Knowledge Representation and Reasoning

Lecture 13: Default Logics II

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**Default:** \[ \delta(x) = \frac{\alpha(x) : \beta_1(x), \ldots, \beta_n(x)}{\gamma(x)} \]

Also written \((\alpha(x) : \beta_1(x), \ldots, \beta_n(x)) / \gamma(x))\).
Recall...

- **Default:** $\delta(x) = \frac{\alpha(x) : \beta_1(x), \ldots, \beta_n(x)}{\gamma(x)}$

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- **Default theory:** a pair $T = (A, \Delta)$, where $A$ is the set of closed first-order (or propositional) formulas and $\Delta$ is the set of defaults.
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Closed defaults — all its formulas are closed (without variables).
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- A closed default \( \delta \) is *applicable* wrt the set \( B \) of formulas iff
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\[ \alpha \in B \]
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Closed defaults — all its formulas are closed (without variables).

A closed default \(\delta\) is **applicable** wrt the set \(B\) of formulas iff

- \(\alpha \in B\)
- \(\neg \beta_1, \ldots, \neg \beta_n \notin B\)
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A closed default \(\delta\) is **applicable** wrt the set \(B\) of formulas iff

- \(\alpha \in B\)
- \(\neg \beta_1, \ldots, \neg \beta_n \notin B\)

If \(\delta\) is applicable wrt \(B\), then after its application \(\gamma \in B\).
Intuitively, an extension of $T = (A, \Delta)$ is a set of formulas derivable from $A$ and all applicable defaults from $\Delta$. 
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Formally:
Let $T = (A, \Delta)$ be a closed let $S$ be a set of sentences. Let $\Gamma(S)$ be the smallest set of sentences satisfying
\[ A \subseteq \Gamma(S) \]
\[ \Gamma(S) = Th(\Gamma(S)) \]
\[ \text{If } (\alpha : \beta_1, \ldots, \beta_n) / \gamma \in \Delta, \alpha \in \Gamma(S), \neg \beta_1, \ldots, \neg \beta_n \notin S, \text{ then } \gamma \in \Gamma(S). \]
The set $E$ of sentences is an extension of $T$ iff $E = \Gamma(E)$. 
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Let $T = (A, \Delta)$ be a closed let $S$ be a set of sentences. Let $\Gamma(S)$ be the smallest set of sentences satisfying
- $A \subseteq \Gamma(S)$
- $\Gamma(S) = Th(\Gamma(S))$
- If $(\alpha : \beta_1, \ldots, \beta_n)/\gamma \in \Delta$, $\alpha \in \Gamma(S)$, $\neg \beta_1, \ldots, \neg \beta_n \notin S$, then $\gamma \in \Gamma(S')$.

The set $E$ of sentences is an extension of $T$ iff $E = \Gamma(E)$.
Some default theories may have no extensions!
Example 13.1

Consider a default theory: $A = \emptyset$, $\Delta = \left\{ \delta = \frac{p}{\neg p} \right\}$. 
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Put \( E_1 = Th(\{\neg p\}) \).

\[ \Gamma(E_1) = Th(\{\ldots\}). \]

\[ \delta \in \Delta, \ \top \in \Gamma(E_1), \ \neg p \in E_1 \implies \neg p \notin \Gamma(E_1). \]
Consider a default theory: \( A = \emptyset, \quad \Delta = \{ \delta = \frac{p}{\neg p} \} \).

Put \( E_1 = Th(\{ \neg p \}) \).

1. \( \Gamma(E_1) = Th(\{ \ldots \}) \).
2. \( \delta \in \Delta, \quad \top \in \Gamma(E_1), \quad \neg p \in E_1 \implies \neg p \notin \Gamma(E_1) \).

Hence \( \Gamma(E_1) = Th(\emptyset) \neq E_1 \), so \( E_1 \) is not an extension of \( (A, \Delta) \).
Example 13.1 (cont.)

\[ A = \emptyset, \quad \Delta = \left\{ \delta = \frac{p}{\neg p} \right\}. \]

Put \( E_2 = Th(\{p\}) \).
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Put \( E_2 = Th(\{p\}). \)

- \( \Gamma(E_2) = Th(\{\ldots\}). \)
- \( \delta \in \Delta, \top \in \Gamma(E_2), \neg p \not\in E_2 \implies \neg p \in \Gamma(E_2) \)
Example 13.1 (cont.)

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Put \( E_2 = Th(\{p\}) \).

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- \( \delta \in \Delta, \top \in \Gamma(E_2), \neg p \notin E_2 \implies \neg p \in \Gamma(E_2) \)

Hence \( \Gamma(E_2) = Th(\{\neg p\}) \neq E_2 \), so this is not an extension of \((A, \Delta)\).
Example 13.1 (cont.)

\[ A = \emptyset, \quad \Delta = \left\{ \delta = \frac{\vdash p}{\neg p} \right\}. \]

Put \( E_3 = Th(\emptyset) \) (the set of all tautologies).
Example 13.1 (cont.)

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Put \( E_3 = Th(\emptyset) \) (the set of all tautologies).

- \( \Gamma(E_3) = Th(\{\ldots\}) \).
- \( \delta \in \Delta, \quad \top \in \Gamma(E_3), \quad \neg p \notin E_3 \implies \neg p \in \Gamma(E_3) \).
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Hence \( \Gamma(E_3) = Th(\{\neg p\}) \neq E_3 \), so \( E_3 \) is not an extension of \((A, \Delta)\).
Example 13.1 (cont.)

\[ A = \emptyset, \quad \Delta = \left\lbrace \delta = \frac{p}{\neg p} \right\rbrace. \]

Put \( E_4 = \mathcal{L}. \)
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\[ A = \emptyset, \quad \Delta = \left\{ \delta = \frac{p}{\neg p} \right\}. \]

Put \( E_4 = \mathcal{L} \).

- \( \Gamma(E_4) = Th(\{\ldots\}) \).
- \( \delta \in \Delta, \quad \top \in \Gamma(E_4), \quad \neg p \in E_4 \implies \neg p \notin \Gamma(E_4) \).
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\[ A = \emptyset, \quad \Delta = \left\{ \delta = \frac{p}{\neg p} \right\}. \]

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\[ \Gamma(E_4) = Th(\{\ldots\}). \]

\[ \delta \in \Delta, \quad \top \in \Gamma(E_4), \quad \neg p \in E_4 \implies \neg p \not\in \Gamma(E_4). \]

Hence \( \Gamma(E_4) = Th(\emptyset) \neq E_4 - E_4 \) is not an extension of \((A, \Delta)\).
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- \( \Gamma(E_4) = Th(\{\ldots\}). \)
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Hence \( \Gamma(E_4) = Th(\emptyset) \neq E_4 - E_4 \) is not an extension of \((A, \Delta)\).

There is no other candidates for an extension of \((A, \Delta)\). \((A, \Delta)\) has NO extension!
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$(A, \Delta)$ has an inconsistent extension iff $A$ is inconsistent.
Basic properties of DL

- A default theory \((A, \Delta)\) may have exactly one extension, several (infinitely many!) extensions, or no extension.
- \((A, \Delta)\) has an inconsistent extension iff \(A\) is inconsistent.
- If \((A, \Delta)\) has an inconsistent extension, this is its only extension.
Basic properties of DL

- A default theory \((A, \Delta)\) may have exactly one extension, several (infinitely many!) extensions, or no extension.

- \((A, \Delta)\) has an inconsistent extension iff \(A\) is inconsistent.

- If \((A, \Delta)\) has an inconsistent extension, this is its only extension.

- If \(E\) and \(F\) are extensions of \((A, \Delta)\) and \(E \subseteq F\), then \(E = F\) (maximality of extensions).
Normal defaults

Normal default:

\[
\frac{\alpha(x) : \beta(x)}{\beta(x)}
\]
Normal defaults

- **Normal default**: 

  \[
  \frac{\alpha(x)}{\beta(x)} : \beta(x)
  \]

- **Normal default theory**: \((A, \Delta)\), where all defaults from \(\Delta\) are normal.
Normal default theories

Existence of extensions:

Every closed normal default theory has an extension.
Normal default theories

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  Every closed normal default theory has an extension.

- **Semi–monotonicity:**
  Let $T_1 = (A, \Delta_1)$ and $T_2 = (A, \Delta_2)$ be closed normal default theories such that $\Delta_1 \subseteq \Delta_2$ and let $E_1$ be the extension of $T_1$. Then $T_2$ has an extension $E_2$ such that $E_1 \subseteq E_2$.
  In other words, adding new defaults makes the set of beliefs larger.
Normal default theories

- **Existence of extensions:**
  Every closed normal default theory has an extension.

- **Semi-monotonicity:**
  Let \( T_1 = (A, \Delta_1) \) and \( T_2 = (A, \Delta_2) \) be closed normal default theories such that \( \Delta_1 \subseteq \Delta_2 \) and let \( E_1 \) be the extension of \( T_1 \). Then \( T_2 \) has an extension \( E_2 \) such that \( E_1 \subseteq E_2 \).
  In other words, adding new defaults makes the set of beliefs larger.

- **Orthogonality of extensions:**
  Let \( E \) and \( F \) be two different extensions of a closed normal default theory \( T \).
  Then \( E \cup F \) is inconsistent.
Representational Issues in DL
Example 13.2: Clever Students

Consider the following sentences:

- Students are usually clever.
- Clever people usually make good career.
- Peter is a student.
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Consider the following sentences:

- Students are usually clever.
- Clever people usually make good career.
- Peter is a student.

Representation in DL:

\[ A = \{ \text{Student}(\text{peter}) \} \]
\[ \Delta = \left\{ \frac{\text{Student}(x) \cdot \text{Clever}(x)}{\text{Clever}(x)} , \frac{\text{Clever}(x) \cdot \text{MakesCareer}(x)}{\text{MakesCareer}(x)} \right\}. \]
Example 13.2 (cont.)

The closure of $T = (A, \Delta)$:

$$A = \{ \text{Student}(\text{peter}) \}$$

$$\Delta = \left\{ \begin{array}{l}
\delta_1 = \frac{\text{Student}(\text{peter}) : \text{Clever}(\text{peter})}{\text{Clever}(\text{peter})}, \\
\delta_2 = \frac{\text{Clever}(\text{peter}) : \text{MakesCareer}(\text{peter})}{\text{MakesCareer}(\text{peter})}
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Example 13.2 (cont.)

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\delta_2 = \frac{\text{Clever}(peter) : \text{MakesCareer}(peter)}{\text{MakesCareer}(peter)} 
\end{array} \right\} .
\]

Application of \( \delta_1 \) gives \( \text{Clever}(peter) \). Then \( \delta_2 \) is applicable, after its application we get \( \text{MakesCareer}(peter) \).
Example 13.2 (cont.)

The closure of $T = (A, \Delta)$:

$$A = \{ \text{Student}(peter) \}$$

$$\Delta = \left\{ \begin{array}{l}
\delta_1 = \frac{\text{Student}(peter) : \text{Clever}(peter)}{\text{Clever}(peter)}, \\
\delta_2 = \frac{\text{Clever}(peter) : \text{MakesCareer}(peter)}{\text{MakesCareer}(peter)}
\end{array} \right\}.$$

Application of $\delta_1$ gives $\text{Clever}(peter)$. Then $\delta_2$ is applicable, after its application we get $\text{MakesCareer}(peter)$.

Here *transitivity of defaults* is desirable.
Example 3: Married Students

Consider the following sentences:

- Mike is a student.
- Students are usually adults.
- Adults are usually married.

Representation in DL:

\[ A = \{ Student(mike) \} \]

\[ \Delta = \left\{ \frac{Student(x) : Adult(x)}{Adult(x)} \right\}, \frac{Adult(x) : IsMarried(x)}{IsMarried(x)} \right\} \]
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Representation in DL:

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A = \{ \text{Student(mike)} \}
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One can easily note that we get \( \text{IsMarried}(mike) \). Moreover, for an arbitrary student we will obtain a similar conclusion!
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One can easily note that we get IsMarried(mike). Moreover, for an arbitrary student we will obtain a similar conclusion!

Here transitivity of defaults is not desirable!
Example 13.3 (cont.)

The previous problem results from the fact that

- although typical students are adults, but they are not the representative group of adults (i.e., only small part of adults are students)
- most adults are married.
In other to block undesirable conclusion, we can use a representation:

\[
\Delta = \left\{ \begin{array}{c}
\text{Student}(x) : \text{Adult}(x) \\
\text{Adult}(x) : \text{IsMarried}(x) \land \neg \text{Student}(x)
\end{array} \right. \\
\text{IsMarried}(x)
\right\}.
\]

In other words, we substitute the second rule by:

*Typical adults are married unless they are students.*

Only the first default is applicable and we get \text{Adult}(mike). The previous conclusion \text{IsMarried}(mike) cannot be derived anymore.
Example 13.4: Reasoning by Case

Consider the following sentences:

- Tweety is a bird or a bee.
- Typical birds can fly.
- Typical bees can fly.

Intuitively, we feel that Tweety can fly.
Example 13.4: Reasoning by Case

Consider the following sentences:

- **Tweety is a bird or a bee.**
- **Typical birds can fly.**
- **Typical bees can fly.**

Intuitively, we feel that **Tweety can fly.**

Representation in DL:

\[
A = \{ \text{Bird(tweety)} \lor \text{Bee(tweety)} \} \\
\Delta = \left\{ \frac{\text{Bird}(x) : \text{CanFly}(x)}{\text{CanFly}(x)}, \frac{\text{Bee}(x) : \text{CanFly}(x)}{\text{CanFly}(x)} \right\}
\]
Example 13.4: Reasoning by Case

Consider the following sentences:

- *Tweety* is a bird or a bee.
- Typical birds can fly.
- Typical bees can fly.

Intuitively, we feel that *Tweety* can fly.

Representation in DL:

\[
A = \{ Bird(tweety) \lor Bee(tweety) \} \\
\Delta = \left\{ \frac{Bird(x)}{CanFly(x)} , \frac{Bee(x)}{CanFly(x)} \right\}
\]

None of two defaults is applicable wrt \( A \), so \( E = Th(A) \). Consequently, we cannot derive a natural conclusion “*Tweety can fly*”. 
Example 13.4 (cont.)

We can use the following representation:

$$\Delta = \left\{ \frac{\text{Bird}(x) \to \text{CanFly}(x)}{\text{Bird}(x) \to \text{CanFly}(x)}, \frac{\text{Bee}(x) \to \text{CanFly}(x)}{\text{Bee}(x) \to \text{CanFly}(x)} \right\}$$
Example 13.4 (cont.)

We can use the following representation:

\[ \Delta = \left\{ \begin{array}{c} \frac{\text{Bird}(x) \rightarrow \text{CanFly}(x)}{\text{Bird}(x) \rightarrow \text{CanFly}(x)} \quad , \quad \frac{\text{Bee}(x) \rightarrow \text{CanFly}(x)}{\text{Bee}(x) \rightarrow \text{CanFly}(x)} \end{array} \right\} \]

Now we get

\[ \text{Bird(tweety)} \rightarrow \text{CanFly(tweety)} \]
\[ \text{Bee(tweety)} \rightarrow \text{CanFly(tweety)}, \]

or equivalently, \( \text{Bird(tweety)} \lor \text{Bee(tweety)} \rightarrow \text{CanFly(tweety)} \).

From A we immediately get: \( \text{CanFly(tweety)} \).
Example 13.5

Consider the following sentences:

- Teenagers are not adults.
- Students are usually adults.
- Peter is a teenager.

Intuitively, we want to conclude: Peter is not a student.

Representation in DL:

\[ A = \left\{ \begin{array}{l}
\text{Teenager}(peter) \\
\forall x. \text{Teenager}(x) \rightarrow \neg \text{Adult}(x)
\end{array} \right\} \]

\[ \Delta = \left\{ \delta(x) = \frac{\text{Student}(x) : \text{Adult}(x)}{\text{Adult}(x)} \right\} \]
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Representation in DL:

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\text{Teenager}(peter) \\
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\end{array} \right\}
\]

\[
\Delta = \left\{ \begin{array}{l}
\delta(x) = \frac{\text{Student}(x) \cdot \text{Adult}(x)}{\text{Adult}(x)}
\end{array} \right\}
\]

From \( A \) we conclude \( \neg \text{Adult}(peter) \). Since \( \delta(peter) \) is not applicable, we cannot derive the expected conclusion!
We can use another representation:

\[ A = \left\{ \begin{array}{l}
\text{Teenager}(peter) \\
\forall x. \text{Teenager}(x) \rightarrow \neg \text{Adult}(x)
\end{array} \right\} \]

\[ \Delta = \left\{ \begin{array}{l}
\delta(x) = \frac{\text{Student}(x) \rightarrow \text{Adult}(x)}{\text{Student}(x) \rightarrow \text{Adult}(x)}
\end{array} \right\} \]
Example 13.5 (cont.)

We can use another representation:

\[ A = \left\{ \begin{array}{l}
   \text{Teenager}(peter) \\
   \forall x. \text{Teenager}(x) \rightarrow \neg \text{Adult}(x)
\end{array} \right\} \]

\[ \Delta = \left\{ \begin{array}{l}
   \delta(x) = : \text{Student}(x) \rightarrow \text{Adult}(x) \\
   \text{Student}(x) \rightarrow \text{Adult}(x)
\end{array} \right\} \]

From \( A \) we get \( \neg \text{Adult}(peter) \).
\( \delta(peter) \) is applicable and after its application we obtain
\( \text{Student}(peter) \rightarrow \text{Adult}(peter) \).
Hence we get \( \neg \text{Student}(peter) \).
Example 13.6: Honest People

Consider the following sentences:

- Typically, people are honest.
- If we know that someone is honest, we can safely lend him money.
- John is a man.

Representation in DL:

\[ A = \{ \text{Man}(j) \} \]

\[ \Delta = \left\{ \frac{\text{Man}(x) : \text{Honest}(x)}{\text{Honest}(x)} , \frac{\text{Honest}(x) : \text{LendMoney}(x)}{\text{LendMoney}(x)} \right\}. \]
Example 13.6: Honest People

Consider the following sentences:

- Typically, people are honest.
- If we know that someone is honest, we can safely lend him money.
- John is a man.

Representation in DL:

\[ A = \{ \text{Man}(john) \} \]
\[ \Delta = \left\{ \frac{\text{Man}(x) : \text{Honest}(x)}{\text{Honest}(x)} , \frac{\text{Honest}(x) : \text{LendMoney}(x)}{\text{LendMoney}(x)} \right\} . \]

Note that we get \( \text{LendMoney}(john) \). Moreover, this conclusion is derivable for an arbitrary man!
However, the previously used method for blocking defaults cannot be applied here, i.e., the default

\[
\text{Honest}(x) : LendMoney \land \neg \text{Man}(x) \\
\frac{}{LendMoney(x)}
\]

makes no sense.
Example 13.6 (cont.)

However, the previously used method for blocking defaults cannot be applied here, i.e., the default

$$\begin{align*}
\text{Honest}(x) : \text{LendMoney} \land \neg \text{Man}(x) \\
\hline \\
\text{LendMoney}(x)
\end{align*}$$

makes no sense.

Problem:

*The prerequisite of the second default should not be a default conclusion, but rather an undeniable fact.*
Example 13.6 (cont.)

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\begin{align*}
\text{Honest}(x) : & \quad \text{LendMoney} \land \neg \text{Man}(x) \\
\hline
\text{LendMoney}(x)
\end{align*}
\]

makes no sense.

Problem:

*The prerequisite of the second default should not be a default conclusion, but rather an undeniable fact.*

This problem can be solved on the basis of three–valued logic!
Default rules (cont.)

Consider the following default rules:

(1) *Typically, if* \( \alpha \) *then* \( \beta \).

(2) *If* \( \alpha \) *then typically* \( \beta \).

They can be represented by the following defaults:

\[
\begin{align*}
(1) \quad & \frac{\alpha \rightarrow \beta}{\alpha} \\
(2) \quad & \frac{\alpha : \beta}{\beta}
\end{align*}
\]
Algorithm for computing extensions
Notational remarks

Notation:

- $\text{PERM}(\Delta)$ – the set of all permutations of defaults from $\Delta$.
- $\text{Mod}(\alpha)$ – the set of all models of $\alpha$.
- $M$ – the set of models (propositional, first–order).
- $\mathcal{M} = \{M_1, \ldots, M_m\}$ – the set of sets of models.
**Algorithm**

**INPUT:** a closed default theory $T = (A, \Delta)$

**OUTPUT:** the set $\mathcal{M} = \{M_1, \ldots, M_m\}$, where $M_i$ is the set of all models of $i$–th extension of $T$. 
Algorithm (cont.)

S1. \( P := PERM(\Delta); \)
Algorithm (cont.)

\textbf{S1.} \quad P := PERM(\Delta);

\textbf{S2.} \quad \text{If } P = \emptyset \text{ then STOP} \iff \mathcal{M} \text{ semantically corresponds to the set of all extensions of } T; \text{ otherwise } M := \text{Mod}(A); \quad J := \emptyset; \quad \mathcal{M} := \emptyset
Algorithm (cont.)

S1. \( P := PERM(\Delta); \)

S2. If \( P = \emptyset \) then \( STOP \) \( \Rightarrow \) \( \mathcal{M} \) semantically corresponds to the set of all extensions of \( T \); otherwise \( M := \text{Mod}(A); \) \( J := \emptyset; \) \( \mathcal{M} := \emptyset \)

S3. Take \( \delta = (\delta_1, \ldots, \delta_k) \in P, \) \( P := P \setminus \{\delta\}; \) \( i := 1 \)
Algorithm (cont.)

S1. \( P := PERM(\Delta); \)

S2. If \( P = \emptyset \) then STOP \( \Rightarrow \) \( M \) semantically corresponds to the set of all extensions of \( T \); otherwise \( M := \text{Mod}(A); \ J := \emptyset; \ M := \emptyset \)

S3. Take \( \delta = (\delta_1, \ldots, \delta_k) \in P, \ P := P \setminus \{\delta\}; \ i := 1 \)

S4. If \( i > k \) then \( M := M \cup \{M\}; \) go to S2; otherwise take \( \delta_i = (\alpha : \beta)/\gamma \) and put \( i := i + 1; \)
Algorithm (cont.)

S1. \( P := \text{PERM}(\Delta); \)

S2. If \( P = \emptyset \) then STOP \( \implies M \) semantically corresponds to the set of all extensions of \( T \); otherwise \( M := \text{Mod}(A); \ J := \emptyset; \ M := \emptyset \)

S3. Take \( \delta = (\delta_1, \ldots, \delta_k) \in P, \ P := P \setminus \{\delta\}; \ i := 1 \)

S4. If \( i > k \) then \( M := M \cup \{M\}; \) go to S2; otherwise take \( \delta_i = (\alpha : \beta)/\gamma \) and put \( i := i + 1; \)

S5 If

- \( M \subseteq \text{Mod}(\alpha) \)
- \( M \cap \text{Mod}(\beta) \neq \emptyset \)

then \( J := J \cup \{\beta\}; \ M := M \cap \text{Mod}(\gamma); \) go to S6; otherwise go to S4
Algorithm (cont.)

S1. \( P := P E R M(\Delta); \)

S2. If \( P = \emptyset \) then STOP \( \Rightarrow \mathcal{M} \) semantically corresponds to the set of all extensions of \( T \); otherwise \( M := \text{Mod}(A); \ J := \emptyset; \ \mathcal{M} := \emptyset \)

S3. Take \( \delta = (\delta_1, \ldots, \delta_k) \in P, \ P := P \setminus \{\delta\}; \ i := 1 \)

S4. If \( i > k \) then \( \mathcal{M} := \mathcal{M} \cup \{M\}; \) go to S2; otherwise take \( \delta_i = (\alpha : \beta)/\gamma \) and put \( i := i + 1; \)

S5 If
\begin{itemize}
  \item \( M \subseteq \text{Mod}(\alpha) \)
  \item \( M \cap \text{Mod}(\beta) \neq \emptyset \)
\end{itemize}
then \( J := J \cup \{\beta\}; \ M := M \cap \text{Mod}(\gamma); \) go to S6; otherwise go to S4

S6 If \( J \) is consistent and \( M \neq \emptyset \), then go to S4; otherwise go to S2.
Example 13.7

Consider the following story:

- *On Sundays* Bill usually goes fishing except when he is tired.
- *If Bill worked hard the day before, then he is usually tired except when he woke up late.*
- *If Bill is on vacations, then he usually wakes up late except when he goes fishing.*
- *Today is Sunday, Bill worked hard yesterday, and he has vacations.*
Example 13.7

Consider the following story:

- **On Sundays** Bill usually goes fishing except when he is tired.
- **If Bill worked hard the day before, then he is usually tired except when he woke up late.**
- **If Bill is on vacations, then he usually wakes up late except when he goes fishing.**
- **Today is Sunday, Bill worked hard yesterday, and he has vacations.**

Intuitively, we should conclude that either

- **Bill went fishing,** or
- **Bill was tired,** or
- **Bill woke up late.**

but none pair of these conclusions can be inferred.
Example 13.7 (cont.)

Denote

\[ s \] – Today is Sunday. \[ v \] – Bill is on vacations.
\[ w \] – Bill worked hard yesterday. \[ g \] – Bill goes fishing.
\[ l \] – Bill wakes up late. \[ t \] – Bill is tired.

Representation in DL:

\[ A = \{ s, w, v \} \]
\[ \Delta = \left\{ \delta_1 = \frac{s : g \land \lnot t}{g}, \delta_2 = \frac{w : t \land \lnot l}{t}, \delta_3 = \frac{v : l \land \lnot g}{l} \right\}. \]
Example 13.7 (cont.)

\[ A = \{ s, w, v \} \]

\[ \Delta = \left\{ \delta_1 = \frac{s : g \land \neg t}{g}, \quad \delta_2 = \frac{w : t \land \neg l}{t}, \quad \delta_3 = \frac{v : l \land \neg g}{l} \right\}. \]
Example 13.7 (cont.)

\[ A = \{ s, w, v \} \]
\[ \Delta = \left\{ \delta_1 = \frac{s \cdot g \land \neg t}{g}, \ \delta_2 = \frac{w \cdot t \land \neg l}{t}, \ \delta_3 = \frac{v \cdot l \land \neg g}{l} \right\}. \]

Take \((\delta_1, \delta_2, \delta_3)\). \(\delta_1\) is applicable, so

\[
M_1 = \left\{ \begin{array}{l}
\{s, w, v, g, l, t\} \\
\{s, w, v, g, l, \neg t\} \\
\{s, w, v, g, \neg l, t\} \\
\{s, w, v, g, \neg l, \neg t\} 
\end{array} \right\}, \quad J_1 = \{g \land \neg t\}, \quad B_1 = Th(A \cup \{g\}).\]
Example 13.7 (cont.)

\[ A = \{s, w, v\} \]

\[ \Delta = \left\{ \delta_1 = \frac{s : g \land \neg t}{g}, \delta_2 = \frac{w : t \land \neg l}{t}, \delta_3 = \frac{v : l \land \neg g}{l} \right\} \]

Take \((\delta_1, \delta_2, \delta_3)\). \(\delta_1\) is applicable, so

\[ M_1 = \left\{ \begin{array}{l} \{s, w, v, g, l, t\} \\ \{s, w, v, g, l, \neg t\} \\ \{s, w, v, g, \neg l, t\} \\ \{s, w, v, g, \neg l, \neg t\} \end{array} \right\}, \quad J_1 = \{g \land \neg t\}, \quad B_1 = Th(A \cup \{g\}). \]

\(\delta_2\) is applicable, so

\[ M_2 = \left\{ \begin{array}{l} \{s, w, v, g, t, l\} \\ \{s, w, v, g, t, \neg l\} \end{array} \right\}, \quad J_2 = \{g \land \neg t, t \land \neg l\}, \quad B_2 = Th(A \cup \{g, t\}). \]

\(J_2\) is inconsistent!
Example 13.7 (cont.)

\[ A = \{ s, w, v \} \]

\[ \Delta = \left\{ \delta_1 = \frac{s}{g} : g \wedge \neg t, \delta_2 = \frac{w}{t} : t \wedge \neg l, \delta_3 = \frac{v}{l} : l \wedge \neg g \right\}. \]

Take \((\delta_1, \delta_3, \delta_2)\). \(\delta_1\) is applicable, so

\[ M_1 = \begin{cases} 
\{ s, w, v, g, l, t \} \\
\{ s, w, v, g, l, \neg t \} \\
\{ s, w, v, g, \neg l, t \} \\
\{ s, w, v, g, \neg l, \neg t \} 
\end{cases}, \quad J_1 = \{ g \wedge \neg t \}, \quad B_1 = Th(A \cup \{ g \}). \]
Example 13.7 (cont.)

\[ A = \{ s, w, v \} \]

\[ \Delta = \left\{ \delta_1 = \frac{s : g \land \neg t}{g}, \delta_2 = \frac{w : t \land \neg l}{t}, \delta_3 = \frac{v : l \land \neg g}{l} \right\} \]

Take \((\delta_1, \delta_3, \delta_2)\). \(\delta_1\) is applicable, so

\[ M_1 = \left\{ \begin{array}{l}
\{ s, w, v, g, l, t \} \\
\{ s, w, v, g, l, \neg t \} \\
\{ s, w, v, g, \neg l, t \} \\
\{ s, w, v, g, \neg l, \neg t \}
\end{array} \right\}, \quad J_1 = \{ g \land \neg t \}, \quad B_1 = Th(A \cup \{ g \}). \]

\(\delta_3\) is not applicable, so \(M_2 = M_1, \ J_2 = J_1, \ B_2 = B_1.\)
Example 13.7 (cont.)

\[ A = \{s, w, v\} \]
\[ \Delta = \{\delta_1 = \frac{s \land \neg t}{g}, \delta_2 = \frac{w \land \neg l}{t}, \delta_3 = \frac{v \land \neg g}{l}\} \]

Take \((\delta_1, \delta_3, \delta_2)\). \(\delta_1\) is applicable, so

\[
M_1 = \begin{cases} 
\{s, w, v, g, l, t\} \\
\{s, w, v, g, l, \neg t\} \\
\{s, w, v, g, \neg l, t\} \\
\{s, w, v, g, \neg l, \neg t\}
\end{cases}, \quad J_1 = \{g \land \neg t\}, \quad B_1 = Th(A \cup \{g\})
\]

\(\delta_3\) is not applicable, so \(M_2 = M_1\), \(J_2 = J_1\), \(B_2 = B_1\).

\(\delta_2\) is applicable, so

\[
M_3 = \begin{cases} 
\{s, w, v, g, t\} \\
\{s, w, v, g, t, \neg l\}
\end{cases}, \quad J_3 = \{g \land \neg t, t \land \neg l\}, \quad B_3 = Th(A \cup \{g, t\})
\]
Example 13.7 (cont.)

\[ A = \{s, w, v\} \]
\[ \Delta = \left\{ \delta_1 = \frac{s : g \land \neg t}{g}, \delta_2 = \frac{w : t \land \neg l}{t}, \delta_3 = \frac{v : l \land \neg g}{l} \right\}. \]

Take \((\delta_2, \delta_1, \delta_3)\). \(\delta_2\) is applicable, so

\[ M_1 = \left\{ \begin{array}{c} \{s, w, v, t, g, l\} \\ \{s, w, v, t, g, \neg l\} \\ \{s, w, v, t, \neg g, l\} \\ \{s, w, v, t, \neg g, \neg l\} \end{array} \right\}, \quad J_1 = \{t \land \neg l\}, \quad B_1 = Th(A \cup \{t\}). \]
Example 13.7 (cont.)

\[ A = \{s, w, v\} \]
\[ \Delta = \left\{ \delta_1 = \frac{s : g \land \neg t}{g}, \delta_2 = \frac{w : t \land \neg l}{t}, \delta_3 = \frac{v : l \land \neg g}{l} \right\}. \]

Take \((\delta_2, \delta_1, \delta_3)\). \(\delta_2\) is applicable, so

\[ M_1 = \left\{ \begin{array}{l}
\{s, w, v, t, g, l\} \\
\{s, w, v, t, g, \neg l\} \\
\{s, w, v, t, \neg g, l\} \\
\{s, w, v, t, \neg g, \neg l\}
\end{array} \right\}, \quad J_1 = \{t \land \neg l\}, \quad B_1 = Th(A \cup \{t\}). \]

\(\delta_1\) is not applicable, so \(M_2 = M_1, J_2 = J_1, B_2 = B_1\).
Example 13.7 (cont.)

\[ A = \{s, w, v\} \]
\[ \Delta = \left\{ \delta_1 = \frac{s : g \land \neg t}{g}, \delta_2 = \frac{w : t \land \neg l}{t}, \delta_3 = \frac{v : l \land \neg g}{l} \right\} \]

Take \((\delta_2, \delta_1, \delta_3)\). \(\delta_2\) is applicable, so

\[ M_1 = \left\{ \begin{array}{ll}
\{s, w, v, t, g, l\} \\
\{s, w, v, t, g, \neg l\} \\
\{s, w, v, t, \neg g, l\} \\
\{s, w, v, t, \neg g, \neg l\}
\end{array} \right\}, \quad J_1 = \{t \land \neg l\}, \quad B_1 = Th(A \cup \{t\}) \]

\(\delta_1\) is not applicable, so \(M_2 = M_1, J_2 = J_1, B_2 = B_1\).

\(\delta_3\) is applicable, so

\[ M_3 = \left\{ \begin{array}{ll}
\{s, w, v, t, l, g\} \\
\{s, w, v, t, l, \neg g\}
\end{array} \right\}, \quad J_3 = \{t \land \neg l, l \land \neg g\}, \quad B_3 = Th(A \cup \{t, l\}) \]
Example 13.7 (cont.)

\[ A = \{s, w, v\} \]

\[ \Delta = \left\{ \delta_1 = \frac{s : g \land \neg t}{g}, \delta_2 = \frac{w : t \land \neg l}{t}, \delta_3 = \frac{v : l \land \neg g}{l} \right\} \]

Take \((\delta_2, \delta_3, \delta_1)\). \(\delta_2\) is applicable, so

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\end{array} \right\}, \quad J_1 = \{t \land \neg l\}, \quad B_1 = Th(A \cup \{t\}). \]
Example 13.7 (cont.)

\[ A = \{s, w, v\} \]
\[ \Delta = \left\{ \delta_1 = \frac{s \cdot g \land \neg t}{g}, \delta_2 = \frac{w \cdot t \land \neg l}{t}, \delta_3 = \frac{v \cdot l \land \neg g}{l} \right\}. \]

Take \((\delta_2, \delta_3, \delta_1)\). \(\delta_2\) is applicable, so

\[ M_1 = \left\{ \begin{align*}
\{s, w, v, t, g, l\} \\
\{s, w, v, t, g, \neg l\} \\
\{s, w, v, t, \neg g, l\} \\
\{s, w, v, t, \neg g, \neg l\}
\end{align*} \right\}, \quad J_1 = \{t \land \neg l\}, \ B_1 = Th(A \cup \{t\}). \]

\(\delta_3\) is applicable, so

\[ M_2 = \left\{ \begin{align*}
\{s, w, v, t, l, g\} \\
\{s, w, v, t, l, \neg g\}
\end{align*} \right\}, \quad J_2 = \{t \land \neg l, l \land \neg g\}, \ B_2 = Th(A \cup \{t, l\}). \]
Example 13.7 (cont.)

\[ A = \{s, w, v\} \]

\[ \Delta = \left\{ \delta_1 = \frac{s : g \land \neg t}{g}, \delta_2 = \frac{w : t \land \neg l}{t}, \delta_3 = \frac{v : l \land \neg g}{l} \right\}. \]

Take (\( \delta_3, \delta_1, \delta_2 \)). \( \delta_3 \) is applicable, so

\[ M_1 = \left\{ \begin{array}{c}
\{s, w, v, l, t, g\} \\
\{s, w, v, l, t, \neg g\} \\
\{s, w, v, l, \neg t, g\} \\
\{s, w, v, l, \neg t, \neg g\}
\end{array} \right\}, \quad J_1 = \{l \land \neg g\}, \quad B_1 = Th(A \cup \{l\}). \]
Example 13.7 (cont.)

\[ A = \{ s, w, v \} \]

\[ \Delta = \left\{ \begin{array}{ll}
\delta_1 = & s : g \land \neg t \\
\delta_2 = & w : t \land \neg l \\
\delta_3 = & v : l \land \neg g 
\end{array} \right\}. \]

Take (\(\delta_3, \delta_1, \delta_2\)). \(\delta_3\) is applicable, so

\[ M_1 = \begin{cases} 
\{ s, w, v, l, t, g \} \\
\{ s, w, v, l, t, \neg g \} \\
\{ s, w, v, l, t, \neg g \} \\
\{ s, w, v, l, \neg t, \neg g \} 
\end{cases} \]

\[ J_1 = \{ l \land \neg g \}, \quad B_1 = Th(A \cup \{ l \}). \]

\(\delta_1\) is applicable, so

\[ M_2 = \begin{cases} 
\{ s, w, v, l, g, t \} \\
\{ s, w, v, l, g, \neg t \} 
\end{cases} \]

\[ J_2 = \{ l \land \neg g, g \land \neg t \}, \quad B_2 = Th(\{ l, g \}). \]
Example 13.7 (cont.)

\[ A = \{ s, w, v \} \]
\[ \Delta = \left\{ \delta_1 = \frac{s : g \land \neg t}{g}, \quad \delta_2 = \frac{w : t \land \neg l}{t}, \quad \delta_3 = \frac{v : l \land \neg g}{l} \right\}. \]

Take \((\delta_3, \delta_2, \delta_1)\). \(\delta_3\) is applicable, so

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\{ s, w, v, l, \neg t, \neg g \} 
\end{array} \right\}, \quad J_1 = \{ l \land \neg g \}, \quad B_1 = Th(A \cup \{l\}). \]
Example 13.7 (cont.)

\[A = \{s, w, v\}\]

\[\Delta = \left\{ \delta_1 = \frac{s : g \land \neg t}{g}, \delta_2 = \frac{w : t \land \neg l}{t}, \delta_3 = \frac{v : l \land \neg g}{l} \right\}.\]

Take \((\delta_3, \delta_2, \delta_1)\). \(\delta_3\) is applicable, so

\[M_1 = \left\{ \begin{array}{l}
\{s, w, v, l, t, g\} \\
\{s, w, v, l, t, \neg g\} \\
\{s, w, v, l, \neg t, g\} \\
\{s, w, v, l, \neg t, \neg g\}
\end{array} \right\}, \quad J_1 = \{l \land \neg g\}, \quad B_1 = Th(A \cup \{l\}).\]

\(\delta_2\) is not applicable, so \(M_2 = M_1, J_2 = J_1, B_2 = B_1.\)
Example 13.7 (cont.)

\[ A = \{s, w, v\} \]
\[ \Delta = \left\{ \delta_1 = \frac{s : g \land \neg t}{g}, \quad \delta_2 = \frac{w : t \land \neg l}{t}, \quad \delta_3 = \frac{v : l \land \neg g}{l} \right\} \]

Take \((\delta_3, \delta_2, \delta_1)\). \(\delta_3\) is applicable, so

\[ M_1 = \left\{ \begin{array}{l}
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\{s, w, v, l, t, \neg g\} \\
\{s, w, v, l, \neg t, g\} \\
\{s, w, v, l, \neg t, \neg g\}
\end{array} \right\}, \quad J_1 = \{l \land \neg g\}, \quad B_1 = \text{Th}(A \cup \{l\}). \]

\(\delta_2\) is not applicable, so \(M_2 = M_1, \quad J_2 = J_1, \quad B_2 = B_1.\)

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\[ M_3 = \left\{ \begin{array}{l}
\{s, w, v, l, g, t\} \\
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Some default theories have no extensions, because the criterion of defaults’ applicability is too weak — it enforces the default to be applicable, but after its application the default is not applicable.
Problems with DL

Some default theories have no extensions, because the criterion of defaults’ applicability is too weak — it enforces the default to be applicable, but after its application the default is not applicable.

The consequent of the default, together with axioms and consequents of previously applied defaults, contradicts some of its own justifications. For example,

\[ A = \emptyset, \quad \Delta = \left\{ \frac{p}{\neg p} \right\}. \]
The consequent of the default, together with axioms and consequents of previously applied defaults, denies some justification of previously applied defaults.

For example,

\[ A = \emptyset, \quad \Delta = \left\{ \frac{p \land r}{p}, \frac{p}{\neg r} \right\}. \]
The consequent of the default, together with axioms and consequents of previously applied defaults, denies some justification of previously applied defaults.

For example,

\[ A = \emptyset, \quad \Delta = \left\{ \frac{p \land r}{p}, \frac{p}{\neg r} \right\}. \]

The consequent of the default contradicts some sentence derivable from axioms and consequents of already applied defaults.

For example,

\[ A = \{p \rightarrow q\}, \quad \Delta = \left\{ \frac{p}{p}, \frac{p}{\neg q} \right\}. \]
Thank you for your attention!