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MULTIVARIATE INFINITELY DIVISIBLE DISTRIBUTIONS WITH THE GAUSSIAN SECOND ORDER CONDITIONAL STRUCTURE

1. Introduction.

Univariate infinitely divisible laws are widely investigated. However the number of papers devoted to the multivariate infinitely divisible distributions is considerably lower. These by Dwass and Teicher [4], Horn and Steutel [5] and Veeh [8] are among the most interesting .

In this note we observe that multivariate infinetly divisible distribution with all the univariate marginals Gaussian is a Gaussian distribution. This, quite simple fact, seems to have wide applications. We use it to simplify a characterization of the multivariate Gaussian law by properties of fourth cumulants obtained by Talwalker [7]. The main result is a characterization of the Gaussian distribution among multivariate infinitely divisible laws with the Gaussian second order conditional structure.

2. Univariate Gaussian marginals.

The characteristic function of a n-variate square integrable infinitely divisible distribution has the form

$$\varphi(\mathbf{t}) = \exp\{i\mathbf{t}'\mathbf{m} - \tfrac{1}{2}\,\mathbf{t}'\Sigma\,\mathbf{t} + \int_{\mathbf{R}^n} (e^{i\mathbf{t}'\mathbf{x}} - 1 - i\mathbf{t}'\mathbf{x})\|\mathbf{x}\|^{-2}\, dK(\mathbf{x})\}\,, \tag{1}$$

where $\mathbf{t}$ and $\mathbf{m}$ are n-dimensional real vector , Σ is a symmetric positive definite $n \times n$ matrix, $\mu_K(\cdot) = \int dK(\mathbf{x})$ is a finite Lebesgue - Stjeltjes measure on the Borel sets of $\mathbf{R}^n$ such that $\mu_K(\{\mathbf{0}\}) = 0$ and $\|\cdot\|$ is the standard Euclidean norm. The triple $(\mathbf{m}, \Sigma, K)$ is uniquely determined by φ. This multivariate version of the Kolmogorov's representation was obtained by Talwalker [7]. We use the above formula since we investigate the case of Gaussian univariate marginals. Consequently the second moments are finite.

Proposition 1. *If $\mathbf{X} = (X_1, \dots, X_n)$ is an infinitely divisible random vector and X_k is a Gaussian random variable for all $k = 1, \dots, n$, then $\mathbf{X}$ is a Gaussian random vector.*

Proof. For any $k = 1, \dots, n$ we put in (1) $t_k = t$ and $t_j = 0$ for all $j \in \{1, \dots, n\} \setminus \{k\}$. From the uniqueness of the Kolmogorov's representation we have

$$\int_{\mathbf{R}^n} (e^{itx_k} - 1 - itx_k)\, \frac{dK(x_1, \dots, x_n)}{x_1^2 + \dots + x_n^2} = 0\,.$$

Let us assume that $\mu_K \not\equiv 0$. Hence $\int_{\mathbf{R}^n} dK = A > 0$ and $G = K/A$ is a n-variate distribution function. The above equation yields

$$\mathbf{E}\left(\frac{e^{itY_k} - 1 - itY_k}{Y_1^2 + \dots + Y_n^2}\right) = 0\,, \qquad k = 1, \dots, n, \tag{2}$$

where G is the distribution function of a random vector $(Y_1, \dots, Y_n)$. Since X_k, $k = 1, \dots, n$ are Gaussian random variables then we can differentiable (2) twice (2) with respect to t. Then we put $t = 0$ and get

$$\mathbf{E}\left(\frac{Y_k^2}{Y_1^2 + \cdots + Y_n^2}\right) = 0\,, \qquad k = 1, \dots, n\,.$$

Consequently, in contradiction to our assumption, we have $\mu_K \equiv 0$. □

Now we apply Proposition 1 to simplify a characterization of the multivariate normal distribution obtained by Talwalker [7]:

If a random vector has infinitely divisible distribution and all its fourth cumulants are equal zero then it is a Gaussian random vector.

It is an extension of the earlier univariate result proved by Borges [1]. As an immediate consequence of the latter characterization we get

Proposition 2. *If $\mathbf{X} = (X_1, \dots, X_n)$ is an infinitely divisible random vector and for any $k = 1, \dots, n$, the fourth cumulant of X_k is equal zero then $\mathbf{X}$ is Gaussian random vector.*

*Proof.*From Borges [1] it follows that X_k is a normal random variable for every $k = 1, \dots, n$. Hence the result is a consequence of Proposition 1. □

3. Gaussian second order conditional structure .

In this Section we investigate multivariate infinitely divisible random vectors with linear conditional expectations and constant conditional variances. Such a conditional structure is a property of the multivariate Gaussian distribution. It explains our title. It is known that a continuous time parameter stochastic process with the Gaussian second order conditional structure is a Gaussian process. Details may be found in Plucińska [6], Wesołowski [9] and Bryc [3]. A similar result holds also for infinite sequences of random variables (see Bryc and Plucińska [3]). However it does not remain true in a finite dimensional case. A bivariatre counter example is given in Bryc and Plucińska [3]. Some other observations are gathered in Bryc [2].

In this Section we show that if we limit a class of multivariate distributions involved to infinitely divisible laws then three-dimensional Gaussian second order conditional structure implies normality.

Let $\mathbf{X} = (X_1, X_2, X_3)$ be a square integrable vector with the following properties

$$\mathbf{E}\left(X_i \mid X_j\right) = a_{i|j} X_j + \alpha_{i|j}, \tag{3}$$

$$\mathbf{Var}\left(X_i \mid X_j\right) = b_{i|j}, \tag{4}$$

$$\mathbf{E}\left(X_i \mid X_j, X_k\right) = a_{j(i|j,k)}\, X_k + + a_{k(i|j,k)}\, X_k + \alpha_{i|j,k}\,, \tag{5}$$

$$\mathbf{Var}\left(X_i \mid X_j, X_k\right) = b_{i|j,k}, \tag{6}$$

where i, j, $k = 1, 2, 3$, $(i \neq j \neq k \neq i)$. Thus $\mathbf{X}$ has the Gaussian second order conditional structure. To avoid the trivial cases we should assume that the components of $\mathbf{X}$ are linearly independent and that they have non-zero correlation in pairs. Our main result is given in

Theorem. *Let $\mathbf{Y} = (Y_1, \ldots, Y_n)$ be an infinitely divisible, square integrable random vector with linearly independent components pairwisely non-zero correlated. If for any $i = 1, \ldots, n$, there are some j, $k = 1, \ldots, n$, $(i \neq j \neq k \neq i)$, such that for the vector*

$$(X_1, X_2, X_3) = (Y_i, Y_j, Y_k)$$

the conditions (3) – (6) *hold, then* $\mathbf{Y}$ *is a Gaussian random vector.*

Proof. Without any loss of generality we can additionaly assume

$$\mathbf{E}\, X_i = 0 \quad \text{and} \quad \mathbf{E}\, X_i^2 = 1; \quad i = 1,\, 2,\, 3.$$

It is easy to observe that then

$$\alpha_{i|j} = \alpha_{i|j,k} = 0\,, \qquad a_{i|j} = \rho_{ij}\,, \qquad b_{i|j} = 1 - \rho_{ij}^2\,,$$

$$a_{j(i|j,k)} = \frac{\rho_{ij} - \rho_{ik}\,\rho_{jk}}{1 - \rho_{jk}^2}\,, \qquad a_{k(i|j,k)} = \frac{\rho_{ik} - \rho_{ij}\,\rho_{jk}}{1 - \rho_{jk}^2}\,,$$

$$b_{i|j,k} = |K| / \left(1 - \rho_{jk}^2\right),$$

where ρ_{ij} is a correlation coefficient of X_i and X_j, i, $j = 1, 2, 3; i \neq j$; and $|K|$ is the determinant of the covariance matrix of the $\mathbf{X}$. Obviously from the assumptions we have $|K| \neq 0$ and $0 < |\rho_{ij}| < 1$.

In Bryc and Plucińska [3] it was proved that (3) and (4) imply existence of the moments of any order of X. We are interested here in the third and fourth moments. At first we compute the conditional moments of the order three.

We apply (3) – (6) to the formulas

$$\mathbf{E}\left(\mathbf{E}\left(X_i^2 \mid X_j, X_k\right) X_j \mid X_k\right) = \mathbf{E}\left(X_i^2 \mathbf{E}\left(X_j \mid X_i, X_k\right) \mid X_k\right).$$

$$\mathbf{E}\left(\mathbf{E}\left(X_i \mid X_j, X_k\right) X_j^2 \mid X_k\right) = \mathbf{E}\left(X_i \mathbf{E}\left(X_j^2 \mid X_i, X_k\right) \mid X_k\right).$$

As a result we have a system of linear equations

$$\text{(7)} \qquad \begin{cases} a_{i(j|i,k)}\, x - a_{j(i|j,k)}^2\, y = \mathbb{P}(x_k)\,, \\ -a_{i(j|i,k)}^2\, x + a_{j(i|j,k)}\, y = \mathbb{Q}(x_k)\,, \end{cases}$$

where $x = \mathbf{E}\left(X_i^3 \mid X_k\right)$, $y = \mathbf{E}\left(X_j^3 \mid X_k\right)$ and $\mathbb{P}, \mathbb{Q}$ are some polynomials of the order three. The determinant of the system takes the form

$$\begin{aligned} W &= a_{i(j|i,k)}\, a_{j(i|j,k)} \left(1 - a_{i(j|i,k)}\, a_{j(i|j,k)}\right) \\ &= \frac{|K| \left(\rho_{ij} - \rho_{ik}\,\rho_{jk}\right)^2}{\left[\left(1 - \rho_{ik}^2\right)\left(1 - \rho_{jk}^2\right)\right]^2}\,. \end{aligned}$$

Now let us observe that from three expressions:

$$\rho_{12} - \rho_{23}\,\rho_{13}, \qquad \rho_{23} - \rho_{13}\,\rho_{12}, \qquad \rho_{13} - \rho_{12}\,\rho_{23}$$

only one may be equal zero. Let us assume that

$$\rho_{12} = \rho_{23}\,\rho_{13} \quad \text{and} \quad \rho_{23} = \rho_{13}\,\rho_{12}\,,$$

say. Hence

$$\rho_{12} = \rho_{12}\,\rho_{13}^2 \quad \text{and} \quad |\rho_{13}^2| = 1$$

which is a contradiction. Consequently (7) has unique solution in one of the following cases: $i = 1$, $j = 2$, $k = 3$ or $i = 1$, $j = 3$, $k = 2$. Without any loss of generality we can consider only the latter case. The uniqueness of the solution of (7) yields the form of $\mathbf{E}\left(X_1^3 \mid X_2\right)$ being as for the Gaussian random vector, i.e.

$$\mathbf{E}\left(X_1^3 \mid X_2\right) = \rho_{12}^3 X_2^3 + 3\rho_{12}\left(1 - \rho_{12}^2\right) X_2. \tag{8}$$

Hence $\mathbf{E}\,X_1^3 = 0$.

Now we compute in the similar way $\mathbf{E}\,X_1^4$. The assumptions (3) – (6) and the equation (8) we apply this time to

$$\mathbf{E}\left(\mathbf{E}\left(X_1^3 \mid X_2\right)X_2\right) = \mathbf{E}\left(X_1^3\mathbf{E}\left(X_2 \mid X_1\right)\right),$$
$$\mathbf{E}\left(\mathbf{E}\left(X_1^2 \mid X_2\right)X_2^2\right) = \mathbf{E}\left(X_1^2\mathbf{E}\left(X_2^2 \mid X_1\right)\right)$$

and get

$$\rho_{12}^2\mathbf{E}\,X_2^4 + 3\left(1 - \rho_{12}^2\right) = \mathbf{E}\,X_1^4, \qquad \mathbf{E}\,X_1^4 = \mathbf{E}\,X_2^4\,.$$

Consequently $\mathbf{E}\,X_1^4 = 3$.

Hence it follows that the fourth cumulant of $Y_i = X_1$ is equal zero since

$$\mathbf{E}\,Y_i = 0, \qquad \mathbf{E}\,Y_i^2 = 1, \qquad \mathbf{E}\,Y_i^3 = 0, \qquad \mathbf{E}\,Y_i^4 = 3\,,$$

for any $i = 1, \ldots, n$. Now the result follows from Proposition 2. □

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