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Bivariate Discrete Measures via a Power Series Conditional Distribution and a Regression Function

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Problems of specifying bivariate discrete distributions by a conditional distribution and a regression function are investigated. A review of the known results, together with new characterizations involving conditional power series laws, is given. Also some remarks on a method making use of marginal and a conditional expectation are enclosed. © 1995 Academic Press, Inc.

1. INTRODUCTION

Problems of conditional specifications of multivariate probability measures have attracted more and more attention in recent years. Those involving the forms of conditional distributions are surveyed in a monograph by Arnold *et al.* [3]. The other method making use of the form of one conditional distribution and some equidistribution conditions was continued in Arnold and Pourahmadi [5], Ahsanullah and Wesolowski [1] and Wesolowski and Ahsanullah [22]. New investigations and a review on determination of a multivariate measure by its second-order conditional structure are contained in Wesolowski [20]. In a very recent paper Arnold *et al.* [4] have rediscovered the conditional distribution—conditional mean specification posing some important consistency and uniqueness questions. Some aspects of the impact of such a conditional structure on the joint distribution is studied in the present paper.

The first contributions in this area go back to the 1970s. In his pioneering paper on regressional specification of a bivariate discrete distribution for a random vector (X, Y) Korwar [12] (actually assuming linear regression) proposed the following two approaches:

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A. Via the conditional expectation $E(X \mid Y)$ and the conditional distribution $Y \mid X$.

B. Via the conditional expectation $E(X \mid Y)$ and the marginal distribution of X.

A successful development for the method A consist of the following special cases of conditional distributions:

1. a binomial law of the form

$$p(y \mid x) = P(Y = y \mid X = x) = \binom{nx}{y} p^y q^{nx-y},$$
$$x \in \mathbf{N} = \{0, 1, \ldots\},\ y \in \{0, 1, \ldots, nx\};$$

2. a Pascal law of the form

$$p(y \mid x) = \binom{-x}{y-x} p^x (-q)^{y-x}, \qquad x \in \{1, 2, \ldots\},\ y \in \{x, x+1, \ldots\};$$

3. a Pascal law of the form

$$p(y \mid x) = \binom{x+y-1}{y} p^x q^y, \qquad x, y \in \mathbf{N};$$

4. a Poisson law of the form

$$p(y \mid x) = \exp(-\lambda x) \frac{(\lambda x)^y}{y!}, \qquad x, y \in \mathbf{N};$$

5. a shifted binomial law of the form

$$p(y \mid x) = \binom{n}{y-x} p^{y-x} q^{n-y+x}, \qquad x \in \mathbf{N},\ y \in \{x, x+1, \ldots, x+n\};$$

6. a shifted Pascal law of the form

$$p(y \mid x) = \binom{n+y-x-1}{n-1} p^n q^{y-x}, \qquad x \in \mathbf{N},\ y \in \{x, x+1, \ldots\};$$

7. a shifted Poisson law of the form

$$p(y \mid x) = \exp(-\lambda) \frac{\lambda^{y-x}}{(y-x)!}, \qquad x \in \mathbf{N},\ y \in \{x, x+1, \ldots\};$$

8. a hypergeometric law of the form

$$p(y \mid x) = \frac{\binom{x}{y}\binom{n-x}{m-y}}{\binom{n}{m}},$$

$$x \in \mathbf{N},\ y \in \{\max(0, m+x-n),$$
$$\max(0, m+x-n)+1, ..., \min(m, x)\};$$

9. a negative hypergeometric law of the form

$$p(y \mid x) = \frac{\binom{-m}{y}\binom{-n+m}{x-y}}{\binom{-n}{x}}, \qquad x \in \mathbf{N},\ y \in \{0, 1, ..., x\};$$

where p, q, n, m are suitable constants.

In Korwar [12] case 1 for $n=1$ and case 2 with linear conditional expectation of the form $m(y) = E(X \mid Y=y) = ay+b$, where a, b are some reals, were treated. The uniqueness question for the cases 1–7 with an arbitrary consistent function m is covered by Cacoullos and Papageorgiou [7]. A recent contribution for case 1 with $m(0)=b$, $m(y)=ay$, $y=1, 2, ...$, and case 2 with $m(y) = (y\theta p/q)((1+\theta p/q)^{y-1}/[(1+\theta p/q)^{y}-1])$, $y=1, 2, ...$, is given in Kyriakoussis and Papageorgiou [16]. Results on uniqueness for cases 8 and 9 with an arbitrary m are contained in Papageorgiou [19]. Kyriakoussis [13] studied uinique determination of a bivariate distribution assuming that the conditional distribution $Y \mid X$ is an X-fold convolution of a discrete measure and m is a polynomial. Some important limitations of this approach were revealed in Johnson and Kotz [9]. As it was pointed earlier an important trial of a more general treatment of the problem including some consistency and uniqueness questions has been performed by Arnold *et al.* [4] lately. Multivariate extensions were considered in Dahiya and Korwar [8], Khatri [10, 11], Xekalaki [23], and Papageorgiou [18, 19].

In the present paper we give some new contributions in this field. Unique determination of a bivariate measure by conditional power series distributions and an arbitrary consistent regression function is investigated in Section 2. The first uniqueness theorem obtained in that section is a generalization of the result obtained by Cacoullos and Papageorgiou [7] for case 4. Section 3 gives some applications of the main results for solution of uniqueness questions in cases of binomial, two versions of Poisson and two versions of geometric conditional distributions. As a consequence new characterizations of the bivariate binomial, geometric, and Poisson conditional distributions introduced in Arnold and Strauss [6] (see also Arnold *et al.* [3]) are obtained.

Let us end this section with two remarks concerning the method B of characterization a bivariate discrete measure by a marginal distribution and a regression function developed after Korwar [12] in Kyriakoussis and Papageorgiou [14–16].

First, some enlightening should be given on what is really going about in this method. The annoucements given in each of these papers, that the conditional distribution $Y \mid X$ is determined by the distribution of X of a special kind and by a suitable form of $E(X \mid Y)$, are quite misleading, since actually it is dealt not with a single random vector (X, Y) there, but with a collection of random vectors (X_θ, Y), $\theta \in \Theta$, where for each $\theta \in \Theta$ the r.v. X_θ has a distribution P_θ and $E(X_\theta \mid Y) = m(\theta, Y)$.

Second, there is a silent, in the first two papers, but important, assumption that for any $\theta \in \Theta$ the conditional distribution $Y \mid X_\theta$ does not depend on θ.

Taking into account what was said above let us state, as an example, a correct version of Theorem 3.1 from Korwar [12].

THEOREM 1. *Let* (X_λ, Y), $\lambda \in (0, \infty)$, *be a family of discrete random vectors with the following properties fulfilled for any* $\lambda \in (0, \infty)$:

i. *the r.v.* X_λ *has a Poisson distribution with the parameter* λ;
ii. $E(X_\lambda \mid Y) = Y + \lambda q$ *for some fixed* $q \in (0, 1)$;
iii. *the conditional distribution* $Y \mid X_\lambda$ *does not depend on* λ.

Then $P(Y = y \mid X_\lambda = x) = \binom{x}{y} (1-q)^y q^{x-y}$ *for any* $x \in \mathbf{N}$, $y \in \{0, 1, ..., x\}$, $\lambda \in (0, \infty)$.

2. CONDITIONAL POWER SERIES DISTRIBUTIONS

Recall that the power series distribution with the parameter θ (PSD(θ)) is defined by the probability function

$$p(\theta, x) = a(x)\theta^x/f(\theta), \qquad x \in S \subset \mathbf{N},$$

where $a \geqslant 0$ is called a coefficient function, and $f > 0$ a series function.

In this section we investigate the case of conditional distribution $Y \mid X$ being the PSD(X). We show that in this case the joint distribution is determined by any consistent regression function of X given Y, provided some slight condition is fulfilled by the function a. By S_X and S_Y denote, respectively, supports of X and Y.

THEOREM 2. *Let (X, Y) be a discrete random vector such that* (a) $S_Y=\{0, 1, ..., n\}$ *for some* $1 \leqslant n < \infty$ *and* $\# S_X \leqslant n+2$ *or* (b) $S_Y = \mathbf{N}$ *and* $S_X \subset \mathbf{N}$, *and* S_X, S_Y *are known. Assume further that for any* $x \in S_X$, $y \in S_Y$

$$P(Y=y \mid X=x)=a(y)x^y/f(x), \tag{1}$$

where $a>0$ *and* $f>0$ *are known. Additionally, if* S_X *is not bounded assume that*

$$\sum_{y \in S_Y} \sqrt[2y]{a(y)} = \infty. \tag{2}$$

Then the distribution of (X, Y) *is uniquely determined by* $E(X \mid Y)$.

Proof. Let $m(y)=E(X \mid Y=y)$, $y \in S_Y$. Then by (1)

$$m(y) \sum_{x \in S_X} x^y P(X=x)/f(x) = \sum_{x \in S_X} x^{y+1} P(X=x)/f(x), \qquad y \in S_Y. \tag{3}$$

Introduce now a r.v. Z with the distribution

$$P(Z=x)=cP(X=x)/f(x), \qquad x \in S_X, \tag{4}$$

where $c^{-1}=\sum_{x \in S_X} P(X=x)/f(x)$. This is a correct definition due to (5) below. Also (5) implies y-integrability of Z for any $y \in S_Y$.

The sequence $E(Z^y)$, $y \in S_Y$, uniquely determines the distribution of Z: In the case of bounded S_X it follows from the fact that $E(Z^{n+1})$ is available additionally and $\#(\operatorname{supp}(Z)) = \#(S_X) \leqslant n+2 = \#(S_Y)+1$ (we have to solve a system of $n+2$ linear equations of full rank). If S_X is not bounded then at first observe that

$$\begin{aligned} E(Z^y) &= \sum_{x \in S_X} x^y cP(X=x)/f(x) \\ &= \frac{c}{a(y)} \sum_{x \in S_X} P(X=x)\, a(y)\, x^y/f(x) \leqslant \frac{c}{a(y)} \end{aligned} \tag{5}$$

for any $y \in \mathbf{N}$. Consequently,

$$\sum_{y=1}^{\infty} 1/\sqrt[2y]{E(Z^y)} \geqslant \sum_{y=1}^{\infty} \sqrt[2y]{a(y)/c} = \infty$$

by (2). Hence the unique solution of the moment problem for Z follows from the Carleman criterion for non-negative r.v.'s (see, for example, [2]).

On the other hand, Eq. (3) implies

$$m(y)\, E(Z^y) = E(Z^{y+1}), \qquad y \in S_Y.$$

Thus

$$E(Z^{y+1}) = \prod_{j=0}^{y} m(j), \qquad y \in S_Y.$$

Hence the distribution of Z is characterized by the function m. Since (4) yields

$$c = \sum_{x \in S_X} P(Z = x)\, f(x)$$

then the distribution of X is also uniquely determined by the regression function $E(X \mid Y)$. Finally the result follows from (1). ∎

Similar results can be obtained for the conditional $Y \mid X$ with the PSD$(1/X)$. However, in this case the point 0 must be excluded from the support of X.

THEOREM 3. *Let (X, Y) be a discrete random vector such that* (a) $S_Y = \{0, 1, ..., n\}$, $\#(S_X) \leqslant n+2$, $S_X \subset (0, \infty)$ *or* (b) $S_Y = \mathbf{N}$ *and* $S_X \subset \{1, 2, ...\}$ *and S_X, S_Y are known. Assume that for any $x \in S_X$, $y \in S_Y$,*

$$P(Y = y \mid X = x) = a(y) x^{-y}/f(x)$$

for some known functions $a > 0$ and $f > 0$. Additionally, if S_X is not bounded assume that (2) *holds. Then the distribution of (X, Y) is uniquely determined by $E(X \mid Y)$.*

Proof. The only modification of the proof of Theorem 2 lies in defining the distribution of r.v. Z by

$$P(Z = 1/x) = cP(X = x)/f(x), \qquad x \in S_X.$$

Hence

$$E(Z^y) = \prod_{j=1}^{y} m^{-1}(j), \qquad y \in S_Y. \quad ∎$$

3. EXAMPLES

EXAMPLE 1. Poisson conditional (first case). Let

$$P(Y = y \mid X = x) = \exp(-\lambda x) \frac{(\lambda x)^y}{y!} \tag{6}$$

for any $y \in \mathbf{N}$, $x \in S_X$, and some $\lambda > 0$. Then $Y \mid X$ is a PSD(X) with $a(y) = \lambda^y/y!$ and $f(x) = \exp(\lambda x)$. Observe that (2) is fulfilled, since

$$\sum_{y=0}^{\infty} \sqrt[2y]{a(y)} \geqslant \sqrt{\lambda} \sum_{y=1}^{\infty} \frac{1}{y} = \infty.$$

Hence, by Theorem 2 the joint distribution of (X, Y) is uniquely determined by the conditional expectation of X, given Y. Such result for $S_X = \mathbf{N}$ is contained in Theorem 3.1 in [7].

If we assume that (6) holds and S_X does not contain the origin then also the conditional moment $E(X^{-1} \mid Y)$ determines the joint distribution. To see that define $U = 1/X$. Then

$$P(Y = y \mid U = u) = \exp(-\lambda/u) \frac{\lambda^y}{u^y y!}$$

for any $y \in \mathbf{N}$ and any $u \in \operatorname{supp}(U) = S_U$. By Theorem 3 the joint distribution of (U, Y) is uniquely determined by $E(U \mid Y) = E(X^{-1} \mid Y)$.

EXAMPLE 2. Geometric conditional (first case). Let

$$P(Y = y \mid X = x) = \frac{x-1}{x} \left(\frac{1}{x}\right)^y \tag{7}$$

for any $x \in \{2, 3, ...\}$ and $y \in \mathbf{N}$. Then $Y|X$ is a PSD($1/X$) with $a(y) = 1$ and $f(x) = x/(x-1)$. Condition (2), obviously, holds. Then by Theorem 3 the regression function of X given Y characterizes the joint distribution.

Again define a r.v. $U = 1/X$. Then

$$P(Y = y \mid U = u) = (1-u)u^y$$

for $u \in S_U$. Hence, by Theorem 2 the conditional distribution (7) and $E(X^{-1} \mid Y)$ determine the distribution of (X, Y).

In Arnold and Strauss [6] a general form of a bivariate measure with conditionals in exponential families was obtained. Some of its discrete consequences that we are going to recall and complement now are reproduced in Arnold *et al.* [3, Chap. 4.4.9].

A r.v. X has a binomial b(n, p) distribution if $P(X = x) = \binom{n}{x} p^x (1-p)^{n-x}$, $x = 0, 1, ..., n$, $p \in (0, 1)$, $n \in \mathbf{N}$. The random vector (X, Y) with the conditionals $X \mid Y$ being b(n_1, $p_1(Y)$) and $Y \mid X$ being b(n_2, $p_2(X)$) has the joint probability function

$$P(X = x, Y = y) = k_B(p_1, p_2, t) \binom{n_1}{x} \binom{n_2}{y} \times p_1^x p_2^y (1-p_1)^{n_1 - x} (1-p_2)^{n_2 - y} t^{xy} \tag{8}$$

for any $x=0, 1, ..., n_1$, $y=0, 1, ..., n_2$ and some $p_1 \in (0, 1)$, $p_2 \in (0, 1)$, $t>0$. This is the bivariate binomial conditional distribution. Then the exact conditional distributions are available: $X \mid Y$ is $b(n_1, p_1 t^Y/(1-p_1+p_1 t^Y))$ and $Y \mid X$ is $b(n_2, p_2 t^X/(1-p_2+p_2 t^X))$. Observe that in this case

$$E(t^{\pm X} \mid Y) = \left(\frac{1-p_1+p_1 t^{Y \pm 1}}{1-p_1+p_1 t^Y}\right)^{n_1}. \tag{9}$$

EXAMPLE 3. Binomial conditional. Let

$$P(Y=y \mid X=x) = \binom{n_2}{y}\left(\frac{p_2 t^x}{1-p_2+p_2 t^x}\right)^y \left(1-\frac{p_2 t^x}{1-p_2+p_2 t^x}\right)^{n_2-y} \tag{10}$$

for any $x=0, 1, ..., n_1$, $y=0, 1, ..., n_2$ and some $n_1 \leqslant n_2+1$, $0<p_1<1$, $0<p_2<1$, $1 \neq t>0$. Then $E(t^X \mid Y)$ uniquely determines the joint distribution.

To prove this fact define $U=t^X$ and observe that $P(Y=y \mid U=u) = a(y)u^y/f(u)$, where

$$a(y) = \binom{n_2}{y} p_2^y(1-p_2)^{n_2-y}, \qquad f(u)=(1-p_2+p_2 t^u)^{n_2}.$$

The result follows now by Theorem 2.

Similarly, (10) together with $E(t^{-X} \mid Y)$, also uniquely determines the bivariate distribution. Now it follows from Theorem 3 by considering a r.v. $U=t^{-X}$. Consequently (10) and any one of the forms of (9) characterize the bivriate binomial conditionals law (8).

A r.v. X has a geometric $g(q)$ distribution if $P(X=x)=(1-q)q^x$, $x=0, 1, ...,$ where $0<q<1$. The bivariate measure with $X \mid Y$ being $g(q_1(Y))$ and $Y \mid X$ being $g(q_2(X))$ has the probability function

$$P(X=x, Y=y) = k_G(q_1, q_2, t)\, q_1^x q_2^y t^{xy}, \tag{11}$$

for all $x, y \in \mathbf{N}$ and some $0<q_1<1$, $0<q_2<1$, $0<t \leqslant 1$. This is the bivariate geometric conditional distribution. The exact forms of the conditionals are the following: $X \mid Y$ is $g(q_1 t^Y)$ and $Y \mid X$ is $g(q_2 t^X)$. Hence,

$$E(t^{\pm X} \mid Y) = \frac{1-q_1 t^Y}{1-q_1 t^{Y \pm 1}}. \tag{12}$$

EXAMPLE 4. Geometric conditional (second case). Let

$$P(Y=y \mid X=x) = (1-q_2 t^x)(q_2 t^x)^y \tag{13}$$

for any $x, y \in \mathbf{N}$ and some $0 < q_2 < 1$, $0 < t < 1$. Then $E(t^X \mid Y)$ uniquely determines the joint distribution.

To see that define a r.v. $U = t^X$. Then $P(Y = y \mid U = u) = a(y)\, u^y / f(u)$, where

$$a(y) = q_2^y, \qquad f(u) = 1/(1 - q_2 u).$$

Observe that

$$\sqrt[2y]{a(y)} = \sqrt{q_2}$$

and condition (2) holds. Now the result follows from Theorem 2.

Similarly as in Example 3 it follows by Theorem 3 that (13), together with $E(t^{-X} \mid Y)$, uniquely determines the bivariate distribution.

Finally, we obtain characterizations of the bivariate geometric conditionals measure (11) by (13) and anyone of the forms of (12).

A r.v. X has a Poisson ($\mathscr{P}(\lambda)$) distribution if $P(X = x) = e^{-\lambda} \lambda^x / x!$, $x = 0, 1, \ldots$. The only bivariate random vector (X, Y) with $X \mid Y$ being $\mathscr{P}(\lambda_1(Y))$ and $Y \mid X$ being $\mathscr{P}(\lambda_2(X))$ has the probability function of the form

$$P(X = x, Y = y) = k_{\mathscr{P}}(\lambda_1, \lambda_2, t)\, \frac{\lambda_1^x \lambda_2^y t^{xy}}{x!\, y!} \tag{14}$$

for any $x, y \in \mathbf{N}$ and some $\lambda_1 > 0$, $\lambda_2 > 0$, $0 < t \leqslant 1$. This is the bivariate Poisson conditionals distribution. (Originally this distribution, was introduced by Obrechkoff [17] and then rediscovered in Arnold and Strauss [6].) Then $X \mid Y$ is $\mathscr{P}(\lambda_1 t^Y)$ and $Y \mid X$ is $\mathscr{P}(\lambda_2 t^X)$, Consequently,

$$E(t^{\pm X} \mid Y) = \exp[\lambda_1 t^Y (t^{\pm 1} - 1)]. \tag{15}$$

EXAMPLE 5. Poisson conditional (second case). Let

$$P(Y = y \mid X = x) = e^{-\lambda_2 t^x} \frac{(\lambda_2 t^x)^y}{y!} \tag{16}$$

for any $x, y \in \mathbf{N}$ and some $\lambda_2 > 0$, $0 < t < 1$. Then $E(t^X \mid Y)$ or $E(t^{-X} \mid Y)$ uniquely determine the joint distribution.

To prove this fact once again consider $U = t^X$ or $U = t^{-X}$. Then we can apply the results of Example 1 to the random vector (U, Y), since $Y \mid U$ is $\mathscr{P}(\lambda_2 U)$.

Finally the characterizations of the bivariate Poisson conditionals distribution (14) by (16) and anyone of the forms of (15) follow.

Remark. Another conditional characterization of the bivariate Poisson conditional law (14) by (16) and a conditional expectation of the form

$$E(X \mid Y) = \lambda_1 t^Y$$

has been discovered in Wesolowski [21] recently.

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