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Normality via conditional normality of linear forms

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Abstract

It is proved that if the conditional distribution of one linear form in two independent (not necessarily identically distributed) random variables given another is normal, then the variables are normal. The result complements a series of characterizations of normal distribution via different properties of linear forms: independence, linearity of regression plus homoscedasticity, equidistribution, conditional symmetry and normality. The method is different from previous ones and is based on properties of densities, not characteristic functions.

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1. Introduction

Conditional specifications of stochastic models have attracted attention of numerous researchers in the field of distribution theory and its applications. A considerable part of this interest is devoted to specifications involving conditional distributions as may be learned from the recent monograph by Arnold et al. (1992). The present contribution is a further development in the case of conditional normality.

Consider a random vector (U, V) such that both the conditional distributions, $\mu_{U|V}$ and $\mu_{V|U}$, are normal, i.e.

$$\mu_{U|V} = \mathcal{N}(m(V), \sigma^2(V)) \quad (1)$$

and

$$\mu_{V|U} = \mathcal{N}(\tilde{m}(U), \tilde{\sigma}^2(U)),$$

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where $m, \tilde{m}, \sigma > 0, \tilde{\sigma} > 0$ are some real Borel functions. Then the joint distribution may not be normal as it was observed in Castillo and Galambos (1989) (see also Bhattacharyya (1943)); it may even be bimodal as a nice picture in Gelman and Meng (1991) reveals. However, if any of the conditional means is linear or any of the conditional variances is non-random then the bivariate distribution of (U, V) is Gaussian. Numerous variations on this theme are gathered in Hamedani (1992). Obviously, normality of the conditional distribution of, say, U given V , even with linearity of the conditional mean $E(U|V)$ and constancy of the conditional variance $\text{Var}(U|V)$, (called homoscedasticity) does not imply normality of (U, V) . However, supplemented by linearity of $E(V|U)$ or by constancy of $\text{Var}(V|U)$, it leads to bivariate Gaussian distribution, as it was observed in Ahsanullah and Wesolowski (1994) (see also Cacoullos and Papageorgiou, 1984).

Here we study the case, where U and V are linear functions of a pair of independent (not necessarily identically distributed) r.v's X, Y and prove that conditional normality of U given V , without any additional conditions on the structure of the parameters of this distribution, implies normality of both X and Y .

This result complements a series of, now classical, characterizations of normality via properties of linear forms in independent r.v's:

- Marcinkiewicz-Linnik theorem on equidistribution of linear forms in i.i.d. r.v's (Marcinkiewicz, 1938; Linnik, 1953);
- Cramèr theorem on normality of a sum of independent r.v's (Cramèr, 1936);
- Darmois–Skitovitch theorem on independence of linear forms (Darmois, 1953; Skitovitch, 1953);
- Lukacs–Laha theorem on constancy of the first two conditional moments of one linear form given another (Lukacs and Laha, 1964);
- Heyde theorem on symmetry of the conditional distribution of one linear form given another (Heyde, 1969).

All these theorems were proved studying functional equations for characteristic functions, which are equivalents of the property under study. The monograph of Kagan et al. (1973) gives a wide perspective on analytical theory of linear forms in independent r.v's and may be consulted for these and other results. In the case considered here it appears that conditional normality of one linear form given another does not allow (without additional assumptions) a convenient analytic equivalent in term of the ch.f's. Instead our proof is based on analyzing the conditional density and relies strongly on the fact that only two r.v's are involved. The general problem of conditional normality of one linear form in n ($n > 2$) independent r.v's given another remains open.

2. Characterization

Let X, Y be independent r.v's. Define

$$U = \alpha X + \beta Y, \quad V = \gamma X + \delta Y,$$

where $\alpha, \beta, \gamma, \delta$ are some real numbers. Obviously, for X and Y normal $\mu_{U|V}$ is normal. Here we study a converse problem; we are interested in the situation when the conditional distribution of U given V is normal, i.e. the condition (1) holds.

Theorem 1. *Assume that $\alpha\delta \neq \beta\gamma$. If the conditional distribution of U given V is normal (with probability 1) then X and Y are normal.*

Proof. Assume that (1) holds. Without loss of generality, we can assume a special form for U and V :

$$U = X + Y, \quad V = X + \alpha Y, \quad \alpha \neq 1.$$

Observe that since $X = (\alpha U - V)/(\alpha - 1)$ and $Y = (U - V)/(1 - \alpha)$, then by (1) it follows that

$$\mu_{X|V} = \mathcal{N}\left(\frac{1}{\alpha - 1}(\alpha m(V) - V), \frac{\alpha^2 \sigma^2(V)}{(1 - \alpha)^2}\right)$$

and

$$\mu_{Y|V} = \mathcal{N}\left(\frac{1}{1 - \alpha}(m(V) - V), \frac{\sigma^2(V)}{(1 - \alpha)^2}\right).$$

Consequently, X and Y have densities, hence (U, V) is also absolutely continuous. (It is well known that, in general, for any random vector, say, (T, Z) , absolute continuity of the conditional distribution of Z given T implies absolute continuity of Z .)

By properties of conditional densities for any real u, v

$$f_{U|V=v}(u)f_V(v) = f_{U,V}(u, v) = \frac{f_X\left(\frac{\alpha u - v}{1 - \alpha}\right) f_Y\left(\frac{v - u}{1 - \alpha}\right)}{|1 - \alpha|}, \tag{2}$$

where $f_{U|V}, f_{U,V}, f_V, f_X, f_Y$ are densities of the conditional distribution of U given V , joint distribution of (U, V) , distribution of V, X, Y , respectively. On the other hand, the conditions of theorem imply that

$$f_{U|V=v}(u) = \frac{1}{\sqrt{2\pi\sigma(v)}} \exp\left(-\frac{(u - m(v))^2}{2\sigma^2(v)}\right), \quad u, v \in \mathbb{R} \tag{3}$$

for some (measurable) functions m and $\sigma > 0$. Observe that combining (2) and (3) it follows that without loss of generality we can assume that the functions f_V, f_X and f_Y do not vanish (since they can be changed on any set of zero Lebesgue measure). Hence, upon taking logarithms we obtain

$$a(\alpha u - v) + b(v - u) = c(v)u^2 + d(v)u + e(v), \quad u, v \in \mathbb{R}, \tag{4}$$

where

$$a(x) = \ln\left(f_X\left(\frac{x}{1 - \alpha}\right)\right), \quad b(x) = \ln\left(f_Y\left(\frac{x}{1 - \alpha}\right)\right), \quad c(x) = -\frac{1}{2\sigma^2(x)},$$

$$d(x) = \frac{m(x)}{\sigma^2(x)}, \quad e(x) = \ln\left(\frac{|1 - \alpha|f_V(x)}{\sqrt{2\pi\sigma(x)}}\right) - \frac{m^2(x)}{2\sigma^2(x)}, \quad x \in \mathbb{R}.$$

Put in (4) $u + x_1$ instead of u and $v + x_2$ instead of v , where x_1, x_2 are arbitrary reals fulfilling $\alpha x_1 = x_2$. Then subtract (4) from what was obtained. It gives

$$\hat{b}(v - u) = \hat{c}(v)u^2 + \hat{d}(v)u + \hat{e}(v), \quad u, v \in \mathbb{R},$$

where $\hat{b}(x) = b(x + x_2 - x_1) - b(x)$ and $\hat{c}, \hat{d}, \hat{e}$ are some functions depending on x_1 and x_2 . Consequently, for $v = 0$ and u changed to $-u$

$$\hat{b}(u) = \hat{c}(0)u^2 - \hat{d}(0)u + \hat{e}(0), \quad u \in \mathbb{R}.$$

Hence, by the definition of \hat{b} for any real u and x

$$b(u + x) - b(u) = C(x)u^2 + D(x)u + E(x),$$

where C, D and E are some functions. It is well known, since it is a Cauchy-like equation, that b must be a polynomial of the order not exceeding three. However due to the fact that e^b is a density function b is at most a quadratic function. Hence, Y has a normal distribution.

The analogous argument holds for the function a (it suffices to take $x_1 = x_2$ this time) and consequently the r.v. X is also normal. \square

Remark 1. Observe that if we try to treat the problem in terms of ch.f.'s, as we are used to in cases involving linear forms in independent r.v.'s, then we arrive at the following strange equation ($U = X + Y, V = X + \alpha Y$)

$$\phi_X(s + t)\phi_Y(s + \alpha t) = E(e^{itV + ism(V) - s^2\sigma^2(V)/2}), \quad s, t \in \mathbb{R},$$

where ϕ_X and ϕ_Y are ch.f.'s of X and Y , respectively. A direct solution of this equation is not known to the authors. However, due to Theorem 1 we know that only the normal case is possible.

Remark 2. The proof of Theorem 1, given above, strongly relies on the fact that only two independent r.v.'s appeared in the linear forms. We conjecture that the result is true also for linear forms in any finite number of independent r.v.'s. It holds if $U = aL_1 + bL_2$ and $V = cL_1 + dL_2$, where L_1 and L_2 are linear forms originating from separate sets of r.v.'s. This is an immediate consequences of our theorem and Cramér's decomposition of the normal law, provided some obvious conditions for the coefficients are fulfilled.

Remark 3. Also another special case may be answered immediately. If $U = \sum_{i=1}^n a_i X_i$ and $V = \sum_{i=1}^n X_i$, where X_i 's are i.i.d. and the condition (1) holds then m is linear. Hence the conditional distribution of the linear form $U - m(V)$ given V is symmetric since then

$$E(\exp(it(U - m(V)))) = \exp(-t^2\sigma^2(V)/2), \quad t \in \mathbb{R}$$

and by the Heyde theorem X_i 's are normal if only suitable conditions are imposed on the coefficients.

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