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Gaussian processes via independence of linear forms

AGNIESZKA PŁUCIŃSKA AND JACEK WESOŁOWSKI

Mathematical Institute, Warsaw University of Technology, Plac Politechniki 1, 00-661 Warsaw, Poland

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Abstract. A simple property of the covariance matrix, which ensures that infinite random sequences with independent linear forms are Gaussian, is discovered. For continuous time stochastic processes it is simplified further by assuming some consistency conditions. The results are parallel to the known characterizations of Gaussian processes by Gaussian conditional structures of the second order.

1. Introduction

Let $X = (X_t)_{t\in T}$, $T = \{0, 1, \ldots, N\}$, where N is finite (or infinite, or $T = [0, \infty)$), be a square integrable zero mean stochastic process defined on a probability space $(\Omega, \mathcal{F}, P)$ with a covariance function K. Additionally assume that for any $0 \leq t_1 < t_2 < \cdots < t_n$, $n = 1, 2, \ldots$

$$K^{(n)}(t_1, \ldots, t_n) = \det[K(t_i, t_j)]_{i,j=1,\ldots,n} > 0, \tag{1}$$

i.e. $X_{t_1}, \ldots, X_{t_n}$ are linearly independent functions on Ω.

We say that X is an ILF process (process with Independent Linear Forms) or it fulfils an ILF condition if for any $0 \leq t_1 < \cdots < t_n$, $n = 1, 2, \ldots$, there exist real functions $c_{ij} = c_{ij}(t_1, \ldots, t_n)$, $i = 1, \ldots, j-1$, $j = 2, 3, \ldots, n$, such that

$$X_{t_1}, \quad X_{t_2} - c_{1,2}X_{t_1}, \quad \ldots, \quad X_{t_n} - c_{n-1,n}X_{t_{n-1}} - \cdots - c_{1,n}X_{t_1} \tag{2}$$

are independent. Our aim in this paper is to find a property of the covariance function K which, together with the ILF condition, sufficiently characterizes the Gaussian process. A suggestion that such assumptions may determine finite dimensional distributions of the process gives the Darmois–Skitovitch Theorem on characterizing the normal law by independence of two linear forms in independent r.v's, but only for those r.v's which are present in both linear forms — see for instance Feller (1971), Ch. III.

Observe that the ILF condition implies linearity of regression, i.e.

$$E(X_{t_n} \mid X_{t_1}, \dots, X_{t_{n-1}}) = c_{1,n} X_{t_1} + \cdots + c_{n-1,n} X_{t_{n-1}}$$

and homoscedasticity, i.e.

$$\mathrm{Var}(X_{t_n} \mid X_{t_1}, \dots, X_{t_{n-1}}) = \mathrm{const}(t_1, \dots, t_n).$$

These two relations strengthened by adding in the conditions also future states are the core of the definition of Gaussian conditional structures of the second order (GCS_2) introduced in Wesolowski (1991). GCS_2 continuous time stochastic processes were first considered by Plucińska (1983), where it was proved that they are Gaussian if and only if the covariance function fulfills some conditions of the Lipschitz type. In other papers: Wesolowski (1984), Bryc (1985), Bryc and Szablowski (1990), it was shown that essentially the continuity of the covariance function is sufficient to characterize the Gaussian process by the GCS_2 condition. However in the case of GCS_2 infinite sequences, Bryc and Plucińska (1985) needed an additional assumption on the form of the covariance function for characterizing the Gaussian measure. Namely, they assumed additionally that

$$c_{n-1,n} \neq 0 \tag{3}$$

for all $0 \le t_1 < \cdots < t_n$, $n = 2, 3, \dots$. Observe that this is essentially a condition imposed on the covariance matrix since

$$c_{i,n} = -\frac{K_{i,n}^{(n)}}{K^{(n-1)}}, \tag{4}$$

where $K_{i,n}^{(n)}$ is the cofactor of the element $K(t_i, t_n)$ in the matrix $[K(t_i, t_j)]_{i,j=1,\dots,n}$, see Plucińska (1984) for the proof of this formula.

In Section 2 we will first consider ILF discrete time processes and give an analogue of condition (3) for the covariance matrix which guarantees that the sequence is Gaussian. As an auxiliary result a characterization of the finite dimensional Gaussian measure is used. Then continuous time processes are investigated. It is proved that some consistency assumptions allows us to weaken the conditions imposed on the covariance matrix in the discrete time case. The proof of the finite dimensional version of the characterization involving laborious computations in terms of cofactors of the covariance matrix is given in Section 3. It is the technical core of the paper.

2. ILF processes with discrete and continuous time

The main result of this section is an analogue of the characterization obtained for GCS_2 random sequences with an additional assumption (3) in Bryc and Plucińska (1985). Instead of linearity of regression and homoscedasticity involving conditioning with respect to past and future states we assume the independence condition leading only to Gaussian second order conditional structures for conditioning with respect to the past.

Theorem 1. *Let $X = (X_n)_{n=1,2\ldots}$ be a square integrable zero mean random sequence fulfilling* (1). *Assume that it has the ILF property. If for any $1 \leq t_1 < \cdots < t_n$, $t_i \in \{1, 2, \ldots\}$, $i = 1, 2, \ldots, n$, $n = 2, 3, \ldots$,*

$$K_{1,n}^{(n)} \neq 0 \tag{5}$$

then X is a Gaussian sequence.

The proof of this result is based on the following characterization of the finite dimensional Gaussian measure making use of a weaker version of the ILF condition. It is similar to the Darmois–Skitovitch Theorem, but without the independence of the components of the involved random vector. While a conjecture that some result of such a type holds true is quite natural since we increase the number of independent linear forms, instead of assuming independence among components, the main difficulty was to discover some convenient properties of the covariance matrix sufficient to obtain the characterization. Another version of the Darmois–Skitovitch Theorem, where weakening of the independence assumption was compensated by considering more than two linear forms, was given in Kagan (1988).

Proposition 1. *Let $(Y_1, \ldots, Y_m)$, $m > 2$, be a square integrable zero mean random vector with a covariance matrix K fulfilling* (1) *for any $n = 1, \ldots, m$. Assume that r.v's*

$$Y_1, \quad Y_2 - a_{1,2}Y_1, \quad \ldots, \quad Y_m - a_{m-1,m}Y_{m-1} - \cdots - a_{1,m}Y_1 \tag{6}$$

are independent for some real numbers $a_{i,j}$, $i = 1, \ldots, j-1$, $j = 1, \ldots, m$ and additionally the linear forms

$$Y_{m-1} - b_{m-2,m-1}Y_{m-2} - \cdots - b_{2,m-1}Y_2, \quad Y_m - b_{m-1,m}Y_{m-1} - \cdots - b_{2,m}Y_2 \tag{7}$$

are independent for some real numbers b_{ij}, $i = 1, \ldots, j-1$, $j = m-1, m$. If condition (5) *holds for all $n = 2, \ldots, m$ then $(Y_1, \ldots, Y_{m-1})$ is a Gaussian random vector.*

The proof of Proposition 1, which essentially is a quite laborious computation involving cofactors of the covariance matrix, is put off until Section 3. Now we will quickly show how Theorem 1 follows from the above result.

Proof of Theorem 1. Fix any integer $m > 2$ and any $0 \leq t_1 < \cdots < t_m$, and let $Y_i = X_{t_i}$, $i = 1, \ldots, m$. Then by the ILF condition we have (6) with $a_{ij} = c_{ij}(t_1, \ldots, t_j)$, $i = 1, \ldots, j-1$, $j = 1, \ldots, m$ and (7) with $b_{ij} = c_{ij}(t_2, \ldots, t_j)$, $i = 2, \ldots, j-1$, $j = m-1, m$. Consequently by Proposition 1 the random vector $(X_{t_1}, \ldots, X_{t_{m-1}})$ is Gaussian for any $0 \leq t_1 < \cdots < t_{m-1}$ and any integer $m > 2$. Hence all finite dimensional distributions of the random sequence X are Gaussian.

Now consider the case when $T = [0, \infty)$. Let us introduce a natural consistency condition of the following form: for any $0 \leq t_1 < \cdots < t_n$, $n \geq 4$

$$\lim_{t_{n-1} \to t_{n-2}} E(X_{t_n} \mid X_{t_1}, \ldots, X_{t_{n-1}}) = E(X_{t_n} \mid X_{t_1}, \ldots, X_{t_{n-2}}), \tag{8}$$

where the limit is understood to be in a quadratic mean or in a probability one. A similar condition proved to be useful in the investigations of the memory of stochastic processes with linearity of regression lead in Plucińska (1988).

Additionally our considerations will be limited to ILF processes with the property: for any $n > 3$ and any $0 \leq t_1 < \cdots < t_{n-1} < t_n$ the following implication holds true

$$c_{1,n}(t_1, \ldots, t_n) = 0 \quad \Rightarrow \quad c_{1,n}(t_1, \ldots, t_{n-2}, s, t_n) = 0, \qquad \forall s \in [t_{n-2}, t_{n-1}], \tag{9}$$

where c's are coefficients of linear forms and, at the same time, regression coefficients (see the formulas in Section 1). Consequently the above implication may be rewritten as

$$E(X_{t_n} \mid X_{t_1}, \ldots, X_{t_{n-1}}) = E(X_{t_n} \mid X_{t_2}, \ldots, X_{t_{n-1}})$$

which implies

$$E(X_{t_n} \mid X_{t_1}, \ldots, X_{t_{n-2}}, X_s) = E(X_{t_n} \mid X_{t_2}, \ldots, X_{t_{n-2}}, X_s), \qquad \forall s \in [t_{n-2}, t_{n-1}].$$

Let us point out that some interesting consequences of the vanishing of some regression coefficients in models with linear regression were discovered in Plucińska (1988).

For continuous time we have the following characterization

Theorem 2. *Let $X = (X_t)_{t \in [0,\infty)}$ be a square integrable zero mean stochastic process fulfilling* (1). *Assume that it has the ILF property and that consistency conditions* (8) *and* (9) *are satisfied. If for any $0 \leq t_1 < t_2 < t_3$*

$$K_{1,n}^{(n)} \neq 0 \tag{10}$$

for $n = 2, 3$ then X is a Gaussian process.

If X is a Markov process then for any $0 \leq r < s < t$

$$K(s,s)K(r,t) = K(r,s)K(s,t)$$

and consequently $K_{1,3}^{(3)} = 0$, hence the method we develop here is not applicable in that case. Observe that for the ILF Markov process fulfilling (10) for $n = 2$ there is a real function $g(t) = K(0,t)K^{-1}(t,t)$ such that $(X_t g^{-1}(t))_{t\in[0,\infty)}$ is a process with independent increments. In a recent paper by Plucińska (1994) it was shown that if in addition linearity of regression with respect to the σ-field generated not only by the past but also by one future state of the process holds, then the process is Gaussian or $(X_t g^{-1}(t))_{t\in[0,\infty)}$, where g is a real positive function with independent increments. Additionally it was assumed that $E(X_t^4) < \infty$, for all $t \in [0,\infty)$. It remains an open question if only these two types of stochastic processes have the ILF property in the case when assumption (10) or the linearity of regression with respect also to future states are replaced by, for instance, continuity of the covariance function. This question seems to be quite natural for continuous time processes since the continuity of the covariance function ensures a characterization of the Gaussian process by the GCS_2 condition — see Section 1.

From the proof given below it follows easily that in the case of continuous covariance function it is sufficient to impose condition (10) only for (t_1, t_2, t_3) from a dense set in $[0,\infty)^3$.

Proof of Theorem 2. By Theorem 1 it is sufficient to prove that assumption (5) from this Theorem holds. We apply induction with respect to n. For $n = 2, 3$, it is obvious. Now assume that (5) holds for some $m \geq 3$, i.e. for any $0 \leq t_1 < \cdots < t_m$ and let $K_{1,m+1}^{(m+1)} = 0$ for some $0 \leq t_1 < \cdots < t_{m+1}$. We will prove that it is impossible. The ILF assumption implies

$$E(X_{t_{m+1}} \mid X_{t_1}, \ldots, X_{t_m}) = c_{1,m+1}(t_1, \ldots, t_{m+1})X_1 + \cdots + c_{m,m+1}(t_1, \cdots, t_{m+1})X_{t_m}, \tag{11}$$

and by (8)

$$\begin{aligned} \lim_{t_m \to t_{m-1}} E(X_{t_{m+1}} \mid X_{t_1}, \ldots, X_{t_m}) &= E(X_{t_{m+1}} \mid X_{t_1}, \ldots, X_{t_{m-1}}) \\ &= c_{1,m+1}(t_1, \ldots, t_{m-1}, t_{m+1})X_{t_1} + \cdots \\ &\quad + c_{m-1,m+1}(t_1, \ldots, t_{m-1}, t_{m+1})X_{t_{m-1}}. \end{aligned} \tag{12}$$

On the other hand by (4) the coefficient $c_{1,m+1}(t_1, \ldots, t_{m+1})$ in (11) equals zero. Consequently

$$E(X_{t_{m+1}} \mid X_{t_1}, \ldots, X_{t_m}) = E(X_{t_{m+1}} \mid X_{t_2}, \ldots, X_{t_m}).$$

Hence by (8) and (9)

$$\lim_{t_m \to t_{m-1}} E(X_{t_{m+1}} \mid X_{t_1}, \ldots, X_{t_m}) =$$
$$= c_{2,m+1}(t_2, \ldots, t_{m-1}, t_{m+1})X_{t_2} + \cdots + c_{m-1,m+1}(t_2, \ldots, t_{m-1}, t_{m+1})X_{t_{m-1}}$$

which contradicts (12) since by the induction assumption

$$c_{1,m+1}(t_1,\ldots,t_{m-1},t_{m+1}) \neq 0.$$

3. Finite dimensional case

This section contains a proof of the main technical result, i.e. Proposition 1. It involves laborious calculations in terms of cofactors of the covariance matrix. One of the tricks we will often use is the following formula for cofactors of a symmetric matrix $K = [K(i,j)]_{i,j=1,\ldots,m}$

$$K^{(m)} K^{(m)}_{i,j,n,n} + K^{(m)}_{i,n} K^{(m)}_{j,n} = K^{(m)}_{i,j} K^{(m)}_{n,n} \tag{13}$$

for any $i \neq n$, $j \neq n$, $i,j,n \in \{1,\ldots,m\}$, $m = 2,3,\ldots$, where $K^{(m)} = \det(K)$, $K^{(m)}_{i,j}$ is a cofactor of the element $K(i,j)$ in the matrix K and $K^{(m)}_{i,j,n,n}$ is a cofactor of the element $K(i,j)$ in a matrix created from K by deleting the n-th row and n-th column. This formula is taken from Plucińska (1984).

Fix an arbitrary integer m and introduce for a covariance $m \times m$ matrix K quantities similar to those from (4)

$$c_{j,k}(1,\ldots,j,k) = -\frac{K^{(j,k)}_{j,k}}{K^{(j,k)}_{k,k}},$$

where $K^{(j,k)}_{j,k}$ is the cofactor of the element $K(j,k)$ in the matrix created from the matrix K by deleting all the rows and columns of numbers $j+1,\ldots,k-1,k+1,\ldots,m$, $j<k$, $K^{(k,k)}_{k,k} = 1$, $k = 1,\ldots,m$. The following identity will be used in the course of the proof later in this section.

Lemma 1. *For any* $j = 1,\ldots,m-1$

$$\sum_{k=j}^{m-1} c_{k,m}(1,\ldots,m)c_{j,k}(1,\ldots,j,k) = c_{j,m}(1,\ldots,j,m). \tag{14}$$

Proof. Consider the Gaussian random vector $(V_1,\ldots,V_m)$ with the covariance matrix K. Then

$$E(V_m \mid V_1,\ldots,V_j) = \sum_{i=1}^{j} c_{i,m}(1,\ldots,j,m)V_i.$$

On the other hand

$$\begin{aligned} E(V_m \mid V_1,\ldots,V_j) &= E(E(V_m \mid V_1,\ldots,V_{m-1}) \mid V_1,\ldots,V_j) \\ &= E\Big(\sum_{i=1}^{m-1} c_{i,m}(1,\ldots,m)V_i \mid V_1,\ldots,V_j\Big) \end{aligned}$$

$$= \sum_{i=1}^{j} c_{i,m}(1,\ldots,m)V_i + \sum_{i=j+1}^{m-1} c_{i,m}(1,\ldots,m)\, E(V_i \mid V_1,\ldots,V_j)$$

$$= \sum_{i=1}^{j} c_{i,m}(1,\ldots,m)V_i + \sum_{i=j+1}^{m-1} c_{i,m}(1,\ldots,m) \sum_{l=1}^{j} c_{l,i}(1,\ldots,j,i)V_l$$

$$= \sum_{i=1}^{j}\Big(\sum_{k=i}^{m-1} c_{k,m}(1,\ldots,m)\, c_{i,k}(1,\ldots,i,k)\Big)V_i.$$

Consequently comparing in both the expressions for $E(V_m \mid V_1,\ldots,V_j)$ the coefficients of V_j we get (14).

The next identity for the covariance matrix K is another important tool in the proof of Proposition 1.

Lemma 2. *For any* $k = 1,2,\ldots,m$

$$K_{1,k}^{(k)}K^{(m)} = K^{(k)}K_{1,k}^{(m)} - \sum_{i=k+1}^{m} K_{k,i}^{(k,i)}K_{1,i}^{(m)}. \tag{15}$$

Proof. We apply backward induction with respect to k. For $k = m$ this is obvious since the sum after the minus sign on the right hand side of (15) is empty. Take now $k = m-1$ and compute the right hand side with the help of (13)

$$\begin{aligned} K_{1,m-1}^{(m)}K^{(m-1)} - K_{m-1,m}^{(m-1,m)}K_{1,m}^{(m)} &= K_{1,m-1}^{(m)}K_{m,m}^{(m)} - K_{m-1,m}^{(m)}K_{1,m}^{(m)} \\ &= K^{(m)}K_{1,m-1,m,m}^{(m)} = K^{(m)}K_{1,m-1}^{(m-1)}. \end{aligned}$$

Now assume that (15) holds for $k+1, k+2,\ldots,m$. We will prove that (15) is also fulfilled for k. To this end observe that

$$\begin{aligned} K_{1,k}^{(k)}K^{(m)} &= K_{1,k}^{(m)}K^{(k)} - K^{(k)}K^{(m)}\sum_{j=k+1}^{m}\left(\frac{K_{1,k}^{(j)}}{K^{(j)}} - \frac{K_{1,k}^{(j-1)}}{K^{(j-1)}}\right) \\ &= K_{1,k}^{(m)}K^{(k)} - K^{(k)}K^{(m)}\sum_{j=k+1}^{m}\frac{K_{1,k}^{(j)}K_{j,j}^{(j)} - K^{(j)}K_{1,k,j,j}^{(j)}}{K^{(j-1)}K^{(j)}}. \end{aligned}$$

Hence by (13) we have

$$K_{1,k}^{(k)}K^{(m)} = K_{1,k}^{(m)}K^{(k)} - K^{(k)}\sum_{j=k+1}^{m}\frac{K_{k,j}^{(j)}}{K^{(j-1)}K^{(j)}}K_{1,j}^{(j)}K^{(m)}.$$

Apply in the above identity the induction assumption for $K_{1,j}^{(j)}K^{(m)}$, $j = k+1,\ldots,m$. Then

$$K_{1,k}^{(k)}K^{(m)} = K_{1,k}^{(m)}K^{(k)} - K^{(k)}\sum_{j=k+1}^{m}\frac{K_{k,j}^{(j)}}{K^{(j-1)}K^{(j)}}$$

$$\left(K_{1,j}^{(m)}K^{(j)} - K_{j,j+1}^{(j,j+1)}K_{1,j+1}^{(m)} - K_{j,j+2}^{(j,j+2)}K_{1,j+2}^{(m)} - \cdots - K_{j,m-1}^{(j,m-1)}K_{1,m-1}^{(m)} - K_{j,m}^{(j,m)}K_{1,m}^{(m)}\right)$$

and after calculation

$$K_{1,k}^{(k)}K^{(m)} = K_{1,k}^{(m)}K^{(k)} -$$

$$- K^{(k)} \sum_{l=k+1}^{m} K_{1,l}^{(m)} \left(\frac{K_{k,l}^{(l)}}{K^{(l-1)}} - \frac{K_{l-1,l}^{(l-1,l)} K_{k,l-1}^{(l-1)}}{K^{(l-2)} K^{(l-1)}} - \cdots - \frac{K_{k+1,l}^{(k+1,l)} K_{k,k+1}^{(k+1)}}{K^{(k)} K^{(k+1)}} \right). \qquad (16)$$

On the other hand for any $1 \leq k < j < l$

$$\frac{K_{k,l}^{(j,l)}}{K^{(j)}} - \frac{K_{j,l}^{(j,l)} K_{k,j}^{(j)}}{K^{(j-1)} K^{(j)}} = \frac{K^{(j,l)} K_{j,j,l,l}^{(j,l)} K_{k,l}^{(j,l)} - K^{(j,l)} K_{k,j,l,l}^{(j,l)} K_{j,l}^{(j,l)}}{K^{(j)} K^{(j-1)} K^{(j,l)}}.$$

Applying in the above formula identity (13) twice we get

$$\frac{K_{k,l}^{(j,l)}}{K^{(j)}} - \frac{K_{j,l}^{(j,l)} K_{k,l}^{(j)}}{K^{(j-1)} K^{(j)}} = \frac{K_{k,l}^{(j,l)} K_{j,j}^{(j,l)} - K_{j,l}^{(j,l)} K_{k,j}^{(j,l)}}{K^{(j-1)} K^{(j,l)}}$$

and now after using once again (13) we obtain

$$\frac{K_{k,l}^{(j,l)}}{K^{(j)}} - \frac{K_{j,l}^{(j,l)} K_{k,j}^{(j)}}{K^{(j-1)} K^{(j)}} = \frac{K_{k,l}^{(j-1,l)}}{K^{(j-1)}}. \qquad (17)$$

Hence we have

$$\frac{K_{k,l}^{(l)}}{K^{(l-1)}} - \sum_{j=k+1}^{l-1} \frac{K_{j,l}^{(j,l)} K_{k,j}^{(j)}}{K^{(j-1)} K^{(j)}} = \frac{K_{k,l}^{(k,l)}}{K^{(k)}} \qquad (18)$$

by applying (17) successively for $j = l-1, l-2, \ldots, k+1$ to the left hand side. Now we end the proof by putting (18) into (16).

Observe that for $k = 1$ formula (15) follows immediately from the Laplace expansion with respect to the first row of the determinant $K^{(m)}$.

We are now ready to prove Proposition 1.

Proof of Proposition 1. Denote by $Z_1, \ldots, Z_m$ the linear forms appearing in (6), respectively, and by U_{m-1}, U_m the linear forms from (7), respectively.

We will prove the result in four steps.

First step. It is sufficient to show that

$$U_n = \sum_{j=1}^{n} \alpha_{j,n} Z_j, \qquad n = m-1, m,$$

where

$$\alpha_{j,n} = -\frac{K_{1,j}^{(j)} K_{1,n}^{(n)}}{K^{(j)} K_{1,1}^{(n-1)}}, \qquad n = m-1, m, \quad j = 1, 2, \ldots, n-1 \qquad (19)$$

and $\alpha_{n,n} = 1$, $n = m-1, m$. Then by (5)

$$K_{1,j}^{(j)} K_{1,m-1}^{(m-1)} K_{1,m}^{(m)} \neq 0, \qquad j = 1, 2, \ldots, m-1$$

and consequently each Z_j, $j = 1, \ldots, m-1$, is a member of both the linear forms U_{m-1}, U_m. Since Z's are independent then from the Darmois–Skitovitch Theorem we conclude that they are normal. Hence the random vector $(Y_1, \ldots, Y_{m-1})$ as a linear transformation of independent normal r.v's has an $(m-1)$-variate Gaussian distribution.

Second step. In this step we will show that for $k = 1, \ldots, m$

$$Y_k = \sum_{j=1}^{k} c_{j,k}(1, \ldots, j, k) Z_j. \tag{20}$$

Apply induction with respect to k.

For $k = 1, 2$ it is obvious since $a_{j,k} = c_{j,k}(1, \ldots, j, k)$. From the definition and the induction assumption for $1, 2, \ldots, k-1$ we have

$$Y_k = \sum_{i=1}^{k-1} c_{i,k}(1, \ldots, m)\Big(\sum_{j=1}^{i} c_{j,i}(1, \ldots, j, i) Z_j\Big) + Z_k =$$

$$= \sum_{j=1}^{k-1}\sum_{i=j}^{k-1} c_{i,k}(1, \ldots, k)\, c_{j,i}(1, \ldots, j, i) Z_j + Z_k.$$

Now the result follows by Lemma 1.

Third step. In this step we will prove that for any $k = 2, \ldots, n-1$, $n = m-1, m$

$$\alpha_{n,k} = \sum_{i=k}^{n-1} c_{k,i}(1, \ldots, k, i)\,[c_{i,n}(1, \ldots, n) - c_{i,n}(2, \ldots, n)]. \tag{21}$$

From (20) and the definition of the U's it follows that

$$U_n = \sum_{j=1}^{n} c_{j,n}(1, \ldots, j, n) Z_j - \sum_{i=2}^{n-1} c_{i,n}(2, \ldots, n)\Big[\sum_{j=1}^{i} c_{j,i}(1, \ldots, j, i) Z_j\Big]$$

and after the calculation we have

$$\begin{aligned} U_n \;&=\; Z_n + \sum_{k=2}^{n-1}\Big[c_{k,n}(1, \ldots, k, n) - \sum_{i=k}^{n-1} c_{i,n}(2, \ldots, n)\, c_{k,i}(1, \ldots, k, i)\Big] Z_k + \\ &+ \Big[c_{1,n}(1, n) - \sum_{i=2}^{n-1} c_{i,n}(2, \ldots, n)\, c_{1,i}(1, i)\Big] Z_1. \end{aligned}$$

Hence for $k = 2, \ldots, n-1$

$$\alpha_{n,k} = c_{k,n}(1, \ldots, k, n) - \sum_{i=k}^{n-1} c_{i,n}(2, \ldots, n)\, c_{k,i}(1, \ldots, k, i).$$

Now apply to $c_{k,n}(1, \ldots, k, n)$ in the above formula the identity from Lemma 1 to obtain (21).

Similarly for $k = 1$ we have

$$\alpha_{n,1} = c_{1,n}(1, \ldots, n) + \sum_{i=2}^{n-1} c_{1,i}(1, i)\,[c_{i,n}(1, \ldots, n) - c_{i,n}(2, \ldots, n)]. \tag{22}$$

Fourth step. In this step it will be proved that (19) follows immediately from Lemma 2.

Observe that for any $i = 1, \ldots, n-1$, $n = m-1, m$,

$$c_{i,n}(1,\ldots,n) - c_{i,n}(2,\ldots,n) = \frac{K^{(n)}_{1,1,i,n} K^{(n-1)} - K^{(n)}_{i,n} K^{(n-1)}_{1,1}}{K^{(n-1)}_{1,1} K^{(n-1)}}.$$

Multiplying the numerator and denominator of the expression on the right hand side by $K^{(n)}$ and applying (13) twice we obtain

$$c_{i,n}(1,\ldots,n) - c_{i,n}(2,\ldots,n) = \frac{K^{(n)}_{1,n}\left[K^{(n)}_{1,n} K^{(n)}_{i,n} - K^{(n)}_{1,i} K^{(n)}_{n,n}\right]}{K^{(n-1)}_{1,1} K^{(n-1)} K^{(n)}}.$$

Once again applying (13) to the above equation we have

$$c_{i,n}(1,\ldots,n) - c_{i,n}(2,\ldots,n) = -\frac{K^{(n)}_{1,n} K^{(n)} K^{(n)}_{1,i,n,n}}{K^{(n-1)}_{1,1} K^{(n-1)}}.$$

Now (21) or (22) and the above identity give for any $k = 1, \ldots, n-1$

$$\alpha_{n,k} = -\frac{K^{(n)}_{1,n} K^{(n-1)}_{1,k}}{K^{(n-1)} K^{(n-1)}_{1,1}} + \sum_{i=k+1}^{n-1} \frac{K^{(k,i)}}{K^{(k)}} \frac{K^{(n)}_{1,n} K^{(n-1)}_{1,i}}{K^{(n-1)} K^{(n-1)}_{1,1}} =$$

$$= -\frac{K^{(n)}_{1,n}}{K^{(k)} K^{(n-1)} K^{(n-1)}_{1,1}}\left[K^{(n-1)}_{1,k} K^{(k)} - \sum_{i=k+1}^{n-1} K^{(k,i)}_{k,i} K^{(n-1)}_{1,i}\right].$$

Hence the result follows from Lemma 2.

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