Journal of Applied Statistical Science
Volume 2 Number 4, pp. 335-343

ISSN 1067-5817

REFINEMENTS OF THE KAGAN-LINNIK-RAO AND KAGAN-RAO THEOREMS

J. Pusz and J. Wesołowski

Institute of Mathematics, University of Technology, 00-661 Warsaw, Plac Politechniki 1

Abstract: The Kagan-Linnik-Rao theorem on characterization of the normal law by constant regression of the sample mean on residuals is extended by widening the class of conditioned random variables. The integrability assumption introduced in Kagan-Rao theorem on characterization of the normal law by constant regression of a polynomial in the sample mean on residuals may be reduced to the minimal necessary condition.

Key words and phrases: characterization of distribution, constant regression, normal law, integrability, residuals.

AMS (1980) subject classification: 62E10.

1. Introduction

At the begining of numerous investigations in characterization of probability distributions by constant regression of some functions of independent r.v's on residuals stands the following celebrated result:

<u>*THEOREM 1*</u> (Kagan, Linnik, Rao (1965)). Let $X_1,\ldots,X_n$ be i.i.d. r.v's with $EX_1 = 0$, $n \geq 3$. If

$$E(\bar{X} \mid X_1 - \bar{X},\ldots,X_n - \bar{X}) = 0 ,$$

where $\bar{X} = (X_1 + \ldots + X_n)/n$, then X_1 is a normal r.v.

This and other results in this area are partial solutions of the following general problem:

Let $X_1,\ldots,X_n$ be independent r.v's and let h be a real measurable function on $\mathbb{R}^n$, such that $E|h(X_1,\ldots,X_n)| < \infty$. Assume that

(1) $$E(h(X_1,\ldots,X_n)|X_1-\bar{X},\ldots,X_n-\bar{X}) = const\ .$$

What are the distributions of X's?

As for as we know the following questions were investigated up to now (details are omitted):

a) If $h=\bar{X}$ then X's are normal.

b) If $h=\bar{X}^2$ then X's are normal.

c) If $h=P(\bar{X})$, where P is a polynomial then X's are normal.

d) If $h=\sum_{k=1}^{n} X_k^{-1}$ then X's are gamma.

e) If $h=\sum_{k=1}^{n} g(X_k)$, where g is locally integrable on $\mathbf{R}$ and characteristic functions have only "short gaps" then X's have a density of the form $f(x)=\exp(\alpha\int_0^x (g(y)-\mu)dy)$.

f) If $h=\prod_{k=1}^{n} X_k$ then X's are Poisson.

g) If $h=\prod_{k=1}^{n} X_k^{-1}$ then X's are Poisson.

h) If $h=\exp(\alpha\bar{X})$ then X's have densities being a product of normal density and some periodic function (if (1) holds for two incommesurable values of α's then X's are normal).

i) If $h=\prod_{k=1}^{n}(1+X_k^{-1})$ then X's are binomial or negative binomial.

j) If $h=\prod_{k=1}^{n}(X_k-\theta_1)-c\prod_{k=1}^{n}(X_k-\theta_2)$ then X's are binomial or negative binomial.

k) If $h=\prod_{k=1}^{n} Z_k-(-1)^n\prod_{k=1}^{n} Z_k^{*}$, where $Z_k=(X_k+i\alpha_k)/(m_k+i\alpha_k)$, and Z_k^{*} is its conjugate then X's are generalized hyperbolic secant .

For details see: Kagan et al.(1965) for (a), Kagan et al.(1973) for (a,d), Wesolowski (1987) for (b), Kagan and Rao (1988) for (c), Khatri and Rao (1968) for (d), Bondesson (1974) for (e), Wesołowski (1989) for (f), Wesołowski (1992) for (g,h,i), Pusz (1992) for (j), and Pusz (1993) for (k).

It should be emphasized that problems of this kind are closely related to the following question of estimation theory: Let $X_1,\dots,X_n$ be a sample from the location family with a distribution function $F = F(\cdot - \theta)$. Assume that $h(X_1,\dots,X_n)$ is an unbiased estimator of some parametric function. What are possible distributions F if h is a minimum variance estimator? For further details in this direction see Kagan et al.(1973), Kagan (1989) or Wesołowski (1992).

In this paper we are concerned with two issues. In Section 2 we deal with a characterization by the condition (1) with $h(x_1,\dots,x_n) = H(x_1)\sum_{k=1}^{n} a_k x_k - G(x_1)$, obtaining a straightforward generalization of Theorem 1. In Section 3 we prove that the integrability condition imposed in the Kagan-Rao characterization, mentioned in (c), may be reduced to the minimal necessary condition.

2. On the Kagan-Linnik-Rao theorem

Let $X_1,\dots,X_n$ be independent r.v's, H, G two real measurable functions and $a_1,\dots,a_n$ some real numbers. Consider (1) in the following version

$$E(H(X_1)\sum_{k=1}^{n} a_k X_k \mid Y) = E(G(X_1) \mid Y), \tag{2}$$

where $Y = (X_1-\bar{X},\dots,X_n-\bar{X})$ is the vector of residuals. Obviously for $H = const$, $G = 0$ and $a_k=1/n$, $k=1,\dots,n$, (2) takes the form of the condition from Theorem 1. Hence the result given in Theorem 2 below is a straightforward extension of that celebrated characterization. From (2) it will follow that $X_2,\dots,X_n$ are normal r.v's and for X_1 we obtain a functional equation in some neighborhood of the origin. This equation is discussed in the remarks following the formulation of Theorem 2. The proof of the main result ends this section.

THEOREM 2. Let $X_1,\ldots,X_n$, $n \geq 3$, be independent r.v's, H, G be real measurable functions and let $a_1,\ldots,a_n$ be real numbers. Assume that $H(X_1)$, $X_1H(X_1)$, $G(X_1), X_2,\ldots,X_n$ are integrable. If $EH(X_1) \neq 0$ and the condition (2) holds then X_k is Gaussian if only $a_k \neq 0$, $k=2,\ldots,n$ and in some neighborhood V of the origin

(3) $$E(a_1H(X_1)X_1 - G(X_1))e^{itX_1} = (At + B)EH(X_1)e^{itX_1},$$

where A and B are constants (B- real, A- possibly complex).

REMARKS:

1^0 Let $H(x) = \gamma x + \delta$ and $G(x) = a_1\gamma x^2 + (a_1\delta - \alpha)x - \beta$, where $\alpha, \beta, \gamma, \delta$ are real constants such that $\alpha^2 + \beta^2 + \gamma^2 + \delta^2 > 0$. Then (3) takes the form $(X = X_1)$

(4) $$E(\alpha X + \beta)e^{itX} = (At + B)E(\gamma X + \delta)e^{itX}.$$

Then three cases are possible: X is degenerate or Gaussian or gamma.

Take first $\alpha\delta = \beta\gamma$. Then (4) yields $E(\alpha X + \beta)\exp(itX) = 0 = E(\gamma X + \delta)\exp(itX)$. Hence it is easy to see that X is degenerate.

Assume now that $\alpha\delta \neq \beta\gamma$, and that X is nondegenerate. Then $\gamma m + \delta \neq 0$, where $m = EX$ (X is integrable by the assumption), and

(5) $$B = \frac{\alpha m + \beta}{\gamma m + \delta}.$$

By the form of the equation (4) we are allowed to differentiate its both sides. Putting then $t = 0$ we find that

(6) $$A = i\sigma^2 \frac{\alpha\delta - \beta\gamma}{(\gamma m + \delta)^2},$$

where $\sigma^2 = Var(X) > 0$.

Take now $\gamma = 0$. Hence $\alpha \neq 0$ and (4) yields for $t \in V$

$$(\ln\varphi(t))' = i\frac{\delta}{\alpha}At + i\frac{\delta B - \beta}{\alpha},$$

where φ is the characteristic function of X. Consequently by (5) and (6) we get $\varphi(t) = \exp(itm - t^2\sigma^2/2)$ in V. And thus X is a normal r.v.

Finally consider the case $\gamma \neq 0$. Then by (4) we obtain in V

$$\ln\varphi(t) = -i\frac{\delta}{\gamma}t - i\frac{\alpha\delta - \gamma\beta}{A\gamma^2}\ln(\gamma At + \gamma B - \alpha) + C,$$

where C is some constant. By (5), (6) and the fact that the gamma distribution is determined by its moments we conclude that X is a gamma type r.v.

2^0 Assume that H is such a function that $\bar{F}(x) = \int_{-\infty}^{x} HdF$ is a distribution function, where F is the distribution function of X. Let $G(x) = (a_1x - e^x)H(x)$ or $G(x) = (a_1x - \frac{1}{x})H(x)$ then (4) takes the form

$$(At + B)\int e^{itx}d\bar{F}(x) = \int e^x e^{itx}d\bar{F}(x) \ or \ \int \frac{1}{x}e^{itx}d\bar{F}(x).$$

It is well known (see Ch.6 in Kagan et al.(1973)) that in both the cases $\bar{F}$ is a gamma type distribution function. Consequently $H(x)dF(x) = \gamma(x)dx$, where γ is a gamma type density.

3^0 For further discussion of some problems connected with equation (4) see Bondesson (1974), where a theoretically useful notion of short gap for characteristic functions is introduced.

Proof of Theorem 2:

Firstly observe that in conditioning the residuals may be replaced by the vector $(X_2 - X_1, \ldots, X_n - X_1)$ (such conditioning was considered in the original KLR theorem). Then from (2) we obtain

$$EH(X_1)\sum_{k=1}^{n} a_kX_k\exp[i(-X_1\sum_{k=2}^{n} t_k + \sum_{k=2}^{n} t_kX_k)] = EG(X_1)\exp[i(-X_1\sum_{k=2}^{n} t_k + \sum_{k=2}^{n} t_kX_k)].$$

Now making use of the assumption $EH(X_1)\neq 0$ and considering $t_2,\dots,t_n$, sufficiently close to zero we have

$$\frac{E(a_1H(X_1)X_1 - G(X_1))\exp(-iX_1\sum_{k=2}^{n}t_k)}{EH(X_1)\exp(-iX_1\sum_{k=2}^{n}t_k)} + \sum_{k=2}^{n} a_k\frac{EX_ke^{it_kX_k}}{Ee^{it_kX_k}} = 0.$$

Since $n\geq 3$ then if only $a_k\neq 0$ by lemma 1.5.1 in Kagan et al.(1973) $\varphi_k'(t)=(A_kt+B_k)\varphi_k(t)$, where φ_k is a characteristic function of X_k, A_k, B_k are constant and $|t|$ is sufficiently small, $k=2,\dots,n$. Hence $\varphi_k(t)=A_kt^2/2+B_kt+C_k$ in some neighborhood of the origin. Consequently $X_2,\dots,X_n$ are normal since this distribution is determined by its moments. At the same time we have got also the equation (4).□

3. On the Kagan-Rao theorem

In Kagan and Rao (1988) the following characterization of the normal distribution is given: Assume that $X_1,\dots,X_n$ are i.i.d. $(k+1)$-integrable r.v's, P is a polynomial of degree $k\geq 2$ and $n\geq 2k$. Then $E(P(\bar{X})\,|\,X_1-\bar{X},\dots,X_n-\bar{X})=const$ iff X_1 is Gaussian. It was also proved in this paper that if $P(\bar{X})\geq 0$ then the assumption of $(k+1)$-integrability may be weakened to the minimal necessary condition $E|X_1|^k<\infty$.

The aim of this section is to prove that it suffices to assume k-integrability also in the general situation. Then the Kagan-Rao theorem takes the form:

THEOREM 3. Let $X_1,\dots,X_n$ be i.i.d. r.v's and P be a polynomial of degree $k\geq 2$. If $n\geq 2k$ and $E|X_1|^k<\infty$ then the condition

(7) $$E(P(\bar{X})|X_1-\bar{X},\dots,X_n-\bar{X}) = const$$

holds iff X_1 is Gaussian.

Proof. Thanks to the original Kagan-Rao theorem it suffices to prove that X_1 is $(k+1)$-

integrable. Similarly as in the proof of the Theorem 2 we have from (7)

$$E(P(\overline{X})\exp(i(t_1X_1+\ldots+t_nX_n))) = c\prod_{j=1}^{n}\varphi(t_j)$$

for $t_1+\ldots+t_n=0$, where φ is a characteristic function of X_1, and c a real constant. Put in the above equation $t_1=s$, $t_2=t$, $t_3=-s-t$, $t_4=\ldots=t_n=0$. Then for sufficiently small $|s|$ and $|t|$ we obtain

$$g^{(k)}(s)+g^{(k)}(t)+g^{(k)}(-s-t) = Q(g^{(1)}(s),g^{(1)}(t),g^{(1)}(-s-t),\ldots,g^{(k-1)}(s),g^{(k-1)}(t),g^{(k-1)}(-s-t)),$$

where $g=\ln\varphi$, $g^{(m)}$ is its m-th derivative and Q is a polynomial.

Integrate the above equation from 0 to $x>0$, sufficiently small, with respect to s

$$g^{(k-1)}(x) - g^{(k-1)}(0) + xg^{(k)}(t) + \int_0^x g^{(k)}(-s-t)ds = \int_0^x Qds.$$

Consequently

$$xg^{(k)}(t) = \int_0^x Qds + g^{(k-1)}(-x-t) - g^{(k-1)}(-t) + g^{(k-1)}(0) - g^{(k-1)}(x).$$

Since X_1 is k-integrable then the right hand side of this identity is differentiable with respect to t (in a neighborhood of the origin). Hence g is $(k+1)$-integrable and $E|X_1|^{k+1}<\infty$.□

In Kagan (1991) a method for solving a class of functional equations arising in some characterization problems (including the Kagan-Rao theorem) was presented. The assumption of $(k+1)$-integrability of the r.v's involved was replaced by the condition of $(k+1)$-differentiability of the characteristic function. Our argumentation from the above proof may be applied directly to reduce this condition to the minimal assumption of existing the derivative of the k-th order. For example Theorem 2.1 from Kagan (1991) takes the following form.

THEOREM 4. Let f be a k-differentiable characteristic function and $g=\ln f$ in a neighborhood of the origin. Consider for any real $t_1,\ldots,t_n$ such that $|t_1|<\varepsilon,\ldots,|t_n|<\varepsilon, \varepsilon>0$ and $b_1t_1+\ldots+b_nt_n=0$ the equation

$$P(S_1,\ldots,S_k) = 0,$$

where

$$S_j = \sum_{m=1}^{n} a_{jm} g^{(j)}(t_m), j=1,\ldots,k,$$

$a_{jm}, j=1,\ldots,k, m=1,\ldots,n$, are some real constants and P is a polynomial of the form

$$P(u_1,\ldots,u_k) = Au_1^k + Bu_k + P_1(u_1,\ldots,u_{k-1}),$$

with $AB\neq 0$ and P_1 being a polynomial of the $(k-1)$ order. If the coefficients a_{jm}, b_m satisfy conditions

$$a_{k1}^2+\ldots+a_{kn}^2>0, a_{11}\ldots a_{1n}\neq 0, b_1\ldots b_n\neq 0$$

then f is a characteristic function of the Gaussian distribution.

REFERENCES

1. BONDESSON L.(1974). Characterization of probability laws through constant regression. *Z. Wahrschein.verw.Geb.* **29**, 95-115.
2. KAGAN A.(1989). Constancy of regression of a polynomial of sample average of positive random variables on their ratios characterizes gamma distribution. In: Statistical Data Analysis and Inference (Y.Dodge,ed.) 309-316.North Holland, Amsterdam-New York-Oxford-Tokyo.
3. KAGAN A.(1991). A method for solving a class of functional equations arising in characterization problems. In: Proceedings of the 1990 Taipei Symposium in Statistics (M.T.Chao and P.E.Cheng, eds), 321-336. Academia Sinica, Taipei.
4. KAGAN A.M., LINNIK Yu.V., RAO C.R.(1965). On a characterization of the normal law based on a property of the sample average. *Sankhyā*,Ser.A **27**, 405-406.
5. KAGAN A.M., LINNIK Yu.V., RAO C.R.(1973). Characterization Problems of Mathematical Statistics. New York, Wiley.

6. KAGAN A., RAO C.R.(1988). Constancy of regression of a polynomial of sample average on residuals characterizes normal distribution. Technical Report No. **88-30**, Center for Multivariate Analysis, Univ. of Pittsburgh (to be published in Proceedings of the Raj Bose Symposium on Probability and Statistics).
7. KHATRI C.G., RAO C.R.(1968). On some characterizations of the gamma distribution. *Sankhyā*, Ser.A **30**, 157-166.
8. PUSZ J.(1992). A characterization of the binomial and negative binomial distribution by regression of products. *Metrika*-to be published
9. PUSZ J.(1993). On a characterization of the generalized hyperbolic secant distribution-preprint.
10. WESOŁOWSKI J.(1987). A regressional characterization of the normal law. *Statistics and Probability Letters* **6**, 11-12.
11. WESOŁOWSKI J.(1989). A regressional characterization of the Poisson distribution. In: Stability Problems for Stochastic Models (Proceedings, Sukhumi, 1987), *Lect. Notes in Math.* **1412**, 349-351. Springer, New York-Berlin-Heidelberg.
12. WESOŁOWSKI J.(1992). Some characterizations by constant regression with respect to residuals. *Statistics and Decisions*-to be published.