Features of objects

Variables
  - Linguistic variables
  - Linguistic hedges

Fuzzy propositions
  - Representation of fuzzy propositions
  - Fuzzy IF-THEN rules
    - Interpretation of fuzzy IF-THEN rules
    - Soundness of IF-THEN rules
  - Fuzzy rule base

Exercises
Features of objects

Let $F$ be a feature of objects, for example $Age$, $Distance$, $Height$, $Size$. Each feature may be

- **numerical** – set of numbers. For example,
  \[
  Age = \{1, 2, \ldots, 120\},
  \]
  \[
  Distance = \mathbb{R}^+. 
  \]
- **linguistic** – set of words. For instance,
  \[
  Age = \{\text{young, old, rather young, teenage, extremely old}\}.
  \]
  \[
  Distance = \{\text{long, close, vicinity, far way}\}.
  \]
A variable is associated with its domain $Dom$ (e.g., a feature of object) and takes values from $Dom$. Depending on $Dom$ we distinguish

- numerical: take numbers as their values (e.g., $Age = 25$)
- linguistic: take words as their values (e.g., $Age = \text{YOUNG}$).
Linguistic variables

For linguistic variables, their values (words) are represented by fuzzy sets in domains they are defined.

Example 8.1

The speed of a car is a variable $x$ taking values from $[0, V_{max}]$. Let \{SLOW, MEDIUM, FAST\} be the set of values of $x$. Define 3 fuzzy sets:

![Figure 8.1](image-url)
A *linguistic variable* is a tuple \((X, T, U, M)\) where

- \(X\) is a name of a variable,
- \(T\) is the set of linguistic values that \(X\) can take,
- \(U\) is the domain in which \(X\) takes its quantitative values,
- \(M\) is the *meaning function* which to every linguistic value in \(T\) assigns a fuzzy set in \(U\)
Example 8.1 (cont.)

In our example,

- $X$ is a SPEED OF A CAR,
- $T = \{\text{SLOW, MEDIUM, FAST}\}$,
- $U = [0, V_{max}]$,
- $M$ relates SLOW, MEDIUM, and FAST with fuzzy sets shown in Figure 8.1.
In general, values of linguistic variables are terms:

\[
\langle \text{term} \rangle ::= \langle \text{atomic term} \rangle \mid \text{NOT} \ \langle \text{term} \rangle \mid \langle \text{term} \rangle \ \text{AND} \ \langle \text{term} \rangle \mid \\
\langle \text{term} \rangle \ \text{OR} \ \langle \text{term} \rangle \mid \langle \text{hedge} \rangle \langle \text{term} \rangle
\]

where \textit{atomic term} is an element of \( T \) and \textit{hedge} is a linguistic modifier which lessen or intensify the meaning of an utterance, e.g., ‘very”, “extremely”, “rather”, “quite”, “more or less”, “fairly”.

Formally

A linguistic hedge is represented by a bijection \( h : [0, 1] \rightarrow [0, 1] \) satisfying \( h(0) = 1 \) and \( h(1) = 1 \). The most commonly used operation is \( h(x) = x^p \), where \( p > 0 \).
Linguistic hedges

In general, values of linguistic variables are terms:

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Definition 8.2 (Zadeh, 1972)

Let $A \in \mathcal{F}(X)$. Then *very* $A$ is a fuzzy set in $X$ defined for every $x \in X$,

$$\text{very}(A)(x) = A^2(x)$$

and *more or less* $A$ is a fuzzy set in $X$ defined for $x \in X$,

$$\text{more or less}(A)(x) = \sqrt{A(x)}.$$
Example 8.3

Let \( X = \{1, 2, 3, 4, 5\} \) and the fuzzy set \( \text{small} \) be defined as

\[
\text{small} = \left\{ \begin{array}{c}
1 \, \tilde{\text{1}}', \quad 2 \, \tilde{\text{0.8}}', \quad 3 \, \tilde{\text{0.6}}', \quad 4 \, \tilde{\text{0.4}}', \quad 5 \, \tilde{\text{0.2}}' \\
\end{array} \right. 
\]

Then

\[
\text{very small} = \left\{ \begin{array}{c}
1 \, \tilde{\text{1.0}}', \quad 2 \, \tilde{\text{0.64}}', \quad 3 \, \tilde{\text{0.36}}', \quad 4 \, \tilde{\text{0.16}}', \quad 5 \, \tilde{\text{0.04}}' \\
\end{array} \right. 
\]

\[
\text{extremely small} = \text{very very small} = \left\{ \begin{array}{c}
1 \, \tilde{\text{1.0}}', \quad 2 \, \tilde{\text{0.4096}}', \quad 3 \, \tilde{\text{0.1296}}', \quad 4 \, \tilde{\text{0.0256}}', \quad 5 \, \tilde{\text{0.0016}}' \\
\end{array} \right. 
\]

\[
\text{more or less small} = \left\{ \begin{array}{c}
1 \, \tilde{\text{1.9}}', \quad 2 \, \tilde{\text{0.8944}}', \quad 3 \, \tilde{\text{0.7746}}', \quad 4 \, \tilde{\text{0.6325}}', \quad 5 \, \tilde{\text{0.4472}}' \\
\end{array} \right. 
\]
Definition 8.3

An *atomic fuzzy proposition* \((\text{atom, for shprt})\) is an expression of the form

\[ x \text{ IS } A \]

where \(x\) is a linguistic variable and \(A\) is a linguistic value. \(A\) is represented by a fuzzy set in \(X\) while \(x \in X\).

E.g.,

- *Temperature IS HIGH*
- *Age IS MEDIUM*
- *Speed IS SLOW.*
Definition 8.4

A *compound fuzzy proposition* (or just *fuzzy proposition*) is a composition of atomic fuzzy propositions using the connectives “and”, “or”, “not”.

For instance,

\[ \text{Speed IS high AND Temperature IS NOT low} \]
\[ \text{Temperature IS high OR Humidity IS fairly low.} \]
Fuzzy propositions are represented by fuzzy relations determined by t-norms, t-conorms, or fuzzy negations:

- for “and” use t-norms $T$. Specifically, if $x$ and $y$ are linguistic variables in $U$ and $V$, respectively, and $A \in \mathcal{F}(U)$, $B \in \mathcal{F}(V)$, then

$$x \text{ IS } A \text{ AND } y \text{ IS } B$$

is interpreted as a fuzzy relation $R \in \mathcal{F}(U \times V)$ with membership function $R(x, y) = T(A(x), B(y))$.

- For “or” use t-conorms $S$. Specifically,

$$x \text{ IS } A \text{ OR } y \text{ IS } B$$

is represented by a fuzzy relation $R \in \mathcal{F}(U \times V)$ with membership function $R(x, y) = S(A(x), B(y))$. 

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For “not” use fuzzy negations $N$, that is,

$$x \text{ IS not } A$$

is represented by $A_N$, where $A_N(z) = N(A(z))$ for every $z$.

**Example 8.4**

The fuzzy proposition

$$(x \text{ IS small AND } x \text{ IS not fast}) \text{ OR } x \text{ IS medium}.$$ 

is represented by a ternary fuzzy relation on $[0, V_{max}]^3$ with membership function

$$R(x_1, x_2, x_3) = S(T(\text{Small}(x_1), N(\text{Fast}(x_2))), \text{Medium}(x_3)).$$
For “not” use fuzzy negations \( N \), that is,

\[
x \text{ IS not } A
\]

is represented by \( A_N \), where \( A_N(z) = N(A(z)) \) for every \( z \).

Example 8.4

The fuzzy proposition

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(x \text{ IS SMALL AND } x \text{ IS not FAST}) \text{ OR } x \text{ IS MEDIUM.}
\]

is represented by a ternary fuzzy relation on \([0, V_{max}]^3\) with membership function

\[
R(x_1, x_2, x_3) = S(T(\text{Small}(x_1), N(\text{Fast}(x_2))), \text{Medium}(x_3)).
\]
Definition 8.5

An **IF-THEN rule** is an expression of the form:

\[ \text{IF } FP_1 \text{ THEN } FP_2 \]

where \( FP_1 \) and \( FP_2 \) are fuzzy propositions.
Interpretation of fuzzy IF-THEN rules

IF \(FP_1\) THEN \(FP_2\).

(1)

Let \(U = U_1 \times \ldots \times U_n\) and \(V = V_1 \times \ldots \times V_k\). If \(FP_1\) and \(FP_2\) are represented by fuzzy relations \(R_1 \in \mathcal{F}(U)\) and \(R_2 \in \mathcal{F}(V)\), then the rule (1) is interpreted as a fuzzy relation \(R \in \mathcal{F}(U \times V)\) defined for all \(u \in U\) and \(v \in V\),

\[ R(u, v) = I(u, v) \]

where \(I\) is a fuzzy implication.

Remark 8.1

1. In practise \(T = T_M\) and \(T = T_P\) are often used to represent fuzzy IF-THEN rules: \(R(u, v) = T(FP_1(u), FP_2(v))\).

2. Zadeh’s approach to represent fuzzy IF-THEN rules:
\[ R_Z(u, v) = \max\{1 - FP_1(v), \min(FP_1(u), FP_2(v))\} \].
Interpretation of fuzzy IF-THEN rules

IF $FP_1$ THEN $FP_2$. \hspace{1cm} (1)

Let $U = U_1 \times \ldots U_n$ and $V = V_1 \times \ldots \times V_k$. If $FP_1$ and $FP_2$ are represented by fuzzy relations $R_1 \in \mathcal{F}(U)$ and $R_2 \in \mathcal{F}(V)$, then the rule (1) is interpreted as a fuzzy relation $R \in \mathcal{F}(U \times V)$ defined for all $u \in U$ and $v \in V$,

$$R(u,v) = I(u,v)$$

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1. In practise $T = T_M$ and $T = T_P$ are often used to represent fuzzy IF-THEN rules: $R(u,v) = T(FP_1(u), FP_2(v))$.

2. Zadeh’s approach to represent fuzzy IF-THEN rules: $R_Z(u,v) = \max\{1 - FP_1(v), \min(FP_1(u), FP_2(v))\}$. 
Let $U = \{1, 2, 3, 4\}$ and $V = \{1, 2, 3\}$. Consider the fuzzy IF-THEN rule

$$\text{IF } x \text{ IS LARGE, THEN } y \text{ IS SMALL.}$$

where

$$\text{Large} = \left\{ \frac{1}{0.0}, \frac{2}{0.1}, \frac{3}{0.5}, \frac{4}{1.0} \right\}, \quad \text{Small} = \left\{ \frac{1}{1.0}, \frac{2}{0.5}, \frac{3}{0.1} \right\}.$$ 

Then

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Relation based on $I_{KD}$

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Relation based on $I_{L}$

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Relation based on $I_{G}$
Soundness of IF-THEN rules

**Definition 8.6**

Let $\sigma \in [0, 1]$. An IF-THEN rule is called $\sigma$-sound for $x \in X$ and $y \in Y$ iff $R(x, y) \geq \alpha$. If $\alpha = 1$, then the rule is totally sound for $x$ and $y$.

**Example 8.5**

- **Deduction:** Which value of $y$ guarantees that the rule is totally sound for $x_0 = 3$?
  
  **Answer:** For $I \in \{I_L, I_G\}$, $y \leq 2$; for $I = I_{KD}$, only $y = 1$.

- **Abduction:** Assume that the output is $y = 2$. Which input value $x$ guarantees that the rule is 0.9-sound?
  
  **Answer:** For $I = I_{KD}$, $x \leq 2$; for $I \in \{I_L, I_G\}$, $x \leq 3$. 
Definition 8.6

Let $\sigma \in [0, 1]$. An IF-THEN rule is called $\sigma$-sound for $x \in X$ and $y \in Y$ iff $R(x, y) \geq \alpha$. If $\alpha = 1$, then the rule is totally sound for $x$ and $y$.

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- **Deduction:** Which value of $y$ guarantees that the rule is totally sound for $x_0 = 3$?
  
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- **Abduction:** Assume that the output is $y = 2$. Which input value $x$ guarantees that the rule is 0.9-sound?
  
  **Answer:** For $I = I_{KD}$, $x \leq 2$; for $I \in \{I_L, I_G\}$, $x \leq 3$. 

A fuzzy rule base is a set of fuzzy IF-THEN rules:

\[
\text{IF } FP_{1i} \text{ THEN } FP_{2i}, \quad i = 1, \ldots, n
\]

represented by a fuzzy relation

\[
R(u, v) = \bigvee_{i=1}^{n} R_i(u, v).
\]

where \(R_i(u, v)\) represents \(i\)-th rule.
Thank you for your attention!
Questions are welcome.
Exercises
Problem 1

Let \( A(x) = \frac{x+1}{3} \) and \( B(y) = 1 - y \), \( x, y \in [0, 1] \). Take \( I = I_L \).

Question 1

Calculate admissible values of \( y \) for which the rule “IF \( x \) IS \( A \), THEN \( y \) IS \( B \)” is totally sound for input \( x \in [0, \frac{1}{3}] \).

Solution

Recall \( I_L(a, b) = 1 \) iff \( a \leq b \).

Then \( R(x, y) = I_L(A(x), B(y)) = 1 \) iff \( \frac{x+1}{3} \leq 1 - y \) iff \( x + 3y \leq 2 \).

Hence \( y \leq \frac{5}{9} \).
Problem 1

Let \( A(x) = \frac{x+1}{3} \) and \( B(y) = 1 - y \), \( x, y \in [0, 1] \). Take \( I = I_L \).

Question 1

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Hence \( y \leq \frac{5}{9} \).
Problem 1 (cont.)

Question 2
Which input values $x$ should be given in order to the rule “IF $x$ IS $A$, THEN $y$ IS $B$” is totally sound for $x$ and any output value $y \in [\frac{1}{3}, \frac{1}{2}]$.

Solution
$R(x, y) = 1$ iff $x + 3y \leq 2$.
For $y \in [\frac{1}{3}, \frac{1}{2}]$ and $x \in [0, 1]$, we get $x \leq \frac{1}{2}$. 
Question 2

Which input values $x$ should be given in order to the rule “IF $x$ IS $A$, THEN $y$ IS $B$” is totally sound for $x$ and any output value $y \in \left[ \frac{1}{3}, \frac{1}{2} \right]$.

Solution

$R(x, y) = 1$ iff $x + 3y \leq 2$.

For $y \in \left[ \frac{1}{3}, \frac{1}{2} \right]$ and $x \in [0, 1]$, we get $x \leq \frac{1}{2}$. 
Problem 2

Consider a fuzzy IF-THEN rule:

\[
\text{IF} \ Speed \ \text{IS} \ \text{SLOW} \ \text{AND} \ \text{Acceleration} \ \text{IS} \ \text{SMALL}, \\
\text{THEN} \ \text{ForceAppliedToAccelerator} \ \text{IS} \ \text{LARGE}. 
\]

Assume that \( X = [0, 100], \ Y = [0, 39], \) and \( Z = [0, 3]. \)

\[
\begin{align*}
\text{slow}(x) &= \begin{cases} 
1 & \text{if } x \leq 35 \\
\frac{55-x}{20} & \text{if } 35 < x \leq 55 \\
0 & \text{if } x > 55 
\end{cases} \\
\text{small}(y) &= \begin{cases} 
1 - \frac{1}{10}y & \text{if } 0 \leq y \leq 10 \\
0 & \text{if } y > 10 
\end{cases} \\
\text{large}(z) &= \begin{cases} 
0 & \text{if } z \leq 1 \\
z - 1 & \text{if } 1 \leq z \leq 2 \\
1 & \text{if } z > 2 
\end{cases}
\]

Calculate:

1. \( FP_1(x, y) \) for \( T = T_P. \)
2. \( R(x, y, z) \) for \( I_{KD}. \)
Problem 3

Consider a rule:

IF a Score is MORE OR LESS GOOD, THEN Scholarship IS AVERAGE.

Assume that GOOD and AVERAGE are represented by the following fuzzy sets:

\[
G(x) = \begin{cases} 
0 & \text{if } 0 \leq x \leq 3.5 \\
 x - 3 & \text{if } 3.5 < x \leq 4.5 \\
1 & \text{if } 4.5 < x \leq 5.0 
\end{cases} \\
A(y) = \begin{cases} 
0 & \text{if } y \leq 400 \text{ or } y \geq 1000 \\
\frac{1}{100} y - 4 & \text{if } 400 < y \leq 500 \\
1 & \text{if } 500 < y \leq 750 \\
-\frac{1}{250} y + 4 & \text{if } 750 < y < 1000 
\end{cases}
\]

Task

1. Taking \( I = I_L \) determine the relation \( R \).
2. Assume that the score is \( x_0 = 4.0 \) and the above rule is totally sound. Determine the admissible scholarship.
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1. Taking \( I = I_L \) determine the relation \( R. \)

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Thank you for your attendance!

See you next week!