Knowledge Representation and Reasoning

Lecture 12: Default Logics

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  - deal with *complete* information
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- Human reasoning
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  - conclusions need not be correct and may be invalidated in view of new evidences
  - is grounded on the concept of *rationality*. 
Rationality is:

- *agent–oriented*: different agents have usually different opinions as to what is/is not rational in a given situation
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- **agent–oriented**: different agents have usually different opinions as to what is/is not rational in a given situation.

- **purpose–oriented**: E.g., one may assume that Bill is honest and lend him 50$, but would be more cautious while considering him as a potential business partner.
Bill is planning to make a trip by car. Given no other evidence, he assumes that
\[ \alpha : \text{his car is where it was parked.} \]
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If Bill ignores all circumstances under which \( \alpha \) could be false and base his actions on the assumption that it is true, then \( \alpha \) is his belief.
Example 12.1 (Winograd, 1980)

*Bill* is planning to make a trip by car. Given no other evidence, he assumes that

\[ \alpha : \text{his car is where it was parked.} \]

- If *Bill* ignores all circumstances under which \( \alpha \) could be false and base his actions on the assumption that it is true, then \( \alpha \) is his belief.

- However, even if he is almost sure that \( \alpha \) is true, but anyway tries to improve his chances and check whether a bus will pass by soon, then he does not regard \( \alpha \) as his belief, but rather as a very likely contingency.
Traditional deduction is *monotonic* – new premises cannot invalidate old conclusion. Formally,

\[ T_1 \subseteq T_2 \implies Th(T_1) \subseteq Th(T_2), \]

where \( Th(T) = \{ \alpha : T \vdash \alpha \} \).

Common-sense reasoning is *non-monotonic* – new premises may invalidate previously derived conclusions.
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Common-sense reasoning is *non-monotonic* – new premises may invalidate previously derived conclusions.

Non-monotonic rule:

*Given \( \alpha \), in the absence of evidence \( \beta \), infer \( \gamma \).*

\( \alpha \) supports the conclusion \( \gamma \), while the lack of \( \beta \) assures its rationality.
Non-monotonicity and RAC

**Frame problem** – all aspects of the world remain invariant except from those explicitly changed by actions. In other words, in the absence of evidence on the contrary, accept only those changes that are caused by performing actions.
Non–monotonicity and RAC

- **Frame problem** – all aspects of the world remain invariant except from those explicitly changed by actions. In other words, in the absence of evidence on the contrary, accept only those changes that are caused by performing actions.

- **Qualification problem** – in the absence of evidence on the contrary, assume that the action succeeds and its execution leads to given effects.
Typology of non-monotonicity

Consider the following rules:

(1) *In the absence of evidence on the contrary, assume that a bird can fly.*
(2) *Unless your name is on a list of winners, assume that you are a looser.*

(1) deals with incomplete information, while (2) refers to complete information (list of winners), but incomplete representation (lack of the list of loosers).
Both rules are non-monotonic:

(1) if we get the information that a bird *Tweety* is a penguin (and we know that all penguins cannot fly), previously derived conclusion (“*Tweety* can fly”) must be rejected.
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(1) if *Bill* is added to the list of winners (broader context), the previous conclusion (“*Bill* is a looser”) is invalidated.
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(1) if *Bill* is added to the list of winners (broader context), the previous conclusion (“*Bill* is a looser”) is invalidated.

(1) is a *default* rule, while (2) is an *autoepistemic* rule.
Types of non-monotonic rules

There are two main types of non-monotonic reasoning:

- Default reasoning.
- Autoepistemic reasoning.
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**Default reasoning:**

- Drawing conclusions from less than conclusive information.
- Incompleteness information makes this inference *defeasible*.
- Defeasibility – any conclusion derived by default is tentative and can be invalidated by giving new evidence.
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- Default reasoning.
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Default reasoning:

- Drawing conclusions from less than conclusive information.
- Incompleteness information makes this inference defeasible.
- Defeasibility – any conclusion derived by default is tentative and can be invalidated by giving new evidence.

Remark: Defeasible reasoning goes back to Aristotle; he considered it in his work Topics.
Autoepistemic reasoning

Reaching conclusions from incomplete representation of virtually complete information.
Autoepistemic reasoning

- Reaching conclusions from incomplete representation of virtually complete information.
- Assumption: since the available information is complete, we would know if the conclusion were false.
Autoepistemic reasoning

- Reaching conclusions from incomplete representation of virtually complete information.
- Assumption: since the available information is complete, we would know if the conclusion were false.
- Non–defeasibility – conclusions cannot be invalidated by new information, but they change non–monotonically wrt the context in which rules are embedded.
### Types of default reasoning

<table>
<thead>
<tr>
<th>Types of reasoning</th>
<th>Explanation of the conclusion $\gamma$</th>
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<tr>
<td>Prototypical reasoning</td>
<td>$\gamma$ describes a typical situation, supported by statistical observations, so there are good chances that $\gamma$ holds.</td>
</tr>
<tr>
<td>No–risk reasoning</td>
<td>If $\neg \gamma$ leads to dangerous consequences, then $\gamma$ should be accepted.</td>
</tr>
<tr>
<td>Best–guess reasoning</td>
<td>In view of the available information, $\gamma$ is the best guess to make.</td>
</tr>
<tr>
<td>Probabilistic default reasoning</td>
<td>Assuming that probabilistic values are sufficiently high, it is sensible to infer $\gamma$.</td>
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Example 12.2 (Prototypical reasoning)

All we know about *Tweety* is that it is a bird. Since most birds (so a *typical* bird) can fly, we can infer that *Tweety* can fly.
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**Example 12.3** (Best–guess reasoning)

There are two shopping centers in the vicinity of our home and all we know that exactly one of them is open on Sundays (but we do not know which one). Assume that now is a Sunday morning and we need to do some shopping. Then it is reasonable (most convenient!) to assume that the shopping center which is now open is the one nearest to our home.
Example 12.4 (No–risk reasoning)
In the absence of evidence to the contrary, assume that an accused is innocent (even if the probability of his innocence is very low).
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Example 12.5 (Probabilistic Default Reasoning)
An airplane has crashed in the mountains. A rescue group superimpose a grid on a map of the area and calculate, for each square in the grid, the probability that the plane has crashed in that square. First they search the area of the highest probability of the accident.
Definition 12.1 (Reiter, 1980) A **default** is an expression of the form

\[ \delta(x) = \frac{\alpha(x) : \beta_1(x), \ldots, \beta_n(x)}{\gamma(x)} \]

where \( \alpha(x), \beta_1(x), \ldots, \beta_n(x), \gamma(x) \) are first–order formulas with free variables \( x = (x_1, \ldots, x_k) \).

\( \alpha(x) \) – **prerequisite**
\( \beta_1(x), \ldots, \beta_n(x) \) – **justifications**
\( \gamma(x) \) – **consequent** of the default \( \delta(x) \).
A default

\[ \delta(x) = \frac{\alpha(x) : \beta_1(x), \ldots, \beta_n(x)}{\gamma(x)} \]

is intuitively read:

*For all individuals* \( x = (x_1, \ldots, x_k) \), *if* \( \alpha(x) \) *is believed and* \( \beta_1(x), \ldots, \beta_n(x) \) *are all consistent with what is believed, then* \( \gamma(x) \) *is to be believed.*
A default $\delta(x)$ is *open* iff it contains at least one free variable; otherwise it is called *closed*.

For example,

\[
\frac{\text{Bird}(x) : \text{CanFly}(x)}{\text{CanFly}(x)} \quad \text{open default}
\]

\[
\frac{\text{Bird(tweety) : CanFly(tweety)}}{\text{CanFly(tweety)}} \quad \text{closed default.}
\]

Open defaults represent universally quantified default rules. By substituting variables by well–founded terms (terms without variables) we obtain closed defaults being *instances* of open ones.
Example 12.2 (cont.) A typical bird can fly.

\[
\text{Bird}(x) : \text{CanFly}(x) \\
\underline{\text{CanFly}(x)}
\]
Example 12.2 (cont.) A typical bird can fly.

\[ \text{Bird}(x) : \text{CanFly}(x) \implies \text{CanFly}(x) \]

Example 12.3 (cont.) If there is no evidence on the contrary, assume that on Sunday the nearest shopping center is open.

\[ \text{Sunday} : \text{NearestIsOpen} \implies \text{NearestIsOpen} \]
Example 12.2 (cont.) A typical bird can fly.

\[
\text{Bird}(x) : \text{CanFly}(x) \\
\text{CanFly}(x)
\]

Example 12.3 (cont.) If there is no evidence on the contrary, assume that on Sunday the nearest shopping center is open.

\[
\text{Sunday} : \text{NearestIsOpen} \\
\text{NearestIsOpen}
\]

Example 12.4 (cont.) If there is no evidence on the contrary, assume that an accused is innocent.

\[
\text{Accused}(x) : \text{Innocent}(x) \\
\text{Innocent}(x)
\]
**Default Theory**

**Definition 12.2** By a *default theory* we mean a pair $T = (A, \Delta)$ where $A$ is the set of closed first–order formulas and $\Delta$ is the set of defaults.

Intuitively, $A$ represents the set of undeniable (known) fact, whereas $\Delta$ represent the set of default rules.

A default theory $T = (A, \Delta)$ is called

- **open** iff at least one default $\delta \in \Delta$ is open; otherwise it is called **closed**.
- **finite** iff $A$ is finite and $\Delta$ is finite.
We assume that all formulas in defaults do not contain existential quantifiers.
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Given an open default theory \((A, \Delta)\), we obtain its \textit{closure} by substituting all open defaults from \(\Delta\) by their instances \textit{wrt} all grounded terms.
We assume that all formulas in defaults do not contain existential quantifiers.

Given an open default theory \((A, \Delta)\), we obtain its *closure* by substituting all open defaults from \(\Delta\) by their instances wrt all grounded terms.

Closures of finite (open) default theories may be infinite – it happens when the set of grounded terms is infinite (when the language contains function symbols).
**Definition 12.3** Let $B$ be a set of closed first–order formulas (the set of beliefs) and let

$$\delta = \frac{\alpha : \beta_1, \ldots, \beta_n}{\gamma}$$

be a closed default. We say that $\delta$ is *applicable wrt* $B$ iff

- $\alpha \in B$
- $\neg \beta_1, \ldots, \beta_n \notin B$.

If $\delta$ is applicable wrt $B$, then after application of $\delta$ to $B$ we get

$$B' = B \cup \{\gamma\}$$
Example 12.6: Italians

Consider the following statements:

Marco, Carlo, and Giuseppe are Italians.
Typically, Italians are Catholics.

Representation in DL:

\[ A = \{ \text{Italian}(m) \land \text{Italian}(c) \land \text{Italian}(g) \} \]

\[ \Delta = \left\{ \frac{\text{Italian}(x) : \text{Catholic}(x)}{\text{Catholic}(x)} \right\} . \]

From \( T = (A, \Delta) \) we can infer that all three Italians are Catholics.
Example 12.6 (cont.)

Assume that we found out that Giuseppe is not a Catholic.

Representation in DL:

\[ A = \left\{ \begin{array}{l} \text{Italian}(m) \land \text{Italian}(c) \land \text{Italian}(g) \\ \neg \text{Catholic}(g) \end{array} \right\} \]

\[ \Delta = \left\{ \begin{array}{l} \text{Italian}(x) \text{Catholic}(x) \\ \text{Catholic}(x) \end{array} \right\} . \]

Now we have to reject our previous conclusion that Giuseppe is a Catholic.
Example 12.6 (cont.)

Assume now that we lean that *Marco is a communist*.

Representation in DL:

\[
A = \left\{ \begin{array}{l}
\text{Italian}(m) \land \text{Italian}(c) \land \text{Italian}(g) \\
\neg \text{Catholic}(g) \\
\text{Communist}(m)
\end{array} \right. \\
\Delta = \left\{ \frac{\text{Italian}(x) \land \text{Catholic}(x)}{\text{Catholic}(x)} \right\}.
\]

We can still derive that *Marco* is a Catholic.
Example 12.6 (cont.)

This problematic conclusion can be blocked in various ways.

(1) All communists are not Catholics.

Representation in DL:

\[
A_1 = \begin{cases} 
\text{Italian}(m) \land \text{Italian}(c) \land \text{Italian}(g) \\
\lnot \text{Catholic}(g) \\
\text{Communist}(m) \\
\forall x. \text{Communist}(x) \rightarrow \lnot \text{Catholic}(x) 
\end{cases}
\]

\[
\Delta = \left\{ \frac{\text{Italian}(x) : \text{Catholic}(x)}{\text{Catholic}(x)} \right\}.
\]

This way we can conclude that Marco is not a Catholic.
(2) Typically, an Italian is a Catholic unless he is a communist.

Representation in DL:

$$A = \left\{ \begin{array}{l}
\text{Italian}(x) \land \text{Italian}(c) \land \text{Italian}(g) \\
\neg \text{Catholic}(g) \\
\text{Communist}(m)
\end{array} \right\}$$

$$\Delta' = \left\{ \text{Italian}(x) : \frac{\text{Catholic}(x) \land \neg \text{Communist}(x)}{\text{Catholic}(x)} \right\}.$$ 

Now we can infer neither $\text{Catholic}(m)$ nor $\neg \text{Catholic}(m)$. 
(3) Typically, communists are not Catholics.

\[
A = \begin{cases} 
\text{Italian}(m) \land \text{Italian}(c) \land \text{Italian}(g) \\
\lnot\text{Catholic}(g) \land \text{Communist}(m)
\end{cases}
\]

\[
\Delta'' = \left\{ \frac{\text{Italian}(x) : \text{Catholic}(x)}{\text{Catholic}(x)}, \frac{\text{Communist}(x) : \lnot\text{Catholic}(x)}{\lnot\text{Catholic}(x)} \right\}
\]

Now there is a conflict:

- From the first default we get \text{Catholic}(m) (as an Italian)
- From the second default we get \lnot\text{Catholic}(m) (as a communist).
(3) Typically, communists are not Catholics.

\[
A = \begin{cases} 
\text{Italian}(m) \land \text{Italian}(c) \land \text{Italian}(g) \\
\neg \text{Catholic}(g) \land \text{Communist}(m) 
\end{cases}
\]

\[
\Delta'' = \begin{cases} 
\text{Italian}(x) : \text{Catholic}(x) \\
\text{Communist}(x) : \neg \text{Catholic}(x)
\end{cases}
\]

Now there is a conflict:

- From the first default we get \text{Catholic}(m) (as an Italian)
- From the second default we get \neg \text{Catholic}(m) (as a communist).

Both defaults cannot be applied simultaneously, but separately. Then there are two sets of beliefs, \(B_1\) and \(B_2\), such that \text{Catholic}(m) \in B_1 \text{ and } \neg \text{Catholic}(m) \in B_2.
Given a default theory $T = (A, \Delta)$, the set of all formulas obtained either by classical deduction or by applying defaults from $\Delta$ is called an extension of $T$. 
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*How to formalize this notion? What are actually conclusions inferred from $T$?*
Extensions (cont.)

Given $T = (A, \Delta)$, the natural requirements imposed on the extension $E(T)$ of $T$ are:

1. $A \subseteq E(T)$
2. $E = Th(E)$ (deductively closed)
3. $E(T)$ should contain all consequents of all applicable defaults, i.e. if

$$\frac{\alpha, \beta_1, \ldots, \beta_n}{\gamma} \in \Delta$$

- $\alpha \in E(T)$
- $\neg \beta_1, \ldots, \neg \beta_n \not\in E(T)$

then $\gamma \in E(T)$.

This conditions are called closure conditions.
Consider the sentences:

- Tweety is a bird.
- A typical bird can fly.

Representation in DL:

\[ A = \{ \text{Bird(tweety)} \} . \]
\[ \Delta = \left\{ \frac{\text{Bird(tweety)} : \text{CanFly(tweety)}}{\text{CanFly(tweety)}} \right\} . \]

Note that there are two sets satisfying closure conditions:

- \( E_1 = \text{Th}(\{ \text{Bird(tweety)}, \text{CanFly(tweety)} \}) \)
- \( E_2 = \text{Th}(\{ \text{Bird(tweety)}, \neg \text{CanFly(tweety)} \}) \),

but only the first one is intuitively justified as a set of beliefs.
Definition 12.4 Let $T = (A, \Delta)$ be a closed default theory and let $S$ be a set of closed formulas. By $\Gamma(S)$ denote the smallest set of formulas satisfying:

- $A \subseteq \Gamma(S)$
- $\Gamma(S) = Th(\Gamma(S))$
- if
  - $\alpha : \beta_1, \ldots, \beta_n \in \Delta$
  - $\gamma \in \Delta$
  - $\alpha \in \Gamma(S)$
  - $\neg \beta_1, \ldots, \neg \beta_n \notin S$

then $\gamma \in \Gamma(S)$.

The set $E$ of formulas is an extension of $T$ iff $E = \gamma(T)$. 
Example 12.7 (cont.)

\[
A = \{Bird(t)\}
\]
\[
\Delta = \left\{ \frac{Bird(t) : CanFly(t)}{CanFly(t)} \right\}.
\]

Consider \( E_1 = Th(\{Bird(t), CanFly(t)\}) \).

\[\Gamma(E_1) = Th(\{\ldots\}), \text{since } E = Th(E)\]
\[\Gamma(E_1) = Th(\{Bird(t), \ldots\}), \text{since } A \subseteq \Gamma(E)\]
\[\frac{Bird(t) : CanFly(t)}{CanFly(t)} \in \Delta, \quad Bird(t) \in \Gamma(E_1), \quad \neg CanFly(t) \notin E_1 \implies CanFly(t) \in \Gamma(E_1)\]

Since \( \Gamma(E_1) = Th(\{Bird(t), CanFly(t)\}) = E_1 \), this is indeed the extension of \((A, \Delta)\).
Example 12.7 (cont.)

\[
A = \{ Bird(t) \} \\
\Delta = \left\{ \frac{ Bird(t) : CanFly(t) }{ CanFly(t) } \right\}.
\]

Consider \( E_2 = Th(\{ Bird(t), \neg CanFly(t) \}) \).

- \( \Gamma(E_2) = Th(\{ Bird(t), \ldots \}) \)
- \( \neg CanFly(t) \in E_2 \)
- Since \( \Gamma(E_2) \) is the smallest set of formulas, \( CanFly(t) \notin \Gamma(E_2) \).

\( \Gamma(E_2) = Th(\{ Bird(t) \}) \neq E_2 \), so \( E_2 \) is not the extension of \((A, \Delta)\).
Example 12.7 (cont.)

\[ A = \{ Bird(t) \} \]
\[ \Delta = \left\{ \frac{Bird(t): CanFly(t)}{CanFly(t)} \right\}. \]

Consider \( E_2 = Th(\{ Bird(t), \neg CanFly(t) \}) \).

- \( \Gamma(E_2) = Th(\{ Bird(t), \ldots \}) \)
- \( \neg CanFly(t) \in E_2 \)
- Since \( \Gamma(E_2) \) is the smallest set of formulas, \( CanFly(t) \not\in \Gamma(E_2) \).

\( \Gamma(E_2) = Th(\{ Bird(t) \}) \neq E_2 \), so \( E_2 \) is not the extension of \((A, \Delta)\).

No other candidates for extensions exist, so \( E_1 \) is the only extension of \((A, \Delta)\).
Example 12.8: Nixon Diamond Problem

Consider the following sentences:

Typically, republicans are not pacifists.
Typically, quakers are pacifists.
Nixon is a quaker an a republican.

Denote:

\[ R(x) \quad – \quad x \text{ is a republican} \]
\[ Q(x) \quad – \quad x \text{ is a quaker} \]
\[ P(x) \quad – \quad x \text{ is a pacifist} \]
Representation in DL:

\[ A = \{ R(n) \land Q(n) \} \]

\[ \Delta = \left\{ \frac{R(x) : \neg P(x)}{-P(x)}, \frac{Q(x) : P(x)}{P(x)} \right\}. \]

We have two extensions of \((A, \Delta)\), \(E_1\) and \(E_2\), such that

\[ P(n) \in E_1 \]

\[ \neg P(n) \in E_2. \]
A = \{ R(n), Q(n) \}, \quad \Delta = \left\{ \delta_1 = \frac{R(n) : \neg P(n)}{\neg P(n)}, \quad \delta_2 = \frac{Q(n) : P(n)}{P(n)} \right\}. 

\( A = \{ R(n), Q(n) \} \), \( \Delta = \left\{ \delta_1 = \frac{R(n) : \neg P(n)}{-P(n)}, \delta_2 = \frac{Q(n) : P(n)}{P(n)} \right\} \).

Consider \( E_1 = Th(\{ R(n), Q(n), \neg P(n) \}) \). Take \( (\delta_1, \delta_2) \).
\[ A = \{ R(n), Q(n) \}, \quad \Delta = \left\{ \delta_1 = \frac{R(n) : -P(n)}{-P(n)}, \quad \delta_2 = \frac{Q(n) : P(n)}{P(n)} \right\}. \]

Consider \( E_1 = Th(\{ R(n), Q(n), -P(n) \}) \). Take \((\delta_1, \delta_2)\).

- \( \Gamma(E_1) = Th(\{ \ldots \}) \), since \( \Gamma(E) = Th(\Gamma(E)) \).
- \( \Gamma(E_1) = Th(\{ R(n), Q(n), \ldots \}) \), since \( A \subseteq \Gamma(E) \).
- \( \delta_1 \in \Delta, R(n) \in \Gamma(E_1), P(n) \not\in E_1 \implies -P(n) \in \Gamma(E_1) \).
- \( \delta_2 \in \Delta, Q(n) \in \Gamma(E_1), -P(n) \in E_1 \implies P(n) \not\in \Gamma(E_1) \) (minimality of \( \Gamma(E) \)).
\[ A = \{R(n), Q(n)\}, \quad \Delta = \left\{ \delta_1 = \frac{R(n) : \neg P(n)}{\neg P(n)}, \quad \delta_2 = \frac{Q(n) : P(n)}{P(n)} \right\}. \]

Consider \( E_1 = Th(\{R(n), Q(n), \neg P(n)\}) \). Take \((\delta_1, \delta_2)\).

- \( \Gamma(E_1) = Th(\{\ldots\}) \), since \( \Gamma(E) = Th(\Gamma(E)) \).
- \( \Gamma(E_1) = Th(\{R(n), Q(n), \ldots\}) \), since \( A \subseteq \Gamma(E) \).
- \( \delta_1 \in \Delta, R(n) \in \Gamma(E_1), P(n) \notin E_1 \implies \neg P(n) \in \Gamma(E_1) \).
- \( \delta_2 \in \Delta, Q(n) \in \Gamma(E_1), \neg P(n) \in E_1 \implies P(n) \notin \Gamma(E_1) \) (minimality of \( \Gamma(E) \)).

Hence \( \Gamma(E_1) = Th(\{R(n), Q(n), \neg P(n)\}) = E_1 \). Hence \( E_1 \) is an extension of \((A, \Delta)\).
A = \{R(n) , Q(n) \}, \quad \Delta = \left\{ \delta_1 = \frac{R(n) : \neg P(n)}{\neg P(n)}, \delta_2 = \frac{Q(n) : P(n)}{P(n)} \right\}
Nixon Diamond Problem (cont.)

\[ A = \{ R(n), Q(n) \}, \quad \Delta = \left\{ \delta_1 = \frac{R(n) : \neg P(n)}{\neg P(n)}, \quad \delta_2 = \frac{Q(n) : P(n)}{P(n)} \right\} \]

Consider \( E_2 = Th(\{ R(n), Q(n), P(n) \}) \). Take \( (\delta_2, \delta_1) \).
Nixon Diamond Problem (cont.)

\[ A = \{R(n), Q(n)\}, \quad \Delta = \left\{ \delta_1 = \frac{R(n) : \neg P(n)}{\neg P(n)}, \delta_2 = \frac{Q(n) : P(n)}{P(n)} \right\} \]

Consider \( E_2 = Th(\{R(n), Q(n), P(n)\}) \). Take \((\delta_2, \delta_1)\).

- \( \Gamma(E_2) = Th(\{\ldots\}) \), since \( \Gamma(E) = Th(\Gamma(E)) \).

- \( \Gamma(E_2) = Th(\{R(n), Q(n), \ldots\}) \), since \( A \subseteq \Gamma(E) \).

- \( \delta_2 \in \Delta, Q(n) \in \Gamma(E_2), \neg P(n) \notin E_2 \implies P(n) \in \Gamma(E_2) \).

- \( \delta_1 \in \Delta, R(n) \in \Gamma(E_2), P(n) \in E_2 \implies \neg P(n) \notin \Gamma(E_2) \) (minimality of \( \Gamma(E) \)).
Nixon Diamond Problem (cont.)

\[ A = \{ R(n), Q(n) \}, \quad \Delta = \left\{ \delta_1 = \frac{R(n) : \neg P(n)}{\neg P(n)}, \delta_2 = \frac{Q(n) : P(n)}{P(n)} \right\} \]

Consider \( E_2 = Th(\{ R(n), Q(n), P(n) \}) \). Take \((\delta_2, \delta_1)\).

- \( \Gamma(E_2) = Th(\{ \ldots \}) \), since \( \Gamma(E) = Th(\Gamma(E)) \).
- \( \Gamma(E_2) = Th(\{ R(n), Q(n), \ldots \}) \), since \( A \subseteq \Gamma(E) \).
- \( \delta_2 \in \Delta, Q(n) \in \Gamma(E_2), \neg P(n) \notin E_2 \implies P(n) \in \Gamma(E_2) \).
- \( \delta_1 \in \Delta, R(n) \in \Gamma(E_2), P(n) \in E_2 \implies \neg P(n) \notin \Gamma(E_2) \) (minimality of \( \Gamma(E) \)).

Hence \( \Gamma(E_2) = Th(\{ R(n), Q(n), P(n) \}) = E_2 \). So \( E_2 \) is an extension of \((A, \Delta)\).
\[ A = \{ r, q \}, \quad \Delta = \left\{ \delta_1 = \frac{r : \neg p}{p}, \quad \delta_2 = \frac{q : p}{p} \right\}. \]
Nixon Diamond Problem (cont.)

\[ A = \{ r, q \}, \quad \Delta = \left\{ \delta_1 = \frac{r : \neg p}{p}, \quad \delta_2 = \frac{q : p}{p} \right\} \]

Determine all models of \( A \):
\[ \mathcal{M}_0 = \{ \{ r, q, p \}, \{ r, q, \neg p \} \}. \]
Nixon Diamond Problem (cont.)

\[ A = \{r, q\}, \quad \Delta = \left\{ \delta_1 = \frac{r : \neg p}{p}, \quad \delta_2 = \frac{q : p}{p} \right\} . \]

Determine all models of \( A \): \( M_0 = \{\{r, q, p\}, \{r, q, \neg p\}\} \).

Application of \( \delta_1 \):

- \( r \) holds in all models from \( M_0 \)
- \( \neg p \) holds in at least one model from \( M_0 \)

\( \delta_1 \) is applicable wrt \( M_0 \), so \( M_1 = M_0 \cap Mod(\neg p) = \{\{r, q, \neg p\}\} \).
Nixon Diamond Problem (cont.)

\[ A = \{r, q\}, \quad \Delta = \left\{ \delta_1 = \frac{r : \neg p}{p}, \quad \delta_2 = \frac{q : p}{p} \right\} . \]

- Determine all models of \( A \): \( M_0 = \{ \{r, q, p\}, \{r, q, \neg p\} \} \).

- Application of \( \delta_1 \):
  - \( r \) holds in all models from \( M_0 \)
  - \( \neg p \) holds in at least one model from \( M_0 \)

\( \delta_1 \) is applicable wrt \( M_0 \), so \( M_1 = M_0 \cap Mod(\neg p) = \{ \{r, q, \neg p\} \} \).

- Application of \( \delta_2 \):
  - \( q \) holds in all models from \( M_1 \)
  - \( p \) does not hold in any model from \( M_1 \)

\( \delta_2 \) is not applicable wrt \( M_1 \), so \( M_2 = M_1 \).
Nixon Diamond Problem (cont.)

\[ A = \{r, q\}, \quad \Delta = \left\{ \delta_1 = \frac{r : \neg p}{p}, \quad \delta_2 = \frac{q : p}{p} \right\}. \]

- Determine all models of \( A \): \( \mathcal{M}_0 = \{\{r, q, p\}, \{r, q, \neg p\}\} \).

- Application of \( \delta_1 \):
  - \( r \) holds in all models from \( \mathcal{M}_0 \)
  - \( \neg p \) holds in at least one model from \( \mathcal{M}_0 \)

  \( \delta_1 \) is applicable wrt \( \mathcal{M}_0 \), so \( \mathcal{M}_1 = \mathcal{M}_0 \cap Mod(\neg p) = \{\{r, q, \neg p\}\} \).

- Application of \( \delta_2 \):
  - \( q \) holds in all models from \( \mathcal{M}_1 \)
  - \( p \) does not hold in any model from \( \mathcal{M}_1 \)

  \( \delta_2 \) is not applicable wrt \( \mathcal{M}_1 \), so \( \mathcal{M}_2 = \mathcal{M}_1 \).

- \( \mathcal{M} = \mathcal{M}_2 \) is the set of all models of the extension of \( (A, \Delta) \).
Nixon Diamond Problem (cont.)

\[ A = \{r, q\}, \quad \Delta = \left\{ \delta_1 = \frac{r : \neg p}{p}, \quad \delta_2 = \frac{q : p}{p} \right\}. \]
Nixon Diamond Problem (cont.)

\[ A = \{ r, q \}, \quad \Delta = \left\{ \delta_1 = \frac{r : \neg p}{p}, \quad \delta_2 = \frac{q : p}{p} \right\}. \]

Determine all models of \( A \): \( \mathcal{M}_0 = \{ \{ r, q, p \}, \{ r, q, \neg p \} \} \).
Nixon Diamond Problem (cont.)

\[ A = \{ r, q \}, \quad \Delta = \left\{ \delta_1 = \frac{r : \neg p}{p}, \delta_2 = \frac{q : p}{p} \right\} . \]

- Determine all models of \( A \): \( \mathcal{M}_0 = \{ \{ r, q, p \}, \{ r, q, \neg p \} \} \).

- Application of \( \delta_2 \):
  - \( q \) holds in all models from \( \mathcal{M}_0 \)
  - \( p \) holds in at least one model from \( \mathcal{M}_0 \)

  \( \delta_2 \) is applicable wrt \( \mathcal{M}_0 \), so \( \mathcal{M}_1 = \mathcal{M}_0 \cap Mod(p) = \{ \{ r, q, p \} \} \).
Nixon Diamond Problem (cont.)

\[ A = \{ r, q \}, \quad \Delta = \left\{ \delta_1 = \frac{r : \neg p}{p}, \quad \delta_2 = \frac{q : p}{p} \right\}. \]

- Determine all models of \( A \): \( \mathcal{M}_0 = \{ \{ r, q, p \}, \{ r, q, \neg p \} \} \).
- Application of \( \delta_2 \):
  - \( q \) holds in all models from \( \mathcal{M}_0 \)
  - \( p \) holds in at least one model from \( \mathcal{M}_0 \)
  \( \delta_2 \) is applicable wrt \( \mathcal{M}_0 \), so \( \mathcal{M}_1 = \mathcal{M}_0 \cap Mod(p) = \{ \{ r, q, p \} \} \).
- Application of \( \delta_1 \):
  - \( r \) holds in all models from \( \mathcal{M}_1 \)
  - \( \neg p \) does not hold in any model from \( \mathcal{M}_1 \)
  \( \delta_1 \) is not applicable wrt \( \mathcal{M}_1 \), so \( \mathcal{M}_2 = \mathcal{M}_1 \).
Nixon Diamond Problem (cont.)

\[ A = \{r, q\}, \quad \Delta = \left\{ \delta_1 = \frac{r : \neg p}{p}, \, \delta_2 = \frac{q : p}{p} \right\}. \]

- Determine all models of \( A \): \( \mathcal{M}_0 = \{\{r, q, p\}, \{r, q, \neg p\}\} \).

- Application of \( \delta_2 \):
  - \( q \) holds in all models from \( \mathcal{M}_0 \)
  - \( p \) holds in at least one model from \( \mathcal{M}_0 \)

  \( \delta_2 \) is applicable wrt \( \mathcal{M}_0 \), so \( \mathcal{M}_1 = \mathcal{M}_0 \cap Mod(p) = \{\{r, q, p\}\} \).

- Application of \( \delta_1 \):
  - \( r \) holds in all models from \( \mathcal{M}_1 \)
  - \( \neg p \) does not hold in any model from \( \mathcal{M}_1 \)

  \( \delta_1 \) is not applicable wrt \( \mathcal{M}_1 \), so \( \mathcal{M}_2 = \mathcal{M}_1 \).

- \( \mathcal{M} = \mathcal{M}_2 \) is the set of all models of the extension of \((A, \Delta)\).
Nixon Diamond Problem (cont.)

- Republican
- Pacifist
- Quaker
- Nixon
- Pacifist, ¬Pacifist

Diagram showing the relationships among Republican, Pacifist, Quaker, and Nixon, with a diamond structure.
Thank you for your attention!