The forbidden poset problem (for consecutive levels)

Gyula O.H. Katona
MTA Rényi Institute
Budapest Pf 127, 1364 Hungary
ohkatona@renyi.hu

Abstract

The family of all subsets of an n-element set forms a poset for the binary relation "subset". This is the Boolean lattice B_n, but we disregard the lattice operations here. Take a "small" poset P with the binary relation (order) $\prec$. An embedding of P into B_n is a mapping $\phi : P \to B_n$ where $a \prec b$ implies $\phi(a) \subset \phi(b)$. Since a family $\mathcal{F}$ is a subset of B_n one can speak about the embedding of P into a family $\mathcal{F}$ as well. We say in this case $\mathcal{F}$ contains a copy of P. The general problem is to determine $\max |\mathcal{F}|$ where $\mathcal{F} \subset 2^{[n]}$ is a family without a copy of P. This maximum is denoted by $\mathrm{La}(n, P)$. If P has two comparable elements, the poset is denoted by I. $\mathrm{La}(n, I)$ is the maximum number of subsets without inclusion. This was determined by Sperner's theorem (the largest binomial coefficient of order n). We will survey results exactly or asymptotically determining $\mathrm{La}(n, P)$. A more general problem is when two small posets (say P_1, P_2) are forbidden. Then the maximum is denoted by $\mathrm{La}(n, P_1, P_2)$. One of the completely solved cases is when P_1 is the so-called Y poset (it has 4 distinct elements with the relations $a < b, b < c, b < d$) and P_2 is its complement (turning the relations back). Then $\mathrm{La}(n, P_1, P_2)$ is attained for the two middle levels of B_n. The area is far from being "completed" as the following example shows. The diamond D is a poset of 4 elements with one minimal and one maximal element a and b where $a \prec c, d \prec b$ (c and d are incomparable). The value of $\mathrm{La}(n, D)$ is not even asymptotically determined. At the end of the lecture we will show new results for the modified problems where the small poset

is forbidden only for consecutive levels of B_n. Let us formulate the problem of Y and its complement for this case in terms of subsets. Find the largest family of subsets of an n-element set so that there are no 4 distinct members A, B, C, D such that $A \subset B, B \subset C, B \subset D, |B - A| = |C - B| = |D - B| = 1$ and there are no 4 distinct members A, B, C, D such that $A \supset B, B \supset C, B \supset D, |A - B| = |B - C| = |B - D| = 1$. The exact maximum has been determined.