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THE LYAPOUNOV CENTRAL LIMIT THEOREM FOR FACTORIZABLE ARRAYS

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ABSTRACT. A sort of the Lyapounov central limit theorem for row-wise factorizable triangular arrays is obtained. Also a new version of the classical Lyapounov theorem for independent random variables, being a tool in the proof of the main result, seems to be of independent interest.

1. INTRODUCTION

The celebrated theorem on decomposition of the normal law proved in Cramér [1] was the beginning of many further investigations that developed in different directions. Some of them were concerned with analytical extensions of the original result and lead in Linnik and Zinger [8] to the well-known α-decomposition theorem (see also Linnik and Ostrovskii [7]). Next Linnik [6] revealed strong relationships between the Darmois-Skitovitch theorem on the characterization of the normal distribution by independence of linear forms in independent random variables and the Carmér theorem; this line was continued by Kagan [2–4]. A culmination of these studies was the following concept of factorizable measures, originally named $\mathscr{D}_{n,k}$ classes, introduced by Kagan in [3].

Definition 1. A random vector $X = (X_1, \dots, X_n)$ (or its distribution) is k-factorizable iff its characteristic function ϕ has the form

$$\phi(t_1, \dots, t_n) = \prod_{1 \le i_1 < \cdots < i_k \le n} R_{i_1 \cdots i_k}(t_{i_1}, \dots, t_{i_k}) \tag{1}$$

for any $(t_1, \dots, t_n) \in \mathbf{R}^n$, where $R_{i_1 \cdots i_k}$ is a continuous complex-valued function such that

$$R_{i_1 \cdots i_k}(0, \dots, 0) = 1 \quad \text{for any } 1 \le i_1 < \cdots < i_k \le n.$$

The random vector X (or its distribution) is locally k-factorizable if the representation (1) holds in some neighborhood of the origin.

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Some examples and many interesting properties of these families including extensions of the Darmois-Skitovitch and Cramér theorems are given in [3]. To become more familiar with the device recall some nice observations from Kagan [3]:

- Gaussian measures are 2-factorizable,
- a k-factorizable random vector with Gaussian k-dimensional marginal distributions is jointly Gaussian,
- a convolution of k-factorizable distributions is again k-factorizable,
- classes of product and 1-factorizable measures are equal.

The last property allows treating the notion as a gradual weakening of independence formulated in a strictly analytical form.

The research was continued in Kagan [5] where a related concept of (n, k)-equivalence was introduced. In Wesolowski [9] a formula expressing the characteristic function of a k-factorizable measure in terms of the characteristic function of its k-dimensional marginals was proved. Also some consequences on determination of the joint k-factorizable distribution by its k-dimensional marginals were given there. The formula was intensively exploited in Wesolowski [10] where relations between factorizable measures and the Gaussian conditional structure of the second order were investigated. It is the technical core of the present paper, too.

Slightly reformulated it may be written in the following form:

If a random vector $(X_1, \dots, X_n)$ is k-factorizable, then its characteristic function has the form

$$\phi(t_1, \dots, t_n) = \prod_{r=1}^{k} \left[\prod_{1 \le i_1 < \cdots < i_r \le n} \phi_{i_1 \cdots i_r}(t_{i_1}, \dots, t_{i_r}) \right]^{a_{n,k,r}} \tag{2}$$

for all points $(t_1, \dots, t_n)$ in which the right-hand side of the above formula is well defined (i.e., for all points in which products of the marginal characteristic functions with negative powers $a_{n,k,r}$ are nonzero), where

$$a_{n,k,r} = \sum_{i=0}^{k-r} \binom{n-r}{i} (-1)^i, \qquad r = 1, \dots, k,$$

and $\phi_{i_1 \cdots i_r}$ denotes the characteristic function of $(X_{i_1}, \dots, X_{i_r})$, $1 \le i_1 < \cdots < i_r \le n$.

The aim of this paper is to prove the Lyapounov version of the central limit theorem for row-wise k-factorizable triangular arrays, where k is a natural constant. It is a new contribution to the research in that it extends the limit theorems by weakening the assumption of independence. Also a new version of the classical Lyapounov theorem (i.e., for row-wise independent arrays) in the case of divergent sums of variances, being essentially a tool for the proof of the main result, seems to be of independent interest.

In dealing with limit problems for infinite triangular arrays it is natural to assume that the number k_n of random variables in the nth row tends to infinity together with the number of the row. Throughout this paper we consider only such arrays.

2. The Lyapounov central limit theorem

Recall the classical Lyapounov central limit theorem for triangular arrays.

Theorem 1 (Lyapounov). *Assume that* $\{X_{n,j}, j = 1, \dots, k_n\}$ *are zero-mean independent random variables,* $n = 1, 2, \dots$. *If*

$$\lim_{n\to\infty} \sum_{j=1}^{k_n} E(X_{n,j}^2) = \sigma^2 > 0, \tag{3}$$

$$\lim_{n\to\infty} \sum_{j=1}^{k_n} E(|X_{n,j}|^3) = 0, \tag{4}$$

then $S_n = X_{n,1} + \cdots + X_{n,k_n}$ *converges in distribution to the normal law with the mean zero and the variance* σ^2 $(S^n \xrightarrow{d} \mathcal{N}(0, \sigma^2))$ *as* $n \to \infty$.

The formula (4) is often called the Lyapounov condition. Different versions of this theorem for some classes of dependent random variables (martingale, mixing, conditioning) are known in the literature. The main result of this paper shows that factorizable families may also be fruitfully applied in central limit problems.

Theorem 2. *Assume that* $(X_{n,1}, \dots, X_{n,k_n})$ *is a zero-mean* k*-factorizable random vector,* $n = 1, 2, \dots$. *If*

$$\lim_{n\to\infty} \sum_{i,j=1}^{k_n} E(X_{n,i}X_{n,j}) = \sigma^2 > 0, \tag{5}$$

$$\lim_{n\to\infty} k_n^{k-1} \sum_{j=1}^{k_n} E(|X_{n,j}|^3) = 0, \tag{6}$$

then $S_n \xrightarrow{d} \mathcal{N}(0, \sigma^2)$ *as* $n \to \infty$.

Observe that for $k = 1$ from Theorem 2 immediately follows the classical result, hence it is a straightforward generalization of the Lyapounov theorem. A much weaker version of the result for $k = 2$ was proved in Wesolowski [11]. It contained an additional assumption $\lim_{n\to\infty} k_n \sum_{j=1}^{k_n} E(X_{n,j}^2) = \tau^2 > 0$.

Before we give the proof of Theorem 2 (see §3), a new version of the classical result with assumption (3) omitted will be presented.

Theorem 3. *Assume that* $\{X_{n,j}, j = 1, \dots, k_n\}$ *are zero-mean independent random variables,* $n = 1, 2, \dots$. *If the Lyapounov condition* (4) *holds, then*

$$\lim_{n\to\infty} \exp\left(\frac{t^2}{2} \sum_{j=1}^{k_n} E(X_{n,j}^2)\right) E(\exp(itS_n)) = 1 \tag{7}$$

for any real t.

The core of this result lies in a special formulation of the thesis. In the other well-known versions of the Lyapounov theorem it takes a weaker form:

$$\lim_{n\to\infty}\left|E(\exp(itS_n)) - \exp\left(-\frac{t^2}{2}\sum_{j=1}^{k_n}E(X_{n,j}^2)\right)\right| = 0.$$

Of course the convergence given in (7) also holds in the assumptions of Theorem 1. Hence it is another straightforward generalization of the Lyapounov theorem. Theorem 3 is another important tool in the proof of the main result (besides the formula (2)).

Unfortunately we are not able to prove the Lindelberg analogue of this theorem. This fact is also the essential obstacle in obtaining the Lindelberg limit theorem for k-factorizable distributions, i.e., the result similar to Theorem 2 but with the Lyapounov type condition (6) replaced by a condition of the Lindelberg type:

$$\lim_{n\to\infty} k_n^{k-1}\sum_{j=1}^{k_n}E(X_{n,j}^2 I(|X_{n,j}| > \varepsilon)) = 0$$

for any $\varepsilon > 0$.

3. Proofs and lemmas

Begin with the proof of Theorem 3. Then four auxiliary lemmas follow. The proof of the main result ends the section.

Proof of Theorem 3. For any r.v. X such that $E(|X|^3) < \infty$ and $E(X) = 0$ define the function ψ_X by

$$\psi_X(t) = \exp\left(\frac{1}{2}t^2E(X^2)\right)E(\exp(itX)), \qquad t \in \mathbf{R}.$$

Observe that $\psi_X'(0) = \psi_X''(0) = 0$ and consequently

$$|\psi_X(t) - 1| = \left|\psi_X(t) - 1 - t\psi_X'(0) - \frac{t^2}{2}\psi_X''(0)\right| \le \frac{|t|^3}{6}|\psi'''(\theta t)| \tag{8}$$

for some $\theta \in (0, 1)$.

Since

$$\max_{1\le j\le k_n} E(X_{n,j}^2) \le \left(\max_{1\le j\le k_n} E(|X_{n,j}|^3)\right)^{2/3} \to 0$$

as $n \to \infty$ by (4),

$$\max_{1\le j\le k_n}\exp\left(\frac{t^2}{2}E(X_{n,j}^2)\right) \le c(t) < \infty$$

for sufficiently large n, where $c(t)$ is nonradom. Let $\psi_{nj} = \psi_{X_{nj}}$. Again by the Lyapounov condition (4) and the above inequality after some easy calculations we have

$$|\psi_{nj}'''(t)| \le C(t)E(|X_{n,j}|^3), \tag{9}$$

for any $j = 1, \dots, k_n$, where $C(t)$ is a real number. Consequently, by the

Lyapounov condition $\max_{1\le j\le k_n} |\psi'''_{n,j}(t)| \to 0$ as $n\to\infty$. Hence by (9)

$$\lim_{n\to\infty} \max_{1\le j\le k_n} |\psi_{n,j}(t)-1| = 0$$

for any t. It allows considering the logarithm of ψ_{S_n} for sufficiently large n. By the elementary inequality $|\ln(z)| \le 2|1-z|$ for $|z-1|\le 1/2$ we have

$$|\ln(\psi_{S_n}(t))| \le \sum_{j=1}^{k_n} |\ln(\psi_{n,j}(t))| \le 2\sum_{j=1}^{k_n} |\psi_{n,j}(t)-1|.$$

Now (8) and then (9) yield

$$|\ln(\psi_{S_n}(t))| \le \frac{|t|^3}{3}\sum_{j=1}^{k_n} |\psi'''_{nj}(\theta t)| \le \frac{|t|^3}{3} C(\theta t)\sum_{j=1}^{k_n} E(|X_{n,j}|^3),$$

and once again applying the Lyapounov condition (4) we arrive at the final result. □

The following nonprobabilistic lemma deals with sums of subsums and binomial coefficients. In the sequel we use the standard convention: $\binom{n}{r}=0$ for $r<0$ or $r>n$, $\binom{x}{0}=1$ for any real x.

Lemma 1. *For any real numbers $a_1,\dots,a_n$ and any $r=1,\dots,n$, $n=1,2,\dots$,*

$$\sum_{1\le i_1<\cdots<i_r\le n}\ \sum_{j=1}^{r} a_{i_j} = \binom{n-1}{r-1}\sum_{i=1}^{n} a_i, \tag{10}$$

$$\sum_{1\le i_1<\cdots<i_r\le n}\left(\sum_{j=1}^{r} a_{i_j}\right)^2 = \binom{n-2}{r-1}\sum_{i=1}^{n} a_i^2 + \binom{n-2}{r-2}\left(\sum_{i=1}^{n} a_i\right)^2. \tag{11}$$

Proof. Begin with (10) and apply induction. For $n=1$ it is obvious. Assume that (10) holds for some n and any $r=1,\dots,n$. We will prove that it is fulfilled for $n=1$ and any $r=1,\dots,n+1$.

First, observe that for $r=1$ and $r=n+1$ it is trivial. Take any $r=2,\dots,n$. Then

$$\begin{aligned}\sum_{1\le i_1<\cdots<i_r\le n+1}\ \sum_{j=1}^{r} a_{i_j} &= \sum_{1\le i_1<\cdots<i_r\le n}\ \sum_{j=1}^{r} a_{i_j} + \sum_{1\le i_1<\cdots<i_{r-1}\le n}\left(\sum_{j=1}^{r-1} a_{i_j} + a_{n+1}\right)\\ &= \binom{n-1}{r-1}\sum_{i=1}^{n} a_i + \binom{n-1}{r-2}\sum_{i=1}^{n} a_i + \binom{n}{r-1}a_{n+1}\end{aligned}$$

by the induction assumption applied to the first and second terms in the middle expression. Hence the result follows by the elementary identity $\binom{n-1}{r-1}+\binom{n-1}{r-2}=\binom{n}{r-1}$.

We use induction to prove (11) too. For $n=1,2$ it is obvious. Assume that it holds for some n and any $r=1,\dots,n$. We will prove that it is fulfilled for $n+1$ and any $r=1,\dots,n+1$. For $r=1$ or $n+1$ it is immediate. Take an arbitrary $r = 2, \ldots, n$. Then

$$\begin{aligned}
&\sum_{1\le i_1<\cdots<i_r\le n+1} \left(\sum_{j=1}^{r} a_{i_j}\right)^2 \\
&= \sum_{1\le i_1<\cdots<i_r\le n} \left(\sum_{j=1}^{r} a_{i_j}\right)^2 + \sum_{1\le i_1<\cdots<i_{r-1}\le n} \left(\sum_{j=1}^{r-1} a_{i_j} + a_{n+1}\right)^2 \\
&= \binom{n-2}{r-1}\sum_{i=1}^{n} a_i^2 + \binom{n-2}{r-2}\left(\sum_{i=1}^{n} a_i\right)^2 + \sum_{1\le i_1<\cdots<i_{r-1}\le n} \left(\sum_{j=1}^{r-1} a_{i_j}\right)^2 \\
&\quad + 2a_{n+1} \sum_{1\le i_1<\cdots<i_{r-1}\le n} \sum_{j=1}^{r-1} a_{i_j} + a_{n+1}^2\binom{n}{r-1} = (*),
\end{aligned}$$

where we have applied above the induction assumption. Now using (10) in the fourth summand above we have

$$\begin{aligned}
(*) &= \left[\binom{n-2}{r-1} + \binom{n-2}{r-2}\right]\sum_{i=1}^{n} a_i^2 + \left[\binom{n-2}{r-2} + \binom{n-2}{r-3}\right]\left(\sum_{i=1}^{n} a_i\right)^2 \\
&\quad + 2\binom{n-1}{r-2} a_{n+1}\sum_{i=1}^{n} a_i + a_{n+1}^2\binom{n}{r-1} \\
&= \binom{n-1}{r-1}\sum_{i=1}^{n} a_i^2 + \binom{n-1}{r-2}\sum_{i=1}^{n} a_i^2 + 2\binom{n-1}{r-2}\sum_{i=1}^{n} a_i a_{n+1} \\
&\quad + \left[\binom{n-1}{r-1} + \binom{n-1}{r-2}\right] a_{n+1}^2 .
\end{aligned}$$

Hence (11) follows immediately. □

In the next two auxiliary lemmas the coefficients

$$a_{n,k,r} = \sum_{i=0}^{k-r} \binom{n-r}{i}(-1)^i, \tag{12}$$

$r = 1, \ldots, k$, $k = 1, \ldots, n$, $n = 1, 2, \ldots$, appearing in the factorization (2) of a characteristic function of a factorizable distribution, are considered. It is not difficult to show that these coefficients fulfill the following useful identity:

$$a_{n,k+1,r} = a_{n,k,r} + (-1)^{k-r+1}\binom{n-r}{k-r+1} \tag{13}$$

for any r, n as above and $k = 1, \ldots, n-1$.

Lemma 2. *For any* $k = 2, \ldots, n$, $n = 2, 3, \ldots$,

$$\sum_{r=1}^{k} a_{n,k,r}\binom{n-2}{r-1} = 0. \tag{14}$$

Proof. Obviously the result is true for $k = 2$. Assume that it holds for some k, $2 \le k \le n-1$. Then

$$\sum_{r=1}^{k+1} a_{n,k+1,r}\binom{n-2}{r-1} = \sum_{r=1}^{k} a_{n,k+1,r}\binom{n-2}{r-1} + \binom{n-2}{k} = (**)$$

since $a_{n,k+1,k+1} = 1$. By (13) and the induction assumption

$$\begin{aligned}(**) &= \sum_{r=1}^{k}(-1)^{k-r+1}\binom{n-r}{k-r+1}\binom{n-2}{r-1} + \binom{n-2}{k} \\ &= \sum_{l=0}^{k}(-1)^{l}\binom{n-k-1+l}{l}\binom{n-2}{k-l} = (***)\end{aligned}$$

for $l = k+1-r$. Consequently,

$$(***) = \frac{1}{n-2}\binom{n-1}{k}\sum_{l=0}^{k}\binom{k}{l}(-1)^l(n-k-1+l) = 0$$

since $\sum_{l=0}^{k} l^i\binom{k}{l}(-1)^l = 0$ for $i = 0, 1$. □

Lemma 3. *For any* $k = 2, \dots, n$, $n = 2, 3, \dots$,

$$\sum_{r=2}^{k} a_{n,k,r}\binom{n-2}{r-2} = 1. \tag{15}$$

Proof. For $k = 2$ it is immediate. Assume that it holds for some k, $2 \le k \le n-1$. Then similarly as in the proof of Lemma 2 by (13) we obtain

$$\begin{aligned}\sum_{r=2}^{k+1} a_{n,k+1,r}\binom{n-2}{r-2} &= 1 + \sum_{r=2}^{k+1}(-1)^{k-r+1}\binom{n-r}{k-r+1}\binom{n-2}{r-2} \\ &= 1 + \binom{n-2}{k-1}\sum_{l=0}^{k-1}\binom{k-1}{l}(-1)^l = 1. \quad \square\end{aligned}$$

Now combining the above three lemmas we obtain an identity, which will be used in the proof of the main theorem.

Lemma 4. *For any* $k = 1, \dots, n$, $n = 1, 2, \dots$,

$$\sum_{r=1}^{k}\left[a_{n,k,r}\sum_{1\le i_1<\cdots<i_r\le n}\left(\sum_{j=1}^{r} a_{i_j}\right)^2\right] = \left(\sum_{i=1}^{n} a_i\right)^2. \tag{16}$$

Proof. For $k = 1$ it is obvious. By (11) of Lemma 1 for any $k = 2, \dots, n$

$$\begin{aligned}&\sum_{r=1}^{k}\left[a_{n,k,r}\sum_{1\le i_1<\cdots<i_r\le n}\left(\sum_{j=1}^{r} a_{i_j}\right)^2\right] \\ &\quad = \left(\sum_{i=1}^{n} a_i^2\right)\sum_{r=1}^{k} a_{n,k,r}\binom{n-2}{r-1} + \left(\sum_{i=1}^{n} a_i\right)^2\sum_{r=2}^{k} a_{n,k,r}\binom{n-2}{r-2}\end{aligned}$$

and the final result follows now by (14) and (15). □

Now we are ready to prove the limit theorem. Recall that the main tools used in the course of the proof are the following: the factorization (2), Theorem 3, and Lemma 4. Also the routine technique of subsequences being a consequence of the Prokhorov theorem will be applied.

Proof of Theorem 2. *Step* 1. Observe that the assumption (5) and the Tchebyshev inequality yield

$$P(|S_n| > \lambda) \leq \lambda^{-2} \sum_{i,j=1}^{k_n} E(X_{n,i}X_{n,j}) \to \lambda^{-2}\sigma^2$$

as $n \to \infty$. Hence the sequence of distributions of S_n, $n = 1, 2, \ldots$, is tight. By the Prokhorov theorem, each subsequence contains a weakly convergent subsequence. Consequently without losing generality we can assume that S_n converges in distribution. Thus, to prove that the limit distribution is normal $\mathcal{N}(0, \sigma^2)$ it suffices to show that

$$\lim_{n\to\infty} E(\exp(itS_n)) = \exp\left(-\frac{t^2}{2}\sigma^2\right) \tag{17}$$

only in some neighborhood of the origin.

Step 2. Take any $r = 1, \ldots, k$. Define independent r.v's

$$Y^{(j)}_{n;i_1,\ldots,i_r} \stackrel{d}{=} X_{n,i_1} + \cdots + X_{n,i_r}$$

for $j = 1, \ldots, |a_{k_n,k,r}|$, where the coefficients $a_{k_n,k,r}$ are defined in (12) (see also (2)), $1 \leq i_1 < \cdots < i_r \leq k_n$. In this step we prove that Theorem 3 holds for the Y's.

Notice that by the definition and an elementary inequality for the third moment of the sum

$$\begin{aligned} S &= \sum_{1\leq i_1<\cdots<i_r\leq k_n} \sum_{j=1}^{|a_{k_n,k,r}|} E(|Y^{(j)}_{n;i_1,\ldots,i_r}|^3) = |a_{k_n,k,r}| \sum_{1\leq i_1<\cdots<i_r\leq k_n} E(|Y^{(j)}_{n;i_1,\ldots,i_r}|^3) \\ &\leq M(r)|a_{k_n,k,r}| \sum_{1\leq i_1<\cdots<i_r\leq k_n} (E(|X_{n,i_1}|^3) + \cdots + E(|X_{n,i_r}|^3)), \end{aligned}$$

where $M(r)$ is a number dependingly only on r. Now by (10) of Lemma 1

$$S \leq M(r)|a_{k_n,k,r}| \binom{k_n - 1}{r - 1} \sum_{j=1}^{k_n} E(|X_{n,j}|^3).$$

Due to (6) to prove the Lyapounov condition for the Y's it suffices to show that

$$\binom{n-1}{r-1} a_{n,k,r} n^{-(k-1)} \to c(k, r) \tag{18}$$

as $n \to \infty$, where $c(k, r)$ is a real number. Apply induction with respect to k. For $k = 1$ (18) is obvious since then $r = 1$ and $a_{n,1,1} = 1$. Now assume that (18) holds for some $k \geq 1$. Then by (13)

$$\binom{n-1}{r-1} a_{n,k+1,r} n^{-k} = \binom{n-1}{r-1} a_{n,k,r} n^{-k} + (-1)^{k-r+1} n^{-k} \binom{n-1}{r-1} \binom{n-r}{k-r+1}.$$

Hence we obtain (18) since the first term tends to zero by the induction assumption and the second is equal to $c(k, r)(n-1)\cdots(n-k)n^{-k}$.

Consequently, Theorem 3 holds for the Y's.

Step 3. Define for any $r = 1, \dots, k$

$$\beta(t, r, n) = \exp\left(\frac{t^2}{2} \sum_{1 \le i_1 < \cdots < i_r \le k_n} E((X_{n,i_1} + \cdots + X_{n,i_r})^2)\right).$$

Then the result of Step 2 may be formulated as follows:

$$\lim_{\to\infty} \left[\beta(t, r, n) \prod_{1 \le i_1 < \cdots < i_r \le k_n} \phi_{n;i_1,\dots,i_r}(t, \dots, t)\right]^{a_{k_n,k,r}} = 1 \tag{19}$$

for any $r = 1, \dots, k$, where $\phi_{n;i_1,\dots,i_r}$ is the characteristic function of the marginal $(X_{n,i_1}, \dots, X_{n,i_r})$. Hence there is a neighborhood V of the origin such that for sufficiently large n and all $t \in V$

$$\prod_{1 \le i_1 < \cdots < i_r \le k_n} \phi_{n;i_1,\dots,i_r}(t, \dots, t) \ne 0,$$

$r = 1, \dots, k$. Now the formula (2) for $t \in V$ and large n yields

$$\phi_{S_n}(t) = \prod_{r=1}^{k} \left[\prod_{1 \le i_1 < \cdots < i_r \le k_n} \phi_{n;i_1,\dots,i_r}(t, \dots, t)\right]^{a_{k_n,k,r}}.$$

Multiplying and dividing the above equation by

$$\prod_{r=1}^{k} [\beta(t, r, n)]^{a_{k_n,k,r}}$$

we obtain

$$\begin{aligned}\phi_{S_n}(t) = \prod_{r=1}^{k} &\left[\prod_{1 \le i_1 < \cdots < i_r \le k_n} \beta(t, r, n)\phi_{n;i_1,\dots,i_r}(t, \dots, t)\right]^{a_{k_n,k,r}} \\ &\cdot \exp\left(-\frac{t^2}{2} E\left\{\sum_{r=1}^{k}\left[a_{k_n,k,r} \sum_{1 \le i_1 < \cdots < i_r \le k_n} (X_{n,i_1} + \cdots + X_{n,i_r})^2\right]\right\}\right).\end{aligned}$$

Applying (16) from Lemma 4 for $a_i = X_{n,i}$ we find that

$$\sum_{r=1}^{k}\left[a_{k_n,k,r} \sum_{1 \le i_1 < \cdots < i_r \le k_n} (X_{n,i_1} + \cdots + X_{n,i_r})^2\right] = \sum_{i,j=1}^{k_n} X_{n,i} X_{n,j}.$$

Hence by (19) and (5) the final result (17) follows. □

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