# Finite segments of the harmonic series

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Dedicated to Paul Erdös 1913-1996

**Abstract**

Let $\sigma(m,k) := \sum_{j=0}^{k-1} 1/(m+j)$. For $\{m,k,m'\} \subset \mathbf{N}$ we define $k'$ by $\sigma(m',k'-1) < \sigma(m,k) \leq \sigma(m',k')$. Extending work by Taeisinger, Kürschák, Erdös, Belbachir, and Khelladi, we prove:

**1.** If $\sigma(m,k)$ is the reciprocal of an integer then $k = 1$.

**2.** $(k \notin [2,m) \vee k' \geq 2(m+k) \vee m' \geq 4m^2) \Rightarrow \sigma(m',k') \neq \sigma(m,k)$.

Conjecture: The function $\sigma : \mathbf{N} \times \mathbf{N} \to \mathbf{Q}^+$ is injective.

## §1. Introduction.

This paper studies the set $H$ of all *harmonic* rationals,

$$\sigma(m,k) := \sum_{j=0}^{k-1} \frac{1}{m+j}\,, \text{ where } \langle m,k\rangle \in \mathbf{N}^2 \text{ and } \mathbf{N} := \{1,2,3,\ldots\}.$$

It focuses on the function $\sigma : \mathbf{N}^2 \to \mathbf{Q}^+$ defined by $\sigma : \langle m,k\rangle \mapsto \sigma(m,k)$.

Let $H_n := H \cap \{\sigma(m,k) : m \geq n\}$. Of course $H = H_1 \supseteq H_2 \supseteq H_3 \supseteq \ldots$

**Theorem 1.** *Let* $n \in \mathbf{N}$. *Then* $H_n$ *is dense in* $[0,\infty)$.

**Proof.** Let $0 < l < r$ be real numbers. There exists an integer $m > n$ such that $1/m < \min\{l, r-l\}$. Since $\lim_{j\to\infty} \sigma(m,j) = \infty$, there exists $k \geq 2$ such that $\sigma(m,k-1) \leq l < \sigma(m,k)$. Hence $l < \sigma(m,k) = \sigma(m,k-1) + 1/(m+k-1) < l + 1/m < l + (r-l)$. So $\sigma(m,k) \in (l,r)$. ∎

This paper partially supports our belief that $\sigma : \mathbf{N}^2 \to \mathbf{Q}^+$ is injective.

*Via* the notion of a prime power that is "sylvester" for a set of integers, we consider the coprime pairs $\langle \nu,\delta\rangle$ such that $\nu/\delta \in H$.

In 1915, L. Taeisinger proved that $\sigma(1,k) \in \mathbf{N}$ only if $k = 1$. In 1918, J. Kürschák showed that $\sigma(m,k)$ is an integer only if $m = k = 1$. P. Erdös, [**1932E**] and Page 157 in [**1998H**], extended Kürschák's theorem to finite arithmetic subseries of the harmonic series:

$$\sum_{j=0}^{k-1} \frac{1}{m+dj} \notin \mathbf{N} \quad \text{if} \quad (m,d) = 1\,.$$

H. Belbachir and A. Khelladi [**2007BK**] generalized the Erdös result thus: For $\alpha_0, \alpha_1, \ldots, \alpha_{k-1}$ any positive integers,

$$\sum_{j=0}^{k-1} \frac{1}{(m+dj)^{\alpha_j}} \notin \mathbf{N} \quad \text{if} \quad (m,d) = 1\,.$$

Let $1/\mathbf{N} := \{1/n : n \in \mathbf{N}\}$. We will prove: $\sigma(m,k) \in 1/\mathbf{N} \Rightarrow k = 1$.

We will write $x \approx y$ to assert that $|x - y|$ is small enough not to invalidate size relationships based upon our approximation to $x = y$.

## §2. Sylvester powers.

For $S$ a finite nonempty set of positive integers, $\sigma(S) := \sum_{j \in S} 1/j$, and $\mu(S) :=$ the least common multiple of the elements in $S$. Observe that

$$\sigma(S) = \frac{\sigma(S)\mu(S)}{\mu(S)} = \frac{\nu(S)}{\delta(S)}, \text{ and that } \{\sigma(S)\mu(S), \mu(S)\} \subset \mathbf{N}, \text{ where}$$

$\langle \nu(S), \delta(S) \rangle$ is a uniquely determined coprime pair of positive integers.

Our interest lies in the intervals $[m, m+k) := \{m, m+1, \ldots, m+k-1\}$ of consecutive positive integers. We write $\sigma([m, m+k))$, $\mu([m, m+k))$, *etc.* simply as $\sigma(m,k)$, $\mu(m,k)$, *etc.*; indeed we may simplify further by writing instead $\sigma$, $\mu$, and so forth when $m$ and $k$ are understood.

We evoke two classic results:

**Bertrand's Postulate/Chebyshev's Theorem.** (See Erdös [**1934E**]) *If* $2 \le n \in \mathbf{N}$ *then there is a prime* $p \in (n, 2n)$.

**Sylvester's Theorem.** ([**1892S**] and [**1934E**]) *If* $m > k$ *then* $p|\mu$ *for some prime* $p > k$.

One writes $x^n \| y$ in order to state that both $x^n | y$ and $\neg\, x^{n+1} | y$.

**Definition.** If $X \subseteq \mathbf{N}$, if $p$ is a prime, and if $v \in \mathbf{N}$, then we call $p^v$ *sylvester* for $X$ iff $p^v \| \mu(X)$ while $p^v | s$ for exactly one element $s \in X$.

$$\mathbf{S}(X) := \{p^v : p^v \text{ is sylvester for } X\}, \text{ and } \mathbf{S}(m,k) := \mathbf{S}([m, m+k)).$$

**Lemma 2.** (Noted also in [**2007BK**]) *Let the prime power $p^v$ be sylvester for a finite nonempty set $F \subseteq \mathbf{N}$. Then $p^v \| \delta(F)$.*

**Proof.** Let $x_p$ be the unique element in $F$ with $p^v \| x_p$. Among the $|F|$ distinct integers $\mu(F)/x$ whose sum is $\mu(F)\sigma(F)$, only $\mu(F)/x_p$ fails to be a multiple of $p$. So $p$ is not a factor of the integer $\mu(F)\sigma(F)$. Thus $p^v \| \delta(F)$, since $p^v \| \mu(F)$ and since $\mu(F)\sigma(F)/\mu(F) = \nu(F)/\delta(F)$. ∎

**Lemma 3.** *For each $\langle m, k\rangle$ with $k \geq 2$, there is a $2^v$ that is sylvester for $[m, m+k)$. Indeed $k/2 < 2^v < m+k$.*

**Proof.** Since $k \geq 2$ we have that $2^v \| \mu(m,k)$ for some $v \geq 1$. Let $s$ be the smallest multiple of $2^v$ in $[m, m+k)$. Plainly $s = 2^v a$ for some odd integer $a$. Then $b := 2^v a + 2^v$ is the smallest multiple of $2^v$ with $b > s$. Since $b = 2^v(a+1)$, and since the integer $a+1$ is even, we have that $2^{v+1} | b$. Hence $b \geq m+k$ since $\neg\, 2^{v+1} | \mu(m,k)$. Thus $s$ is the only multiple of $2^v$ in $[m, m+k)$. So $2^v$ is sylvester for $[m, m+k)$. The first claim is established. The second claim follows from the facts that $2^{v+1} | \mu(m,k)$ if $2^v \leq k/2$, and that $2^v < m+k$ since $2^v | \mu(m,k)$. ∎

**Corollary 4.** *If $\langle m,k\rangle \neq \langle 1,1\rangle$ then $\mathbf{S}(m,k) \neq \emptyset$.* ∎

**Remark.** If $p > 2$ is prime then $\neg\,(p^v \| \mu(F) \Rightarrow p^v \in \mathbf{S}(F))$. *E.g.*, $5^2$ is not sylvester for $[25, 51) := \{25, 26, \ldots, 50\}$ although $5^2 \| \mu(25, 26)$.

**Kürschák's Theorem.** ([**1918K**]) *If $\sigma(m,k) \in \mathbf{N}$ then $m = k = 1$.*

**Proof.** Let $\langle m,k\rangle \neq \langle 1,1\rangle$. Then Lemmas 2 and 3 imply that $\delta(m,k)$ is even. So $\nu(m,k)/\delta(m,k)$ is not an integer. ∎

**Corollary 5.** *If $a/b \in H$ with $(a,b) = 1$ and with $a > 1$, then $2|b$.* ∎

**Corollary 6.** *Let* $a/b \in H$ *with* $(a,b) = 1$. *Then there is a pair* $\langle m,k \rangle$ *such that* $b|\mu(m,k)$, *and such that* $\mu(m,k)/b = \sigma(m,k)\mu(m,k)/a$.

**Proof.** There exists $\langle m,k \rangle \in \mathbf{N}^2$ for which $\langle a,b \rangle = \langle \nu(m,k), \delta(m,k) \rangle$. But $\nu/\delta$ is the lowest-terms form of the rational number $\sigma\mu/\mu$. It follows that $\mu/b = q = \sigma\mu/a$ for some $q \in \mathbf{N}$. ∎

**Corollary 7.** *The integer* $\mu/\delta = \sigma\mu/\nu$ *is a product of powers* $p^n$ *of odd primes* $p \leq k$. ∎

The next result engenders our guess that $\sigma : \mathbf{N}^2 \to \mathbf{Q}^+$ is injective.

**Theorem 8.** *If* $k \geq 2$ *then* $\sigma(m,k) \notin 1/\mathbf{N}$.

**Proof.** Pretend the theorem fails for a given $k \geq 2$ and $m \geq 1$. That is, we pretend that $\sigma = 1/\delta$.

If $k\delta < m$ then $1/k\delta > 1/m > 1/(m+1) > \cdots > 1/(m+k-1)$, whence $1/\delta = k \cdot 1/k\delta > 1/m + 1/(m+1) + \cdots + 1/(m+k-1) =: \sigma$. Similarly, if $k\delta > m+k-1$ then $1/\delta < \sigma$. Hence $m \leq k\delta \leq m+k-1$ if $\sigma = 1/\delta$.

From $k \geq 2$ we get $1/m < \sigma = 1/\delta$. Thus $2 \leq \delta < m$. So $k\delta \leq m+k-1$ implies $k \leq k(\delta-1) \leq m-1 < m$, giving us $m > k$. Sylvester's theorem implies that there is a prime $p > k$ and $v \in \mathbf{N}$ such that $p^v \in \mathbf{S}(m,k)$. Then $p^v \| x_p$ for exactly one $x_p \in [m, m+k)$, and $(p,x) = 1$ for every $x \in [m,m+k) \setminus \{x_p\}$. Hence $(\mu/x_p, p) = 1$, while $p|\mu/x$ for each $x \in [m,m+k) \setminus \{x_p\}$. So $(\mu\sigma, p) = 1$ while $p^v \| \mu$. Therefore $p^v \| \delta$, since $\mu\sigma/\mu = 1/\delta$. We must infer that $x_p = k\delta$.

Define $\sigma' := \sigma - 1/k\delta = c/d$ with $(c,d) = 1$. Then $\sigma' = 1/\delta - 1/k\delta = (k-1)/k\delta$. Now $p|k\delta$, but $(k-1,p) = 1$ since $p > k$. So $p|d$. On the other hand, $\sigma' = (\mu\sigma - \mu/k\delta)/\mu$, and $p^v|\,(\mu\sigma - \mu/k\delta)$ since $p^v\|\,\mu/x$ for every $x \in [m,m+k) \setminus \{k\delta\}$. Thus, recalling that $p^v\|\,\mu$ we infer that $(p,d) = 1$, reaching a contradiction: $p|d \wedge (p,d) = 1$. So $\sigma \neq 1/\delta$. ∎

Notice that Theorem 8 implies that $H_n \supset H_{n+1}$ for all $n \in \mathbf{N}$.

Henceforth fix $\langle m,k,m' \rangle \in \mathbf{N}^3$ with $k \geq 2$ and $m' \geq m+k$; define $k' = \min\{j : \sigma(m',j) \geq \sigma(m,k)\}$. Observe that $0 \leq \sigma(m',k') - \sigma(m,k) < \sigma(m',k') - \sigma(m',k'-1) = 1/(m'+k'-1) \leq 1/(m+2k)$.

**Theorem 9.** *Let* $m' \leq k'$. *Then* $\sigma(m',k') \neq \sigma(m,k)$.

**Proof.** Since $2m' \leq m' + k'$, by Bertrand's Postulate there is a largest prime $p$ with $m' \leq (m'+k')/2 < p < m'+k'$, and $2p \notin [m', m'+k')$. Surely $p^1$ is sylvester for $[m', m'+k')$, whence $p|\delta(m', k')$ by Lemma 2. But $p > i$ for every $i \in [1, m'-1]$. Thus, $\neg\, p|j$ for all $j \in [m, m+k) \subseteq [1, m'-1]$, and so $\neg\, p|\mu(m, k)$ whence $\neg\, p|\delta(m, k)$. Therefore $\sigma(m, k) \neq \sigma(m', k')$. ∎

The following lemma utilizes a tightening of the fact that

$$\sigma(m, k) \approx \int_m^{m+k-1} \frac{dx}{x} = \ln(1 + \frac{k-1}{m}).$$

**Lemma 10.** *Let* $k \geq 2$. *Then* $1 + k/m \leq m'/m < (k'-1)/(k-1)$.

**Proof.** We can view $\sigma(m, k)$ as a step function above $1/x$, and note that

$$\sigma(m, k-1) > \ln(1 + \frac{k-1}{m}).$$

Therefore, since $\sigma(m, k) = \sigma(m, k-1) + 1/(m+k-1)$, we infer that

$$\frac{1}{m+k-1} < \sigma(m, k) - \ln(1 + \frac{k-1}{m}).$$

Next, viewing $\sigma(m+1, k-1)$ as a step function under $1/x$, we see that

$$\sigma(m+1, k-1) < \ln(1 + \frac{k-1}{m})$$

and hence, since $\sigma(m, k) = \sigma(m+1, k-1) + 1/m$, we infer that

$$\sigma(m, k) - \ln(1 + \frac{k-1}{m}) < \frac{1}{m}.$$

Combining inequalities, we get that

$$\frac{1}{m+k-1} < \sigma(m, k) - \ln(1 + \frac{k-1}{m}) < \frac{1}{m}.$$

Similarly we infer that

$$-\frac{1}{m'} < \ln(1 + \frac{k'-1}{m'}) - \sigma(m', k') < -\frac{1}{m'+k'-1}.$$

From the immediately preceding two sets of inequalities we get that

$$\frac{1}{m+k-1} - \frac{1}{m'} < \ln(1+\frac{k'-1}{m'})/(1+\frac{k-1}{m}) + \sigma - \sigma' \,,$$

where $\sigma$ and $\sigma'$ are shorthand for $\sigma(m,k)$ and $\sigma(m',k')$, respectively. Recall that $\sigma - \sigma' \leq 0$. Since $m+k \leq m'$, we have that

$$0 < L := \frac{1}{m+k-1} - \frac{1}{m'} < \ln(1+\frac{k'-1}{m'})/(1+\frac{k-1}{m}) \,.$$

Therefore

$$1 < e^L < (1+\frac{k'-1}{m'})/(1+\frac{k-1}{m}) \,, \quad \text{whence}$$

$$1+\frac{k'-1}{m'} > 1+\frac{k-1}{m}.$$

The lemma follows. ∎

**Theorem 11.** *Let* $m < k$. *Then* $\sigma(m,k) \neq \sigma(m',k')$.

**Proof.** Lemma 10 gives us that $1 \leq (k-1)/m < (k'-1)/m'$. So $m' < k'-1$ whence $\sigma(m,k) \neq \sigma(m',k')$ by Theorem 9. ∎

**Corollary 12.** *Let* $\sigma > 1/m + \ln 2$. *Then* $\sigma' \neq \sigma$.

**Proof.** In our proof of Lemma 10 we saw that $\sigma < 1/m + \ln(1+(k-1)/m)$. So $1/m + \ln 2 < \sigma < 1/m + \ln(1+(k-1)/m)$ by hypothesis. It follows that $\ln 2 < \ln(1+(k-1)/m)$. Thus $m < k-1$. So $\sigma' \neq \sigma$ by Theorem 11. ∎

Given $\langle m,k,m' \rangle \in \mathbf{N}^3$, recall that $k'$ is determined, and that if $\sigma(m',j) = \sigma(m,k)$ then $j = k'$. By Theorem 11, $\sigma(m',k') \neq \sigma(m,k)$ when the interval $[m, m+k)$ is "long"; *i.e.*, when $m < k$. Theorem 8 implies that for no $m$ is there a $\langle m',k' \rangle$ such that $\sigma(m',k') = \sigma(m,1)$.

We have not established the injectivity of $\sigma$ for $2 \leq k \leq m$. But we now show that, for each $m$, there are at most finitely many $m' > m$ for which the possibility $\sigma(m',k') = \sigma(m,k)$ will not yet have been eliminated.

**Theorem 13.** *Let* $k' \geq 2(m+k)$. *Then* $\sigma(m',k') \neq \sigma(m,k)$.

**Proof.** By Lemma 3, $2^v \in \mathbf{S}(m,k)$ for some $v \in \mathbf{N}$. But $2^v \notin \mathbf{S}(m',k')$ since $2^{v+1} | \mu(m',k')$. So $2^{v+1} | \delta(m',k')$ by Lemma 2, while $2^v \| \delta(m,k)$. Since the fractions $\nu(m,k)/\delta(m,k)$ and $\nu(m',k')/\delta(m',k')$ are in lowest terms, we have that $\sigma(m',k') \neq \sigma(m,k)$. ∎

Given $m$ and $k$, clearly the integers $m'$ and $k'$ determine each other. For $2 \leq k < m$ this implies that $k' < 2(m+k) \Rightarrow m' < 4m^2$. In summary, $\sigma(m',k') = \sigma(m,k)$ remains unprecluded by Theorems 8, 11, and 13 only when all three of the following conditions obtain: $m \geq 3$ and $2 \leq k < m$ and $k' < 2(m+k)$. Note that $m'/m \approx k'/k$, with $k' \geq m'k/m > k' - 1$.

Thus, to prove $\sigma$ injective, we need depose for each $m \geq 3$ fewer than $m^2 + m$ possible counterexamples.

We have shown that $\mathbf{S}(X) \neq \mathbf{S}(Y) \Rightarrow \sigma(X) \neq \sigma(Y)$, But regretably $\neg\, [\mathbf{S}([x,y)) = \mathbf{S}([w,z)) \Rightarrow [x,y) = [w,z)]$. For example, $\mathbf{S}(\{5,6,7\}) = \{2,3,5,7\} = \mathbf{S}(\{14,15\})$. But $\mathbf{S}(\{14,15,16\}) = \{2^4,3,5,7\} \neq \mathbf{S}(\{5,6,7\})$. Indeed, $\mathbf{S}(\{5,6,7\}) \neq \mathbf{S}([14,14+k))$ for every $k \geq 3$.

Echoing Sylvester, upon its verification the following narrowly germane guess would immediately establish that $\sigma : \mathbf{N}^2 \to \mathbf{Q}^+$ is injective.

**Conjecture.** If $m+2 \leq m+k \leq m'$ then $\mathbf{S}(m',k') \neq \mathbf{S}(m,k)$.

**Definition.** We call a family $\mathcal{D}$ of finite subsets of $\mathbf{N}$ *distinguished* iff when $X \neq Y$ are elements in $\mathcal{D}$ then $\mathbf{S}(X \setminus Y) \neq \mathbf{S}(Y \setminus X)$.

From our work thus far, the following two theorems are easy exercises.

**Theorem 14.** *If $\mathcal{D}$ is distinguished then $\sigma : \mathcal{D} \to \mathbf{Q}^+$ is injective.* ∎

We do not allege the converse of Theorem 14.

When $B \subset \mathbf{N}$ and $n \in \mathbf{N}$, let $B^n := \{j^n : j \in B\}$. And, for $\mathcal{D}$ a family of finite nonempty subsets of $\mathbf{N}$, let $\mathcal{D}^{(n)} := \{B^n : B \in \mathcal{D}\}$.

**Theorem 15.** *A family $\mathcal{D}$ of finite nonempty subsets of $\mathbf{N}$ is distinguished if and only if $\mathcal{D}^{(n)}$ is distinguished for every $n \in \mathbf{N}$. Hence, if $\mathcal{D}$ is distinguished then $\sigma \restriction \mathcal{D}^{(n)}$ is injective for every $n$.* ∎

**Problem.** Identify the maximal distinguished subfamilies of the collection $\mathcal{P}(\mathbf{N})$ of all subsets of $\mathbf{N}$?

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