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A Martingale Characterization of the Poisson Process

by

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Summary. The Poisson process is characterized by the martingale property of certain polynomials of the process of order up to three.

The well-known characterization of the Poisson process in [4] states that if $(X_t - t, F_t)_{t\geqslant 0}$ is a purely discontinuous martingale all of whose jumps equal $+1$ and $((X_t - t)^2 - t, F_t)_{t\geqslant 0}$ is also a martingale then $(X_t)_{t\geqslant 0}$ is a Poisson process. In the same paper Meyer proved that if $(X_t)_{t\geqslant 0}$ is a purely discontinuous process whose all jumps equal $+1$ and $(X_t - t, F_t)_{t\geqslant 0}$ is a martingale then $(X_t)_{t\geqslant 0}$ is also a Poisson process. In the both results mentioned above quite strong assumptions were imposed directly on the trajectories of the process. In our note we prove that these assumptions may be replaced by taking into account the conditional moments of the order three of the process. We identify processes with respect to their finite dimensional distributions.

We consider a process $X = (X_t)_{t\geqslant 0}$ adapted to a nondecreasing family of σ-fields $(F_t)_{t\geqslant 0}$. Let us define a process $Y = (Y_t)_{t\geqslant 0}$ by the equation $Y_t = X_t - t$, $t \geqslant 0$. Our main result is the following

THEOREM 1. *Let*

$$(Y_t, F_t)_{t\geqslant 0},$$
$$(Y_t^2 - t, F_t)_{t\geqslant 0},$$
$$(Y_t^3 - 3tY_t - t, F_t)_{t\geqslant 0}$$

be martingales. If X is a non-decreasing process then it is a Poisson process.

Before we give the proof of our theorem let us show how it may be applied to obtain a slight extension of a characterization of the Poisson process by conditional moments obtained in [3].

THEOREM 2. *Let $X = (X_t)_{t\geqslant 0}$ be a square integrable process such that for any $0 \leqslant r_1 \leqslant \ldots \leqslant r_n \leqslant r < s < t$, $n \geqslant 1$*

$$E(X_s \mid X_{r_1}, \ldots, X_{r_n}, X_r) = X_r + s - r\,, \tag{1}$$

$$E(X_s \mid X_{r_1}, \ldots, X_{r_n}, X_r, X_t) = (t-r)^{-1}((t-s)X_r + (s-r)X_t)\,, \tag{2}$$

$$Var(X_s \mid X_{r_1}, \ldots, X_{r_n}, X_r, X_t) = (t-r)^{-2}(t-s)(s-r)(X_t - X_r)\,. \tag{3}$$

Then X is a Poisson process.

In [3], additionally, the following condition was assumed:

$$\mathrm{Var}(X_s \mid X_{r_1}, \ldots, X_{r_n}, X_r) = s - r\,. \tag{4}$$

The technical core of the proof lied in counting all the conditional moments of the process and it was quite complicated.

Proof of Theorem 2. We prove that the assumptions of Theorem 1 are fulfilled. From (1) it follows that (Y_t, F_t) is a martingale, where $F_t = \sigma(X_s, s \leqslant t)$.

Let us denote $Z = (X_{r_1}, \ldots, X_{r_n})$ for any $r_1, \ldots, r_n$. From (1) and (2) we get

$$\begin{aligned} E(X_s^2 \mid Z, X_r) &= E(E(X_s^2 \mid Z, X_r, X_t) \mid Z, X_r) \\ &= (1 - ((s-r)/(t-r))^2)X_r^2 \\ &\quad + [(t-s)(s-r)/(t-r)](2X_r + 1) \\ &\quad + ((s-r)/(t-r))^2 E(X_t^2 \mid Z, X_r)\,. \end{aligned} \tag{5}$$

Since

$$\begin{aligned} E(X_s X_t \mid Z, X_r) &= E(E(X_s \mid Z, X_r, X_t)X_t \mid Z, X_r) \\ &= E(X_s E(X_t \mid Z, X_r, X_s) \mid Z, X_r)\,, \end{aligned}$$

then from (1) and (2) we get

$$\begin{aligned} E(X_s^2 \mid Z, X_r) = [(t-s)/(t-r)]X_r^2 - (t-s)(s-r) \\ +[(s-r)/(t-r)]E(X_t^2 \mid Z, X_r)\,. \end{aligned} \tag{6}$$

The equations (5) and (6) imply

$$E(X_s^2 \mid Z, X_r) = X_r^2 + 2(s-r)X_r + (s-r)^2 + s - r\,. \tag{7}$$

Consequently $((X_t - t)^2 - t, F_t)_{t\geqslant 0}$ is a martingale.

From Corollary 2 in [2] it follows that $E(X_t^3)$ exists for any $t \geqslant 0$. Now we apply (1)–(3) and (7) to $E(X_s^2 X_t \mid Z, X_r)$ and $E(X_s X_t^2 \mid Z, X_r)$ and obtain two linear equations on $E(X_s^3 \mid Z, X_r)$ and $E(X_t^3 \mid Z, X_r)$. Easy computations lead to the formula:

$$E(X_s^3 \mid Z, X_r) = (X_r + s - r)^3 + 3(s-r)(X_r + s - r) + s - r\,.$$

And thus

$$((X_t - t)^3 - 3t(X_t - t) - t, F_t)_{t \geqslant 0}$$

is also a martingale. The condition (3) implies that the process is non-decreasing. Consequently, it follows from Theorem 1 that X is a Poisson process.

From Theorem 1 we obtain the following easy characterization of the Poisson process in the class of processes with independent increments:

COROLLARY. *Let $X = (X_t)_{t\geqslant 0}$ be a non-decreasing process with independent increments such that:*

$$EX_t = t, \quad EX_t^2 = t^2 + t, \quad EX_t^3 = t^3 + 3t^2 + t, \quad t \geqslant 0.$$

Then X is a Poisson process.

Proof of Theorem 1. Let us consider a sequence of divisions of the interval $\langle s, t\rangle$ for any $s < t$:

$$s = t_{n0} < t_{n1} < \ldots < t_{nn} = t$$

such that

$$\lim_n \max\{t_{nj} - t_{n,j-1},\, j = 1, \ldots, n\} = 0\,.$$

We define double sequences of random variables $\{Y_{nk}, k = 1, \ldots, n; n \geqslant 1\}$ and of σ-fields $\{F_{nk}, k = 1, \ldots, n; n \geqslant 1\}$ by the equations:

$$Y_{nk} = X_{t_{nk}} - X_{t_{n,k-1}}, \quad F_{nk} = F_{t_{nk}}, \quad k = 0, 1, \ldots, n; \quad n \geqslant 1\,.$$

Now we apply the following version of the limit theorem obtained in ([1] – see Corollary 5):

THEOREM [1]. *Let $\{Y_{nk}, k = 1, \ldots, n; n \geqslant 1\}$ be a double sequence of non-negative random variables adapted to a row-wise increasing sequence of σ-fields $\{F_{nk}, k = 1, \ldots, n; n \geqslant 1\}$ and $F_{n0} \subseteq F_{n1}, n \geqslant 1$, satisfying the conditions:*

(8) $$\max_{1\leqslant k\leqslant n} E(Y_{nk} \mid F_{n,k-1}) \xrightarrow{P} 0\,,$$

(9) $$\sum_{k=1}^{n} E(Y_{nk} \mid F_{n,k-1}) \xrightarrow{P} a > 0\,,$$

(10) $$\sum_{k=1}^{n} E(Y_{nk} I(|Y_{nk} - 1| > b \mid F_{n,k-1}) \xrightarrow{P} 0, \quad b > 0\,.$$

Then the conditional distributions of the sums $S_n = \sum_{k=1}^{n} Y_{nk}$ conditioned in respect to F_{n0} are weakly convergent to the Poisson distribution with the parameter a.

In our case $S_n = X_t - X_s$. From (1) we get

$$E(Y_{nk} \mid F_{n,k-1}) = t_{nk} - t_{n,k-1}\,. \tag{11}$$

Consequently, the conditions (8) and (9) with $a = t - s$ are fulfilled. On the other hand (2) and (3) imply

$$E(Y_{nk}^2 \mid F_{n,k-1}) = (t_{nk} - t_{n,k-1})^2 + t_{nk} - t_{n,k-1}\,, \tag{12}$$

$$E(Y_{nk}^3 \mid F_{n,k-1}) = (t_{nk} - t_{n,k-1})^3 + 3(t_{nk} - t_{n,k-1})^2 + t_{nk} - t_{n,k-1}\,. \tag{13}$$

Applying the formulas (11)–(13) to the inequality

$$\sum_{k=1}^{n} E(Y_{nk} I(|Y_{nk} - 1| > b) \mid F_{n,k-1}) \leqslant b^{-2} \sum_{k=1}^{n} E(Y_{nk}(Y_{nk} - 1)^2 \mid F_{n,k-1})$$

we get

$$\sum_{k=1}^{n} E(Y_{nk} I(|Y_{nk} - 1| > b \mid F_{n,k-1})$$

$$\leqslant b^{-2} \sum_{k=1}^{n} [(t_{nk} - t_{n,k-1})^3 + (t_{nk} - t_{n,k-1})^2]$$

$$\leqslant 2b^{-2} \max_{1 \leqslant k \leqslant n} (t_{nk} - t_{n,k-1})(t - s)\,.$$

Consequently the condition (10) is also fulfilled and thus the X is a Poisson process.

REMARK. The similar characterization of the Wiener process may be formulated as follows

THEOREM 3. *Let $X = (X_t)_{t \geqslant 0}$ be a process such that*

$$(X_t, F_t)_{t \geqslant 0}\,, \tag{14}$$

$$(X_t^2 - t, F_t)_{t \geqslant 0}\,, \tag{15}$$

$$(X_t^3 - 3tX_t, F_t)_{t \geqslant 0}\,, \tag{16}$$

$$(X_t^4 - 6tX_t^2 + 3t^2, F_t)_{t \geqslant 0}\,,$$

are martingales. Then the X is a Wiener process.

The proof of Theorem 3 is immediate since the martingale properties imply the equality

$$E(X_t - X_s)^4 = 3(t - s)^2$$

for any $s < t$. Consequently the trajectories of the process are continuous functions and the result follows from the well-known Lévy theorem.

An open question is if the assumption that only the processes (14)–(16) are martingales, similarly as in the case of the Poisson process, suffices to characterize the Wiener process.

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