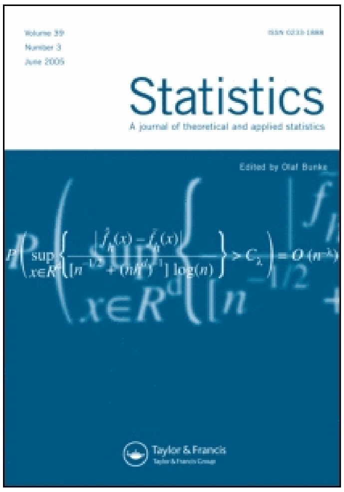

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SOME REMARKS ON THE MULTIVARIATE LIOUVILLE DISTRIBUTION

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Summary. Relations between the univariate gamma, multivariate Dirichlet and Liouville distributions are investigated.

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Key words: Liouville distribution, gamma distribution, Dirichlet distribution, quotient of random variables, beta distribution of the second kind, characterization of probability laws.

1. INTRODUCTION

A random variable X has a gamma distribution with a scale parameter α ($\alpha > 0$) and a shape parameter β ($\beta > 0$) if its density function f has the form

$$f(x) = \begin{cases} \alpha^{\beta} x^{\beta-1} e^{-\alpha x}/\Gamma(\beta) & \text{for } x > 0, \\ 0 & \text{for } x \leq 0. \end{cases}$$

Then we write $X \stackrel{d}{=} G(\alpha, \beta)$. (By '$\stackrel{d}{=}$' we denote identical distribution.)

Consider independent random variables $X_1, \ldots, X_n$ such that $X_i \stackrel{d}{=} G(\alpha, \beta_i)$, $i = 1, \ldots, n$, $n = 1, 2, \ldots$ The distribution of the random vector $Y = (Y_1, \ldots, Y_n)$, where $Y_i = X_i/(X_1 + \cdots + X_n)$, $i = 1, \ldots, n$, is called the n-variate Dirichlet distribution with the parameter $\beta = (\beta_1, \ldots, \beta_n)$; it is denoted by $D_n(\beta)$. This probability law, concentrated on a simplex, is applied nowadays with increasing frequency in statistical modelling, distribution theory and Bayesian inference (see e.g. Dickey and Chen (1985), Aitchison (1986)). It is also a base for defining the multivariate Liouville distribution thoroughly investigated in Gupta and Richards (1987).

Following Fang *et al.* (1990) we say that a random vector Z in $\mathbb{R}_+^n = (0, \infty)^n$ has a Liouville distribution if $Z \stackrel{d}{=} RY$, where $Y \stackrel{d}{=} D_n(\beta)$ and R is a random variable (univariate) independent of Y. Consequently

$$\mathscr{A}: \begin{cases} \text{A random vector } Z \text{ in } \mathbb{R}_+^n \text{ has a Liouville distribution iff} \\ \mathbf{Z} = \left(\dfrac{Z_1}{Z_1 + \cdots + Z_n}, \ldots, \dfrac{Z_n}{Z_1 + \cdots + Z_n}\right) \stackrel{d}{=} D_n(\beta) \\ \text{for some } \beta \in \mathbb{R}_+^n \text{ and } \mathbf{Z} \text{ is independent of } Z_1 + \cdots + Z_n. \end{cases}$$

This and other interesting properties of the multivariate Liouville law are collected in Chapter 6 of Fang *et al.* (1990). (Obviously the Dirichlet distribution is a special case of the Liouville measure.)

Take now a Borel function $\phi:\mathbb{R}_+\to\mathbb{R}_+$ and define a random vector $V=(V_1,\ldots,V_n)$ by $V_i=X_i/\phi(X_1+\cdots+X_n)$, $i=1,\ldots,n$. Then

$$\begin{aligned}\mathcal{V}&=\left(\frac{V_1}{V_1+\cdots+V_n},\ldots,\frac{V_n}{V_1+\cdots+V_n}\right)\\&=\left(\frac{X_1}{X_1+\cdots+X_n},\ldots,\frac{X_n}{X_1+\cdots+X_n}\right)=Y\stackrel{d}{=}D_n(\beta)\end{aligned}\qquad(1)$$

and $\mathcal{V}$ $(=Y)$ by $\mathcal{A}$ is independent of $V_1+\cdots+V_n=(X_1+\cdots+X_n)/\phi(X_1+\cdots+X_n)$. From $\mathcal{A}$ again we conclude that V is a Liouville random vector.

In this note we are interested in the converse result. Assume that V, defined as above for some independent random variables $X_1,\ldots,X_n$, has a Liouville distribution. We try to answer the question if the only possible distribution of the X's is the gamma one. The question for $\phi\equiv\text{const}$ is solved in Th. 6.15 of the monograph Fang *et al.* (1990)—see also Gupta and Richards (1987). We prove that in the general case for $n\geq 3$ the answer is affirmative. The problem for $n=2$ is studied in Section 3.

2. THE CASE $n\geq 3$

Consider independent positive random variables $X_1,\ldots,X_n$, $n\geq 3$, and a Borel function $\phi:\mathbb{R}_+\to\mathbb{R}_+$. Define a random vector V as in Section 1. Then we have the following characterization of the gamma law:

Theorem. *The random vector V has a Liouville distribution iff X's are gamma random variables with a common scale parameter.*

Proof. By the definition and (1) V has a Dirichlet distribution. It is well known that for independent gamma variables a scale invariant statistic and sum are independent. Both these facts by ($\mathcal{A}$) prove sufficiency.

To prove necessity take any $i, j, k=1,\ldots,n$, $i\neq j\neq k\neq i$, and consider the random vector $W=(V_i/V_k, V_j/V_k)$. Observe that W is a scale-invariant statistic. Hence it does not depend on the distribution of the R from the decomposition $(V_i, V_j, V_k)\stackrel{d}{=}RY$ (marginals of the Liouville distribution are also Liouville). We choose $R\equiv 1$. Since $Y\stackrel{d}{=}D_3(\beta)$ then

$$(V_i, V_j, V_k)=\left(\frac{U_1}{U_1+U_2+U_3},\frac{U_2}{U_1+U_2+U_3},\frac{U_3}{U_1+U_2+U_3}\right)$$

for some independent gamma random variables U_1, U_2, U_3 with a common scale parameter and shape parameters β_1, β_2, β_3.

The definition of V together with the last observation leads to the equation

$$W=(X_i/X_k, X_j/X_k)\stackrel{d}{=}(U_1/U_3, U_2/U_3).$$

On the other hand it may be easily checked that the random vector $(U_1/U_3, U_2/U_3)$ has the bivariate beta distribution of the second kind given by the density function

$$g(x, y) = \begin{cases} \dfrac{\Gamma(\beta_1 + \beta_2 + \beta_3)}{\Gamma(\beta_1)\Gamma(\beta_2)\Gamma(\beta_3)} \cdot x^{\beta_1 - 1} y^{\beta_2 - 1}(1 + x + y)^{\beta_1 + \beta_2 + \beta_3}, & x, y > 0 \\ 0, & \text{elsewhere.} \end{cases}$$

Consequently the result follows from Theorem 1 of Kotlarski (1967). □

3. THE CASE $n = 2$

For the bivariate distribution the necessity part of the Theorem is no longer valid for an arbitrary function ϕ. To prove that begin with the following easy consequence of Theorem 3.2 of Gupta and Richards (1987).

Corollary. *If X and Y are such random variables that $X + Y = 1$ a.e. and X/Y has the beta distribution of the second kind then (X, Y) is a Dirichlet random vector.*

Now we are ready to give a counterexample to our original question in the bivariate case for $\phi(x) \equiv x$ i.e. to find such non-gamma random variables X, Y for which the random vector $(X/(X + Y), Y/(X + Y))$ has a Liouville (in this case Dirichlet) distribution. (The function $\phi(x) \equiv x$ is of special interest due to the definition of the Dirichlet law-see Section 1.) To this end consider independent random variables X, Y with a common density ($p > 0$)

$$f(x) = \begin{cases} (\Gamma(p))^{-1} x^{-p-1} e^{-1/x}, & \text{for } x > 0, \\ 0, & \text{for } x \le 0. \end{cases}$$

It is well known—see Kotlarski (1962)—that in this case the quotient of the random variables has the beta density of the second kind (β_1, $\beta_2 > 0$)

$$g(x) = \begin{cases} (B(\beta_1, \beta_2))^{-1} x^{\beta_1 - 1}(1 + x)^{\beta_1 + \beta_2}, & \text{for } x > 0, \\ 0, & \text{for } x \le 0, \end{cases}$$

Hence by Corollary it follows that $(X/(X + Y), Y/(X + Y))$ is a Dirichlet random vector.

Consider now $\phi(x) \equiv x/\psi(x)$, with the function ψ fulfilling the condition $\sigma(X + Y) = \sigma(\psi(X + Y))$. Assume that V has a Liouville distribution. Then by ($\mathscr{A}$) the random variables X/Y and $X + Y$ are independent. Now the Theorem follows from the famous Lukacs (1955) characterization of the gamma distribution.

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References

Aitchison, J. (1986). *The Statistical Analysis of Compositional Data*. Chapman and Hall, London/New York.

Dickey, J. M. and Chen, C. H. (1985). Direct subjective probability modelling using ellipsoidal distributions. *Bayesian Statistics,* 2 (eds. J. M. Bernardo *et al.*), Elsevier Sci. Publ. and Valencia Univ. Press, 157–182.

Gupta, R. D. and Richards, D. S. P. (1987). Multivariate Liouville distributions. *J. Multivar. Anal.* **23,** 233–256.

Fang, K. T., Kotz, S. and Ng, K. W. (1990). *Symmetric Multivariate and Related Distributions.* Chapman and Hall, London/New York.

Kotlarski, I. (1962). On pairs of independent random variables whose quotients follow some known distributions. *Colloq. Math.* **9,** 151–162.

Kotlarski, I. (1967). On characterizing the gamma and the normal distributions. *Pacific J. Math.* **20,** 69–76.

Lukacs, E. (1955). A characterization of the gamma distribution. *Ann. Math. Stat.* **26,** 319–324.