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DISTRIBUTIONAL PROPERTIES OF SQUARES OF LINEAR STATISTICS

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ABSTRACT: Relations between distributions of squares of linear forms in independent random variables and distributions of the parent random variables are studied. Some new characterizations of the normal law involving chi-square distribution are obtained.

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1. INTRODUCTION

In a series of papers: Geisser and Roberts (1966), Roberts (1971), Geisser (1973), relations between the normal distribution of random variables (r.v's) and the chi-square distribution of linear forms in them were investigated – see also 4.2.4 in Patel and Read (1982) and Ahsanullah (1989). The main result of these papers states that if some linear statistics in independent r.v's are $\chi^2(1)$ (chi-square with one degree of freedom) then the parent distribution is normal.

This note is a complement to the papers mentioned above. In Section 2 we show that some of the results for the $\chi^2(1)$ law may be generalized not only to the gamma distribution – this case is considered in Geisser and Roberts (1966) – but to an arbitrary absolutely continuous distribution on $(0, +\infty)$ – Theorem 1. A new characterization of symmetry together with an extension of Theorem 5 from Roberts (1971) follows next. In Section 3 we consider the special case of linear forms distributed according to the $\chi^2(1)$ law. Some new characterizations of the normal law complementary to those from Roberts (1971), Geisser (1973) and Ahsanullah (1989) are obtained.

A new contribution to this theme, involving the concept of sub-independence, is contained in Ahsanullah et al. (1992).

2. GENERAL SCHEME AND SYMMETRY

The scheme used for the gamma law in Geisser and Roberts (1966) may also be applied for an arbitrary positive absolutely continuous distribution.

For a random variable W define its symmetrization W_s by the formula

$$W_s = U\,W,$$

where U is independent of W and P(U=1)=P(U=-1)= 1/2. The following distributional relations involving W, W_s and W^2 are obvious:

Let F_2 and G be distribution functions of W_s and W^2, respectively. Then

$$F_s(x)=\begin{cases}(1-G(x^{2-}))/2 & \text{for } x<0\\ (1+G(x^2))/2 & \text{for } x\ge 0,\end{cases} \tag{1}$$

$$G(x)=2F_s(\sqrt{x})-1 \quad \text{for } x\in R. \tag{2}$$

Denote by ϕ and ϕ_s characteristic functions of W and W_s, respectively. Obviously ϕ_s is determined by the distribution of W^2 (by (1)). On the other hand

$$\frac{1}{2}[\phi(t)+\phi(-t)]=\phi_s, \quad t\in R. \tag{3}$$

If W^2 is absolutely continuous with the density g then W and W_s are also absolutely continuous and their densities, f and f_s, respectively, fulfill the relations

$$\frac{1}{2}[f(x)+f(-x)]=f_s, \quad x\in R, \tag{4}$$

$$f_s(x)=|x|g(x^2), \quad x\in R(by\,(1)). \tag{5}$$

Consequently, we have the following result:

Theorem 1. If W^2 is absolutely continuous with density g then density f of W has the form

$$f(x)=h(x)|x|g(x^2), \quad h(x)+h(-x)=2, \quad x\in R, \tag{6}$$

and its characteristic function ϕ fulfills the equation

$$\phi(t)+\phi(-t)=2\int_R e^{itx}|x|g(x^2)dx, \quad t\in R. \tag{7}$$

Conversely, if the characteristic function ϕ of W fulfils (7) or its density f fulfils (6), where g is a non-negative function, then g is the density of W^2.

Proof: Assume that g and f are the densities of W^2 and W, respectively. Define

$$h(x)=\begin{cases} f(x)/[|x|g(x^2)] & \text{if only } |x|g(x^2)\neq 0\\ 1 & \text{if only } |x|g(x^2)=0.\end{cases}$$

Then by (4) we have

$$f_s(x)=\frac{1}{2}|x|g(x^2)[h(x)+h(-x)], \quad x\in R$$

Now (6) follows from (5) and the above formula. Equation (7) is an immediate consequence of (3) and (5).

Assume conversely that (6) holds. Then (4) yields

$$f_s(x)=\frac{1}{2}|x|g(x^2)[h(x)+h(-x)]=|x|g(x^2),\quad x\in R.$$

Similarly from (7) and (3) we have (5), too. Consequently by (2)

$$P(W^2\le x)=G(x)=2\int_{-\infty}^{\sqrt{x}} f_s(u)du-1=$$

$$=2\int_{-\infty}^{0}(-u)g(u^2)du+2\int_{-\infty}^{\sqrt{x}} ug(u^2)du-1=\int_0^x g(u)du,$$

for any positive x. Hence g is the density of W^2. ❑

Immediate consequences of Theorem 1 are Theorem and Corollary 1 from Geisser and Roberts (1966) and Corollaries 1, 2 from Roberts (1971).

Now we work towards an extension of Theorem 5 from Roberts (1971). The assumption of absolute continuity is relaxed.

Theorem 2. Let X, Y be i.i.d. r.v's with the characteristic function ϕ. Then

$$[\mathrm{Re}\,\phi(t)]^2=[\psi_1(t)+\psi_2(t)]/2, \tag{8}$$

$$[\mathrm{Im}\,\phi(t)]^2=[\psi_1(t)-\psi_2(t)]/2,\quad t\in R \tag{9}$$

where ψ_1 and ψ_2 are the characteristic functions of $(X+Y)_s$ and $(X-Y)_s$, respectively.

Proof: Observe that $\phi^2(t)$ and $\phi(t)\phi(-t)$ are the characteristic functions of X+Y and X-Y, respectively. Consequently by (3)

$$\phi^2(t)+\phi^2(-t)=2\psi_1(t), \tag{10}$$

$$\phi(t)\phi(-t)=\psi_2(t), \tag{11}$$

$t\in R$. Hence

$$4[\mathrm{Re}\,\phi(t)]^2=[\phi(t)+\phi(-t)]^2=2[\psi_1(t)+\psi_2(t)]$$
$$4[\mathrm{Im}\,\phi(t)]^2=[\phi(t)-\phi(-t)]^2=2[\psi_1(t)-\psi_2(t)]$$

$t\in R$. The above equations imply (8) and (9).

As a consequence of Theorem 2 we obtain an extension of Theorem 5 from Roberts (1971).

Corollary 1. Let X, Y be i.i.d. r.v's. The r.v's $(X+Y)^2$ and $(X-Y)^2$ are equidistributed iff X is symmetric.

Proof: Since $(X+Y)^2 \stackrel{d}{=} (X-Y)^2$ then also $(X+Y)_s \stackrel{d}{=} (X-Y)_s$. Consequently in Theorem 2 we have $\psi_1=\psi_2$ and by (9) the characteristic function of X is real. Hence X has a symmetric distribution.

If X is symmetric then (10) and (11) imply equidistribution of $(X+Y)_s$ and $(X-Y)_s$. By (2) $(X+Y)^2$ and $(X-Y)^2$ are also identically distributed.

Remark. Assume that $(X+Y)^2 \stackrel{d}{=} (X-Y)^2$. Observe that if the distribution of $(X+Y)_s$ is uniquely determined by its moments or its characteristic function is non-vanishing then by Theorem 2 the distribution of the X is uniquely determined by that of $(X+Y)^2$. On the other hand the distribution of X determines that of $(X+Y)^2$ and $(X-Y)^2$ even without the symmetry assumption.

In Theorem 5 from Roberts (1971) the special case

$$(X+Y)^2 \stackrel{d}{=} (X-Y)^2 \stackrel{d}{=} x^2(1)$$

was considered. Another straightforward generalization of this result is given in Section 3.

3. THE NORMAL AND x^2 DISTRIBUTIONS.

In this section we study relations between the $x^2(1)$ and normal distributions via properties of squares of linear statistics in independent random variables. Only upon introducing $x^2(1)$ law can we sharpen some of the previously known results.

A useful tool in such investigations is a result of Roberts (1971) (it is also a consequence of our Theorem 1).

Proposition 1. The r.v. $W^2 \stackrel{d}{=} x^2(1)$ iff the characteristic function ϕ of W satisfies

$$\phi^2(t)+\phi^2(-t)=2\exp(-t^2/2), \quad t \in R. \quad \square$$

The following extension of the Theorem from Geisser (1973) and Theorem 2 from Ahsanullah (1989) relaxes the assumption of equidistribution of the r.v's involved.

Theorem 3. Let X and Y be independent r.v's. Then $W_1^2=(aX+bY)^2/(a^2+b^2) \stackrel{d}{=} x^2(1) \stackrel{d}{=} W_1^2=(aX-bY)^2/(a^2+b^2)$ for some a, $b \neq 0$ iff at least one of the pair X, Y is N(0,1) and $X^2 \stackrel{d}{=} Y^2 \stackrel{d}{=} x^2(1)$.

Proof: As in Geisser (1973) we have

$$\exp[-(a^2+b^2)t^2/2] = \phi_{X_s}(at)\phi_{Y_s}(bt), \quad t \in R \tag{12}$$

Hence by the Cramer theorem $X_s \stackrel{d}{=} Y_s \stackrel{d}{=} N(0,1)$. This fact implies $X^2 \stackrel{d}{=} Y^2 \stackrel{d}{=} x^2(1)$. Since $W_1^2 \stackrel{d}{=} x^2(1)$ then by Proposition 1

$$2\exp[-(a^2+b^2)t^2/2] = \phi_X(at)\phi_Y(bt)+\phi_X(-at)\phi_Y(-bt)$$

$t \in R$. Now by (3) applied to X_s and Y_s from (12)

$$[\phi_X(at) - \exp(-a^2t^2)][\phi_Y(bt) - \exp(-b^2t^2)] = 0, \quad t \in R$$

Consequently, there is a sequence (t_n), t_n 0, such that

$$\phi_X(t_n) = \exp(-t_n^2/2) \quad or \quad \phi_Y(t_n) = \exp(-t_n^2/2), \quad n \geq 1$$

By Corollary 1 to Lemma 1.2.1 in Kagan et al. (1973) at least one of the pair X, Y is N(0,1) r.v.

The sufficiency part: Assume that $X \stackrel{d}{=} N(0,1)$ and $Y^2 \stackrel{d}{=} x^2(1)$. Then by Proposition 1

$$\phi_X(at)\phi_Y(bt) + \phi_X(-at)\phi_Y(-bt) =$$
$$= \exp(-a^2t^2/2)[\phi_Y(bt) + \phi_Y(-bt)] = \exp(-a^2t^2/2)$$

for $t \in R$. Consequently once again applying Proposition 1 we find that W_1^2 is $x^2(1)$ r.v. Similar computation proves the result for W_2^2. ❑

Corollary 3 from Geisser (1973) is a natural complement of Theorem 3. However the proof of this Corollary seems to be unsatisfactory since the equation $f(t)g(t)=0, t \in R$, does not yield $f \equiv 0$ or $g \equiv 0$, in general. Fortunately the argument based on a sequence (t_n), t_n 0, from the proof of Theorem 3 works.

Also in Geisser (1973) the following interesting problem was stated: `Assume that X and Y are i.i.d. r.v's and $(X+Y)^2/2$ has the $x^2(1)$ distribution. Is the distribution of X normal N(0,1)? Seemingly slight modification: "–" instead of "+" makes the problem very easy (see also Ahsanullah (1989)).

Proposition 2. Assume that X, Y are i.i.d. r.v's. The r.v. $(X-Y)^2/2$ is $x^2(1)$ iff X is N(0,1).

Observe that it is a straightforward generalization of Theorem 5 from Roberts (1971), where it is assumed that $(X+Y)^2/2$ is $x^2(1)$ additionally.

Proof: By Proposition 1

$$\phi(t)\phi(-t) = \exp(-t^2), \quad t \in R$$

and the result follows from the Cramer theorem. The sufficiency is obvious.

The original Geisser conjecture is answered positively in Ramachandran and Lau (1991).

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