on $\{1,\dots,n\}$ and $\leq_\sigma$ the order defined on $\{1,\dots,n\}$ by $x \leq_\sigma y$ if $\sigma(x) \leq \sigma(y)$. Viewed as a relational structure, B_σ is what I will call a *bichain* (in fact, this is a canonical representative of an isomorphic type of bichain). Next, if $\sigma, \tau \mathfrak{S}$, set $\sigma \leq \tau$ if B_σ is embeddable into B_τ. This define an ordering between permutations and allows to define hereditary classes of permutations.

There are many results on concrete classes of structures, especially of permutations. For an example, Balogh, Bollobás, Saks and Sós (2009) prove that

Theorem 6. *If $\mathcal{C}$ is a hereditary class of finite graphs, then either $\varphi_{\mathcal{C}}$ is polynomial in the sense that $\varphi_{\mathcal{C}}(n) = cn^k + O(n^{k-1})$ or $\varphi_{\mathcal{C}}$ is bounded below by the partition function, that is $\varphi_{\mathcal{C}}(n) \geq \mathfrak{p}(n)$ for all n.*

In the case of tournaments, ordered graphs and permutations, they show that the jump goes from polynomial growth to exponential growth, generalizing a Kaiser-Klazar result on permutations. There is a very nice survey paper by Klazar (2008) (on Arxiv). In his survey, Klazar ask for a link between the results on the profile of relational structures and the profile of hereditary classes.

The link Let P be a poset or even a quasi ordered set. An *initial segment* of P is any subset I such that $x \in I$ and $y \leq x$ imply $y \in I$. An *ideal* is any non-empty initial segment of P which is up-directed (that is $x, y \in J$ imply $x, y \leq z$ for some $z \in J$).

The class of relational structures can be quasi-ordered by embeddability: say that R is *embeddable* into R' if R is isomorphic to an induced substructures of R'. On the set Ω_μ made of finite structures of type μ and considered up to isomorphy this induces an order. Hereditary classes of finite relational structures of a given type, say μ, are simply initial segment of this poset. What about ideals?

The class $\mathcal{A}(R)$ of finite relational structures (considered up to isomorphy) which can be embedded into a given relational structure R, the age of R, is an ideal.

Theorem 7. *R. Fraïssé (1948). Every countable ideal is the age of a (countable) relational structure.*

Note that if the arity μ is finite, the countability condition is superflous. If μ is infinite there are counterexamples, in fact every ideal of Ω_μ, with $\mu := (m_i)_{i \in I}$, is the age of a relational structure if and only if $m_i = 1$ for all i but finitely many, and for those, $m_i = 2$ (necessity in Delhommé, Sauer, Sagi, and I, 2009, sufficiency by Kabil and I, 1992).

How relate initial segments and ideals?
Theorem 8. *Erdös-Tarski (1941). A poset with no infinite antichain is a finite union of ideals.*

A poset is *well-quasi-ordered* (wqo) if it has no infinite descending chain and no infinite antichain.

There is no infinite descending chain in Ω_μ, thus a hereditary class $\mathcal{C} \subseteq \Omega_\mu$ is wqo iff it has no infinite antichain.

In a poset P with no infinite descending chain, an initial segment I is determined by the minimal elements of $P \setminus I$. If $P := \Omega_\mu$, $I := \mathcal{C}$, these minimal elements are the *bounds* of $\mathcal{C}$, that is the relational structures R which does not belong to $\mathcal{C}$ but such that for element x in the domain E of R, the induced relation on $E \setminus \{x\}$ belongs to $\mathcal{C}$.

Lemma 9. *Let $\mathcal{C}$ be a hereditary class of finite structures. If $\mathcal{C}$ is not wqo then there is an age $\mathcal{A} \subseteq \mathcal{C}$ which is wqo and has infinitely many bounds in $\mathcal{C}$.*

Illustration. Let $\mathcal{C}$ be a hereditary class. Either there is an age $\mathcal{A} \subseteq \mathcal{C}$ whose profile grows faster than every polynomial -in which case $\varphi_{\mathcal{C}}$ grows faster than every polynomial- OR NOT.

I claim that in this case $\mathcal{C}$ is wqo and $\varphi_{\mathcal{C}}$ is polynomial. This is a straightforward consequence of what I proved in 1978 namely:

Theorem 10. *The profile of an age $\mathcal{A}$ is either polynomial, in which case $\mathcal{A}$ is wqo, with finitely many bounds or $\varphi_{\mathcal{C}}$ grows faster than every polynomial.*

From this it turns out that $\mathcal{C}$ is wqo. Since $\mathcal{C}$ is a finite union of ages and each has polynomial profile, $\varphi_{\mathcal{C}}$ has polynomial profile.

Furthermore, $\mathcal{C}$ has finitely many bounds.

In the case of polynomial of degree 0, this says that if $\mathcal{C}$ has bounded profile then it is wqo with finitely many bounds (Fraïssé and I, 1971). The finiteness of the number of bounds relies on the work of Frasnay (1965). I addressed the degree 1 in 1981, proving that if the profile is unbounded it is a least linear.

For more about ages which are wqo, see Sobrani and I (2001) and for the study of chains of ideals in posets, Zaguia and I (1985),Chakir and I (2005, 2007).

FUTURE? I see two lines of research.

One consists to extends to relational structures and particularly to binary relational structures, the results obtained for ordered graphs, graphs and permutations. This is the subject of a forthcoming thesis by Djamila Oudrar.

An other is the algebraic approach designed by Peter J. Cameron.

The Age Algebra of Cameron P. J. Cameron associates to $\mathcal{A}(R)$, the age of a relational structure R, its *age algebra*, a graded commutative algebra $\mathbb{K}.\mathcal{A}(R)$ over a field $\mathbb{K}$ of characteristic zero. He shows that if φ_R takes only finite values, then the dimension of $\mathbb{K}.\mathcal{A}(R)_n$, the homogeneous component of degree n of $\mathbb{K}.\mathcal{A}(R)$, is $\varphi_R(n)$. Hence, in this case, the generating series of the profile is simply the Hilbert series of $\mathbb{K}.\mathcal{A}(R)$.

P.J Cameron mentions several interesting examples of algebras which turn to be age algebras. The most basic one is the *shuffle algebra* on the set A^* of words on a finite alphabet A [0]. Indeed, as mentionned at the end of Subsection 2.3, A^* is the age of the relational structure $(\mathbb{Q}, (U_a)_{a \in A})$ where the U_a's are unary relations forming a coloring of $\mathbb{Q}$ into distinct colors, in such a way that between two distinct rational numbers, all colors appear. And the shuffle algebra is isomorphic to the age algebra of $(\mathbb{Q}, (U_a)_{a \in A})$.

The Set Algebra

Let E be a set and let $[E]^{<\omega}$ be the set of finite subsets of E (including the empty set $\emptyset$). Let $\mathbb{K}$ be a field and $\mathbb{K}^{[E]^{<\omega}}$ be the set of maps $f : [E]^{<\omega} \to \mathbb{K}$. Endowed with the usual addition and scalar multiplication of maps, this set is a vector space over $\mathbb{K}$. Let $f, g \in \mathbb{K}^{[E]^{<\omega}}$ and $Q \in [E]^{<\omega}$. Set

$$fg(Q) = \sum_{P \in [Q]^{<\omega}} f(P)g(Q \setminus P) \qquad (2)$$

. With this operation added, the above vector space becomes a $\mathbb{K}$-algebra. This algebra is commutative and it has a unit, denoted by 1. This is the map taking the value 1 on the empty set and the value 0 everywhere else. The *set algebra* is the subalgebra made of the maps such that $f(P) = 0$ for every $P \in [E]^{<\omega}$ with $|P|$ large enough. This algebra is graded, the homogeneous component of degree n being made of maps which take the value 0 on every subset of size different from n.

Let $\equiv$ be an equivalence relation on $[E]^{<\omega}$. A map $f : [E]^{<\omega} \to \mathbb{K}$ is $\equiv$-*invariant*, or briefly, *invariant*, if f is constant on each equivalence class. Invariant maps form a subspace of the vector space $\mathbb{K}^{[E]^{<\omega}}$.

if R is a relational structure with domain E, set $F \equiv_R F'$ for $F, F' \in [E]^{<\omega}$ if the restrictions $R\restriction_F$ and $R\restriction_{F'}$ are isomorphic. The resulting equivalence on $[E]^{<\omega}$ is such that the invariant maps form a subalgebra . Let $\mathbb{K}.\mathcal{A}(R)$ be the intersection of the subalgebra of $\mathbb{K}^{[E]^{<\omega}}$ made of invariant maps with the set algebra. This is a graded algebra, the *age algebra* of Cameron.

The name comes from the fact that this algebra depends only upon the age of R.

If φ_R takes only integer values, $\mathbb{K}.\mathcal{A}(R)$ identifies with the set of (finite) linear combinations of members of $\mathcal{A}(R)$. This explain the fact that, in this case, $\varphi_R(n)$ is the dimension of the homogeneous component of degree n of $\mathbb{K}.\mathcal{A}(R)$. In a special case, we have
Theorem 11. *[0] If R has a monomorphic decomposition into finitely many blocks $E_1, \ldots, E_k$, all infinite, then the age algebra $\mathbb{K}.\mathcal{A}(R)$ is a*

polynomial algebra, isomorphic to a subalgebra $\mathbb{K}[x_1, \ldots, x_k]^R$ of $\mathbb{K}[x_1, \ldots, x_k]$, the algebra of polynomials in the indeterminates $x_1, \ldots, x_k$.

Behavior of the Profile

In the frame of its age algebra, Cameron gave the following proof of the fact that the profile does not decrease.

Let R be a relational structure on a set E, let $e := \sum_{P \in [E]^1} P$ (that we could view as the sum of isomorphic types of the one-element restrictions of R) and U be the subalgebra generated by e. Members of U are of the form $\lambda_m e^m + \cdots + \lambda_1 e + \lambda_0 1$ where 1 is the isomorphic type of the empty relational structure and $\lambda_m, \ldots, \lambda_0$ are in $\mathbb{K}$. Hence U is graded, with U_n, the homogeneous component of degree n, equals to $\mathbb{K}.e^n$.

Theorem 12. *If R is infinite then for every $u \in \mathbb{K}.\mathcal{A}(R)$, $eu = 0$ if and only if $u = 0$*

This innocent looking result implies that φ_R is non decreasing. Indeed, the image of a basis of $\mathbb{K}.\mathcal{A}(R)_n$ by multiplication by e^m is an independent subset of $\mathbb{K}.\mathcal{A}(R)_{n+m}$.

Finite generation

If a graded algebra A is finitely generated, then, since A is a quotient of the polynomial ring $\mathbb{K}[x_1, \ldots, x_k]$, its Hilbert function is bounded above by a polynomial. In fact, as it is well known, its Hilbert series is a fraction of form $\frac{P(x)}{(1-x)^d}$, thus of the form given in (1) of subsection . Moreover, one can choose a numerator with non-negative coefficients whenever the algebra is Cohen-Macaulay. Due to Problem 2, one could be tempted to conjecture that these sufficient conditions are necessary in the case of age agebras. Indeed, from Theorem 12 one deduces easily:

Theorem 13. *The profile of R is bounded if and only if $\mathbb{K}.\mathcal{A}(R)$ is finitely generated as a module over U. In particular, if one of these equivalent conditions holds, then $\mathbb{K}.\mathcal{A}(R)$ is finitely generated*

But this case is exceptional. Indeed, on one hand there are tournaments whose profile has arbitrarily large polynomial growth rate. On an other hand, with N.Thiéry we proved:

Theorem 14. *The age algebra of a tournament is finitely generated if and only if the profile is bounded.*

The Behavior of the Hilbert Function; a Conjecture of Cameron

Cameron [0] made an important observation about the behavior of the Hilbert fonction.

Theorem 15. *Let A be a graded algebra over an algebraically closed field of characteristic zero. If A is an integral domain the values of the Hilbert function h_A satisfy the inequality*

$$h_A(n) + h_A(m) - 1 \leq h_A(n+m) \qquad (3)$$

for all non-negative integers n and m.

In 1981, he made the conjecture that if R codes a permutation groups with no finite orbits then the age algebra over C is an integral domain. I solved it positively in a slightly more general setting:

Theorem 16. *Let R be a relational structure with possibly infinitely many non isomorphic types of n-element substructures. If the kernel of R is empty, then $\mathbb{K}.\mathcal{A}(R)$ is an integral domain.*

Since the kernel of a relational structure R encoding a permutation group G is the union of its finite orbits, if G has no finite orbit, the kernel of R is empty. Thus from this result, $\mathbb{K}.\mathcal{A}(R)$ is an integral domain, as conjectured by Cameron.

At the core of the solution is this lemma:

Lemma 17. *Let m, n be two non negative integers. There is an integer t such that for every set E, every field $\mathbb{K}$ with characteristic zero, every pair of maps $f : [E]^m \to \mathbb{K}$, $g : [E]^n \to \mathbb{K}$, if $fg(Q) := \sum_{P \in [Q]^m} f(P)g(Q \setminus P) = 0$ for every $Q \in [E]^{m+n}$ but f and g are not identically zero, then f and g are zero on $[E \setminus S]^m$ and $[E \setminus S]^n$ where S is a finite subset of E with size at most t*

If the age is inexhaustible, then in order to prove that there is no zero divisor, the only part of the lemma we need to apply is the assertion that S is finite.

The fact that S can be bounded independently of f and g, and the value of the least upper bound $\tau(n,m)$, seem to be of independent interest. The only exact value we know is $\tau(1,n) = 2n$, a fact which amounts to a weighted version of a theorem of Gottlieb and Kantor on incidence matrices. Our existence proof of $\tau(m,n)$ yields astronomical upper bounds. For example, it gives $\tau(2,2) \leq 2(R_k^2(4)+2)$, where $k := 5^{30}$ and $R_k^2(4)$ is the Ramsey number equals to the least integer p such that for every colouring of the pairs of $\{1,\ldots,p\}$ into k colors there are four integers whose all pairs have the same colour. The only lower bound we have is $\tau(2,2) \geq 7$ and more generally $\tau(m,n) \geq (m+1)(n+1)-2$. We cannot preclude a extremely simple upper bound for $\tau(m,n)$, eg quadratic in $n+m$.

For example, our 1971 proof of Theorem 3 consisted to show that $\varphi_R(n) \leq \varphi_R(n+1)$

provided that E is large enough, the size of E being bounded by some Ramsey number, whereas, according to the result of Gottlieb and Kantor, $2n + 1$ suffices [0].

M.H. Albert and M.D. Atkinson, *Simple permutations and pattern restricted permutations. Discrete Mathematics*, **300** (2005) 1–15.

M.H. Albert, M.D. Atkinson and V. Vatter, *Subclasses of the separable permutations.* ArXiv: 1007.1014V1 [math.co], 6 Jul 2010.

J. Balogh, B. Bollobás, M. Saks and V. T. Sós, *The unlabelled speed of a hereditary graph property. Journal of combinatorial theory*, **series B** 99 (2009) 9–19.

J. Berstel, C. Retenauer, *Les séries rationnelles et leurs langages.* Études et recherches en Informatique. Masson, Paris, 1984, 132 pp.

Y. Boudabbous, M. Pouzet, The morphology of infinite tournaments. Application to the growth of their profile. Oct. 2005, Revised Nov. 2006. 20 pp.

P.J. Cameron, Transitivity of permutation groups on unordered sets, *Math. Z.* **48** (1976), pp. 127–139.

P.J. Cameron, Orbits of permutation groups on unordered sets. II. *J. London Math. Soc.* **23(2)** (1981), pp. 249–264.

P.J. Cameron, *Oligomorphic permutation groups*. Cambridge University Press, Cambridge, 1990.

P.J. Cameron, The algebra of an age. In *Model theory of groups and automorphism groups (Blaubeure 1995)*, pages 126–133. Cambridge Univ. Press, Cambridge, 1997.

P.J. Cameron, Sequences realized by oligomorphic permutation groups. *J. Integer Seq.* **3(1)**, Article 00.1.5, 1 HTML document (electronic), 2000.

P.J. Cameron, Some counting problems related to permutation groups. *Discrete Math.* **225(1-3)** (2000), pp. 77–92. Formal power series and algebraic combinatorics (Toronto, ON, 1998).

P.J. Cameron, On an algebra related to orbit-counting. *J. Group Theory* **1(2)** (1998), pp. 173–179.

D.H.J. de Jongh and R. Parikh, Well-partial orderings and hierarchies. *Nederl. Akad. Wetensch. Proc. Ser. A* **80**=*Indag. Math.* **39(3)** (1977), pp. 195–207.

C. Delhommé, M. Pouzet, N. Sauer, G. Sági, Representation of ideals of relational structures, preprint, Calgary, July 2006, 18pp. http://arxiv.org/m

R. Fraïssé, *Sur quelques classifications des systèmes de relations*, Thèse, Paris (1953), *Alger-Math.* **1** (1954), pp. 35–182.

R. Fraïssé, Sur l'extension aux relations de quelques propriétes des ordres, *Ann. Sci. Ecole Norm. Sup.* **71** (1954), pp. 361–388.

R. Fraïssé, *Theory of relations*. Second edition, North-Holland Publishing Co., Amsterdam, 2000.

R. Fraïssé and M. Pouzet, Interprétabilité d'une relation pour une chaîne. *C. R. Acad. Sci. Paris Sér. A-B* **272** (1971), pp. 1624–1627.

R. Fraïssé, *Cours de logique mathématique. Tome 1: Relation et formule logique*. Gauthier-Villars Éditeur, Paris, 1971.

C. Frasnay, *Quelques problèmes combinatoires concernant les ordres totaux et les relations* *monomorphes*. Thèse. Paris. Annales Institut Fourier Grenoble **15** (1965), pp. 415–524.

A.M. Garsia and N. Wallach, Qsym over Sym is free. *J. of Comb. Theory, Series A* **104** (2003), pp. 217–263.

P. Gibson, M. Pouzet, and R. Woodrow, Relational Structures Having Finitely Many Full-cardinality Restrictions. *Discrete Mathematics* **291** (2005), pp. 115–134.

D.H. Gottlieb, A class of incidence matrices, *Proc. Amer. Math. Soc.* **17** (1966), pp. 1233–1237.

G. Higman, Ordering by divisibility in abstract algebras. *Proc. London Math. Soc.* **(3)** (1952), pp. 326–336.

I.M. Hodkinson, H.D. Macpherson, Relational structures determined by their finite induced substructures. *J. Symbolic Logic* **53(1)** (1988), pp. 222–230.

W.M. Kantor, On incidence matrices of finite projective and affine spaces, *Math. Z.* **124** (1972), pp. 315–318.

J.B. Kruskal, The theory of well-quasi-ordering: a frequently discovered concept. *J. Com. Th.* **13** (1972), pp. 297–305.

D. Livingstone, A. Wagner, Transitivity of finite permutation groups on unordered sets. *Math. Z.* **90** (1965), pp. 393–403.

M. Lothaire, *Combinatorics on words.* Vol. 17 of Encyclopedia of Mathematics and its Applications. Addison-Wesley, Reading, Mass. Reprinted in the Cambridge Mathematical Library, Cambridge University Press, U.K. 1997.

H.D. Macpherson, Growth rates in infinite graphs and permutation groups. *J. London Math. Soc.* **35(2)** (1987), pp. 276–286.

H.D. Macpherson, Permutation groups of rapid growth. *Proc. London Math. Soc.* **51(2)** (1985), pp. 285–294.

H.D. Macpherson, M. Pouzet, and R.E. Woodrow, Countable structures of a given age. *The Journal of Symbolic Logic* **57** (1992), pp. 992–1010.

T. Kaizer, M. Klazar *On growth rates of closed sets of hereditary permutation classes. electr. J. Combin.*, 9 (2002/2003), Research Paper 10, 20 pp.

M. Klazar, *Overview of general results in combinatorial enumeration. ArXiv: 0803.4292V1* [math.co], 29 Mar 2008.

A. Marcus, G. Tardos *Excluded permutation matrices and the Stanley-Wilf conjecture*, *J. Combin. Theory*, Ser. A 107 (2004), 153–160.

E.C. Milner, Basic wqo- and bqo-theory. In *Graphs and order, I.Rival Ed. (Banff, Alta., 1984)*, pages 487–502. Reidel, Dordrecht, 1985.

M. Pouzet, Un belordre d'abritement et ses rapports avec les bornes d'une multirelation. *Comptes rendus Acad. Sci. Paris* **274(A)** (1972), pp. 1677–1680.

M. Pouzet, Application d'une propriété combinatoire des parties d'un ensemble aux groupes et aux relations. *Math. Z.* **150(2)** (1976), pp. 117–134.

M. Pouzet, *Sur la théorie des relations*. Thèse d'état, Université Claude-Bernard, Lyon 1, pp. 78–85, 1978.

M. Pouzet, Relation minimale pour son âge. *Z. Math. Logik Grundlag. Math.* **25** (1979), pp. 315–344.

M. Pouzet, The asymptotic behavior of a class of counting functions. *Combinatorics 79 Part 2. M.DEZA and I.G.ROSENBERG Eds., Annals of Discrete Math.* **9** (1980), pp. 223–224.

M. Pouzet, Relation impartible. *Dissertationnes* **103** (1981), pp. 1–48.

M. Pouzet, Application de la notion de relation presque-enchaînable au dénombrement des restrictions finies d'une relation. *Z. Math. Logik Grundlag. Math.* **27(4)** (1981), pp. 289–332.

M. Pouzet, Applications of well quasi-ordering and better quasi-ordering. In *Graphs and order (Banff, Alta., 1984)*, pages 503–519. Reidel, Dordrecht, 1985.

M. Pouzet and I.G. Rosenberg, Sperner properties for groups and relations, *Europ. J. Combinatorics* **7** (1986), pp. 349–370.

M. Pouzet and M. Sobrani, Sandwiches of ages. *Ann. Pure Appl. Logic* **108(3)** (2001), pp. 295–326.

M. Pouzet and M. Sobrani, Ordinal invariants of an age. *Preprint*, 2002.

M. Pouzet and N. Thiéry, Some relational structures with polynomial growth and their associated algebras. May 10th, 2005, 19pages, presented at FPSAC for the 75 birthday of A.Garsia. http://arxiv.org/math.CO/0601256.

M. Pouzet, The profile of relations. Glob. J.Pure Appl. Math. Vol 2 (2006) 237-272.

M. Pouzet, When the orbit algebra of group is an integral domain? Proof of conjecture of Cameron. arXiv:0704.1548.

Y.Boudabbous, M.Pouzet. The morphology of infinite tournaments. Applications to the growth of their profile, preprint, 2007.

D.E. Radford, A natural ring basis for the shuffle algebra and an application to group schemes, *J. Algebra* **58** (1979), pp. 432–454.

F.P. Ramsey, On a problem of formal logic, *Proc. London Math. Soc.* **30** (1930), pp. 264–286.

C. Ryll-Nardzewski, On categoricity in power $\leq \aleph_\alpha$. *Bull. Acad. Pol. Sci. Sér. Math. Astr. Phys.* **7** (1959), pp. 545–548.

J. Schmerl, Coinductive $\aleph_0$-categorical theories *The Journal of Symbolic Logic* **55** (1990), pp. 1130–1137.

R.P. Stanley, *Enumerative combinatorics*. Volume 1, Wadsworth - Brooks/Cole Advanced Books and Software, Belmont, Calif. 1986.

M. Sobrani, *Sur les âges de relations et quelques aspects homologiques des constructions D+M*. Thèse de Doctorat d'état, Université S.M. Ben Abdallah-Fez, Fez, 2002.

E.S. Wolk, Partially well-ordered sets and partial ordinals. *Fun. Math.* **60** (1967), pp. 175–185.