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BIVARIATE DISTRIBUTIONS VIA THE SECOND-KIND BETA CONDITIONAL DISTRIBUTION AND A REGRESSION FUNCTION

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The uniqueness of the specification of a bivariate distribution by the second-kind beta conditionals and a consistent regression function is investigated. New characterizations of the bivariate second-kind beta-conditional distribution are obtained.

1. Introduction

Specifications of the distribution of a random vector (X, Y) by the conditional distribution $\mu_{Y|X}$ and the conditional mean $\mathbf{E}(X \mid Y)$ have been intensively studied in recent years. Research in this field has covered a wide variety of conditional distributions, including the following ones: binomial and Pascal in [10], binomial, Pascal, and Poisson type in [5], hypergeometric and negative hypergeometric in [13], X-fold convolution of a discrete measure in [11] (discussed also in [9]), again binomial and Pascal in [12], exponential in [2], normal in [1], power series in [14], Poisson in [15], Pareto in [16]. The investigations were concerned with the uniqueness question in the presence of any consistent regression function and/or characterizations when the regression function is explicitly specified. This paper is a new contribution which introduces the second-kind beta conditional in this area.

Denote by $\mathcal{B}2(p, q; \sigma)$ the second-kind beta distribution with density function

$$f(x) = \begin{cases} \dfrac{\sigma^q x^{p-1}}{\beta(p,q)(\sigma + x)^{p+q}}, & x > 0, \\ 0, & x < 0. \end{cases}$$

Following the specification of the bivariate Pareto-conditional distribution discovered by Arnold [2], a similar result involving both conditionals distributed according to the second-kind beta law was obtained by Castillo and Sarabia [6] (see also §5.3.1 in [3]). Let us recall the following result:

For a random vector (X, Y), denote by $\mu_{X|Y}$ and $\mu_{Y|X}$ the conditional distributions of X given Y and of Y given X, respectively. Consider a random vector (X, Y) with

$$\mu_{X|Y} = \mathcal{B}2(p, q; \sigma_1(Y)), \qquad \mu_{Y|X} = \mathcal{B}2(p, q; \sigma_2(X)), \tag{1}$$

where σ_1 and σ_2 are some positive functions.

Then

$$\sigma_1(Y) = \frac{a + Y}{b + cY}, \qquad \sigma_2(X) = \frac{a + bX}{1 + cX},$$

and the joint distribution, the bivariate second-kind beta-conditional distribution, has a density of the form

$$f(x, y) = \frac{K x^{p-1} y^{p-1}}{(a + bx + y + cxy)^{p+q}}, \qquad x > 0, \qquad y > 0, \tag{2}$$

$f(x, y) = 0$ otherwise, K is a normalizing constant, $a \geq 0$, $b > 0$, $c \geq 0$. If $a = 0$, then $q < p$; if $c = 0$, then $1 < p < q$. In the latter case it is the bivariate inverted Dirichlet-type distribution — see [8]. Also the possibility of identification of both the special cases of the density (2) by (1) and exact forms of conditional expectations $\mathbf{E}(X \mid Y)$ and $\mathbf{E}(Y \mid X)$ was indicated in the paper (however, the exact statements of the results are incorrect due to the wrong formulas for conditional moments). Another conditional specification of the inverted Dirichlet distribution involving the concept of neutrality is given in [7].

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In this paper, we complement the results of Castillo and Sarabia [6] by considering one second-kind beta-conditional distribution, $\mu_{Y|X}$ say, and one conditional expectation $\mathbf{E}(X \mid Y)$; however, we assume the exact form of the scale parameter of the conditional distribution. The results of the paper generalize theorems and the method of the proofs develop some ideas from [16], where the Pareto case was treated.

2. Uniqueness and Characterization

Let (X, Y) be a random vector with nonnegative components. As was pointed out in the introduction, in this paper we consider possible distributions of (X, Y) with $\mu_{Y|X}$ as in the case of the bivariate second-kind beta-conditional distribution, i.e., we assume that the conditional density of the measure $\mu_{Y|X}$ has the form

$$f_{Y|X=x}(y) = \frac{(a+bx)^q(1+cx)^p y^{p-1}}{\beta(p,q)(a+bx+y+cxy)^{p+q}}, \qquad y > 0, \qquad x \in S_X, \tag{3}$$

or equivalently

$$\mu_{Y|X} = \mathcal{B}2\left(p, q; \frac{a+bX}{1+cX}\right), \tag{4}$$

where $a \geq 0$, $b > 0$, $c \geq 0$ (if $a = 0$, then $q < p$; if $c = 0$, then $1 < p < q$), $S_X = [0, \infty)$.

Additionally we assume $ac \neq b$. Observe that in the case $ac = b$ by (3)

$$f_{Y|X=x}(y) = \frac{a^q y^{p-1}}{\beta(p,q)(a+y)^{q+p}}$$

for any $y > 0$, $x \in S_X$. Consequently Y has the second-kind beta $\mathcal{B}2(p, q; a)$ distribution and X, Y are independent. Hence $\mathbf{E}(X \mid Y) = \mathbf{E}(X)$ and this is the only restriction imposed on the distribution of X.

Our aim in this section is to prove that under the assumptions given above the conditional expectation $\mathbf{E}(X \mid Y)$ uniquely determines the joint distribution.

Theorem 1. *Let (X, Y) be a random vector satisfying* (4). *Then its distribution is uniquely determined by* $\mathbf{E}(X \mid Y)$.

Proof. Denote by m the regression function of X given Y:

$$m(y) = \mathbf{E}(X \mid Y = y) = \int_{S_X} x \, dF_{X|Y=y}(x), \qquad y > 0,$$

where $F_{X|Y=y}$ is the conditional distribution function. From (3) it follows that Y has a density, say f_Y. Then the obvious identity

$$f_Y(y)\, dF_{X|Y=y}(x) = f_{Y|X=x}(y)\, dF_X(x),$$

where F_X denotes the distribution function of X, yields for any $y > 0$

$$m(y) \int_{S_X} f_{Y|X=x}(y)\, dF_X(x) = \int_{S_X} x f_{Y|X=x}(y)\, dF_X(x). \tag{5}$$

Define a generalized distribution function H by the formula

$$dH(x) = (a+bx)^q(1+cx)^p\, dF_X(x), \qquad x \geq 0.$$

Then (3) and (5) imply

$$m(y)\varphi_{p+q}(y) = \int_{S_X} x(a+bx+y+cxy)^{-(p+q)}\, dH(x), \tag{6}$$

where

$$\varphi_\gamma(y) = \int_{S_X} (a+bx+y+cxy)^{-\gamma}\, dH(x)$$

for $\gamma \in \{p+q-1, p+q\}$. Observe that from the definition of φ it follows that for any $y > 0$

$$(b + cy) \int_{S_X} x(a + bx + y + cxy)^{-(p+q)}\, dH(x) = \varphi_{p+q-1}(y) - (a + y)\varphi_{p+q}(y). \tag{7}$$

Now we join (6) and (7) to obtain

$$[(b + cy)m(y) + a + y]\varphi_{p+q}(y) = \varphi_{p+q-1}(y), \qquad y > 0. \tag{8}$$

Observe that φ_{p+q-1} is differentiable and its derivative can be expressed in terms of φ_{p+q-1} and φ_{p+q}:

$$\varphi'_{p+q-1}(y) = \frac{p+q-1}{b+cy}[(ac-b)\varphi_{p+q}(y) - c\varphi_{p+q-1}(y)], \qquad y > 0. \tag{9}$$

Hence by the assumption $ac \neq b$ after some easy algebra, (8) and (9) yield

$$[(b + cy)m(y) + a + y]\varphi'_{p+q-1}(y) = -(p + q - 1)(cm(y) + 1)\varphi_{p+q-1}(y), \qquad y > 0.$$

Consequently

$$\varphi_{p+q-1}(y) = K \exp\left(-(p + q - 1) \int \frac{cm(y) + 1}{(b + cy)m(y) + a + y}\, dy, \qquad y > 0,\right.$$

where K is a positive constant.

Hence by (8) $\varphi_{p+q} = KG$, where G is a function uniquely determined by a, b, c. Let

$$C = \mathbf{E}\left(\frac{1 + cX}{a + bX}\right)^p$$

and

$$d\hat{H}(x) = C^{-1}\left(\frac{1 + cx}{a + bx}\right) dF_X(x).$$

Then $\hat{H}$ is a distribution function. Let $\hat{Z}$ be a r.v. with the d.f. $\hat{H}$ and define $Z = (1 + c\hat{Z})/(a + b\hat{Z})$. Then

$$\varphi_{p+q}(y) = KG(y) = C\mathbf{E}(1 + yZ)^{-(p+q)}.$$

The function φ_{p+q} is k-differentiable for any $k = 1, 2, \dots$, in each point $y \geq 0$ (in $y = 0$ we consider the right-hand-side derivatives). Since

$$KG^{(k)}(0) = \varphi^{(k)}_{p+q}(0) = C(-1)^k \frac{\Gamma(p + q + k)}{\Gamma(p + q)} \mathbf{E}(Z^k), \qquad k = 0, 1, 2, \dots,$$

we have $\mathbf{E}(Z^k) = \hat{K}G^{(k)}(0)$, where G depends only on a, b, c, and $\hat{K}$ is a constant. To prove that the distribution of X is uniquely determined by a, b, c, consider X_1 satisfying the assumptions of the theorem. Then, similarly as for Z we have for its analogue Z_1 built on X_1: $\mathbf{E}(Z_1^k) = \hat{K}_1 G^{(k)}(0)$, $k = 1, \dots$, with the same function G. Hence $\hat{K}_1\mathbf{E}(Z^k) = \hat{K}\mathbf{E}(\hat{Z}_1^k)$, $k = 1, 2, \dots$. Consequently for $L = \hat{K}/\hat{K}_1$

$$\tau(s) = L\tau_1(s) + 1 - L, \qquad s \geq 0,$$

where τ and $\hat{\tau}$ are the Laplace–Stieltjes transforms of Z and Z_1, respectively. Since Z and Z_1 are positive a.s., both the transforms vanish as $s \to \infty$. Thus $L = 1$ and $\hat{K} = \hat{K}_1$. This yields a unique determination of the distribution of Z and finally by the definition of Z the distribution of X is also uniquely characterized by a, b, c. Q.E.D.

By Theorem 1, a bivariate distribution can be specified by the conditional second-kind beta distribution and a regression function. Now we use this result to characterize the bivariate second-kind beta-conditional distribution introduced in [6]. Observe that the mean of the second-kind beta $\mathcal{B}2(p, q, \sigma)$ distribution exists only for $q > 1$ and is equal to $p\sigma/(q - 1)$. Hence for (X, Y) with density (2) and $q > 1$

$$\mathbf{E}(X \mid Y) = \frac{p(a + Y)}{(q - 1)(b + cY)}. \tag{10}$$

COROLLARY 1. *Let (X, Y) be a random vector satisfying* (4) *and* (10). *Then (X, Y) has the bivariate second-kind beta-conditional distribution with density* (2). *If $\mathbf{E}(X \mid Y)$ is linear, i.e., $c = 0$, then (X, Y) has an inverted Dirichlet-type distribution.*

Now we are going to replace the conditional mean $\mathbf{E}(X \mid Y)$ by $\mathbf{E}[(a + bX + Y + cXY)^{-1} \mid Y]$. It appears that in this case we can also obtain a uniqueness result like Theorem 1 using similar methods.

THEOREM 2. *Let (X, Y) be a random vector satisfying* (4). *Then the distribution of (X, Y) is uniquely determined by $\mathbf{E}[(a + bX + Y + cXY)^{-1} \mid Y]$.*

Proof. Adopting the notations from the proof of Theorem 1 and additionally denoting

$$\zeta(y) = \mathbf{E}[(a + bX + Y + cXY)^{-1} \mid Y = y], \qquad y > 0,$$

we have

$$\zeta(y)\varphi_{p+q}(y) = \varphi_{p+q+1}(y), \qquad y > 0.$$

Now (9) with $p + q$ changed into $p + q + 1$ yields

$$(ac - b)\zeta(y)\varphi_{p+q}(y) = \frac{b + cy}{p + q}\varphi'_{p+q}(y) + c\varphi_{p+q}(y), \qquad y > 0.$$

Hence

$$\varphi_{p+q}(y) = \frac{K}{(b + cy)^{p+q}} \exp\left[(p + q)(ac - b)\int \frac{\zeta(y)}{b + cy}\,dy\right], \qquad y > 0.$$

Consequently $\varphi_{p+q} = KG$, where G is a function uniquely determined by a, b, c. Now it suffices to follow the final steps of the proof of Theorem 1. Q.E.D.

Theorem 2 can also be used to characterize the bivariate second-kind beta-conditional distribution. Observe that for a r.v. X with the second-kind beta $\mathcal{B}2(p, q, \sigma)$ distribution, $\mathbf{E}(\sigma/(\sigma + X)) = q/(p + q)$. Consequently (X, Y) with density (2) (and conditional distributions given in (1))

$$\mathbf{E}[(a + bX + Y + cXY)^{-1} \mid Y] = \frac{q}{(p + q)(a + Y)}. \tag{11}$$

COROLLARY 2. *Let (X, Y) be a random vector satisfying* (4) *and* (11). *Then it has the bivariate Pareto-conditional distribution with density* (2). *If $c = 0$ in* (4) *and* (11), *then it is an inverted Dirichlet distribution.*

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