

## A new conditional specification of the bivariate Poisson conditionals distribution<sup>1</sup>

J. Wesolowski\*

*Mathematical Institute, Warsaw University of Technology,  
Plac Politechniki 1, 00-661 Warsaw, Poland*

The bivariate Poisson conditionals distribution is characterized by the form of one of the conditional distributions and one of the conditional expectations.

*Key Words & Phrases:* conditional specification, regression function, conditional Poisson law, bivariate Poisson conditionals distribution.

Problems of identification of a bivariate distribution of a random vector  $(X, Y)$  in terms of the conditional distribution  $Y|X$  and the conditional mean  $E(X|Y)$  have been intensively studied recently. A new strong stimulus in this direction has been given in ARNOLD et al. (1993) recently. Such a method of specification of bivariate measures goes back to KORWAR (1975), where binomial and Pascal conditional distributions together with a conditional expectation of the form  $m(y) = E(X|Y = y) = ay + b$ , for some real constants  $a, b$ , were considered. The uniqueness problem for binomial type, Pascal type and Poisson type conditional distributions and an arbitrary consistent function  $m$  was investigated in CACOULOS and PAPAGEORGIU (1983), while the same question for the hypergeometric and negative hypergeometric conditional laws was studied in PAPAGEORGIU (1985). KYRIAKOUSSIS (1988) obtained a result on unique determination of a bivariate distribution assuming that the conditional  $Y|X = x$  is an  $x$ -fold convolution of an arbitrary discrete measure and  $m$  is a polynomial (discussed also in JOHNSON and KOTZ, 1992). Logarithmic series distributions are characterized by binomial conditional distribution and

$$m(0) = b, \quad m(y) = ay, \quad y = 1, 2, \dots$$

and by a Pascal conditional distribution and

$$m(y) = \frac{\theta p y}{q} \frac{(1 + \theta p/q)^y - 1}{[(1 + \theta p/q)^y - 1]}.$$

<sup>1</sup> This research was partially supported by the KBN 511/2/91 grant.

\* wesolo@alpha.im.pw.edu.pl

More recent contributions in this theme are: ARNOLD et al. (1993) uniqueness result for an exponential conditional distribution and a consistent  $m$ , WESOŁOWSKI (1994a) involving a Pareto conditional, AHSANULLAH and WESOŁOWSKI (1994) characterization of the bivariate normality by the normal conditional distribution and the linear  $m$ , WESOŁOWSKI (1994b) uniqueness theorems for power series conditional distribution and a consistent  $m$ . In this note we give new contributions to this kind of conditional specification taking into account a Poisson conditional law. Then a power regression function implies a bivariate Poisson conditionals distribution while a constant regression yields independence.

ARNOLD and STRAUSS (1991) obtained a general result on the form of a bivariate measure with conditionals in exponential families. As a corollary they observed that the only bivariate law with both conditional distributions Poisson has a probability function

$$P(X = x, Y = y) = k(\lambda_1, \lambda_2, \lambda_3) \frac{\lambda_1^x \lambda_2^y \lambda_3^{xy}}{x!y!}, \quad x, y \in \{0, 1, \dots\} = \mathbb{N} \tag{1}$$

where  $\lambda_1 > 0$ ,  $\lambda_2 > 0$ ,  $0 < \lambda_3 \leq 1$  and  $k(\lambda_1, \lambda_2, \lambda_3) > 0$  are constants. This is the bivariate Poisson conditionals distribution. The conditional  $Y|X$  is Poisson with parameter  $\lambda_2 \lambda_3^x$  and  $X|Y$  is Poisson with parameter  $\lambda_1 \lambda_3^y$ . Consequently

$$E(X|Y) = \lambda_1 \lambda_3^y \tag{2}$$

The details are given also in ARNOLD et al. (1992, Ch. 4.4.9).

In this section first we show that to characterize the bivariate Poisson conditionals distribution the Poisson conditional  $Y|X$  and the conditional expectation of  $X$  given  $Y$  of the above forms are sufficient. Next a constant conditional mean will be considered.

**THEOREM 1.** *Assume that  $(X|Y)$  is a random vector with  $\text{supp}(X) = \text{supp}(Y) = \mathbb{N}$ . If for any  $x \in \mathbb{N}$*

$$Y|X = x \sim \text{Poisson}(\lambda_2 \lambda_3^x) \tag{3}$$

*and  $E(X|Y)$  is given by (2), where  $\lambda_1 > 0$ ,  $\lambda_2 > 0$  and  $0 < \lambda_3 < 1$  are some constants, then  $(X, Y)$  has a bivariate Poisson conditionals distribution with a probability function (1).*

Other regressional characterizations of the bivariate Poisson conditionals law are given in WESOŁOWSKI (1994b) where (3) together with one of regressions

$$E(\lambda_3^{\pm X} | Y) = \exp[\lambda_1 \lambda_3^Y (\lambda_3^{\pm 1} - 1)]$$

was considered. The result was a consequence of a general uniqueness theorem involving power series distributions. The proofs were based on direct expressions for the conditional moments and we do not know how to adapt them to the present situation. Instead we apply the moment generating function of the r.v.  $Y$ . Let us point out that till now no general method for identifying a bivariate distribution by a conditional and a regression function has been discovered. A trial of more universal

approach presented in ARNOLD et al. (1993) revealed some fundamental difficulties in this area.

PROOF: Observe that (3) implies for any real  $s$

$$E(s^Y|X) = \exp(-\lambda_2 \lambda_3^X) \sum_{y=0}^{\infty} \frac{(\lambda_2 \lambda_3^X)^y}{y!} = \exp[\lambda_2 \lambda_3^X (s - 1)]$$

Consequently  $E(s^Y|X)$  is finite, hence  $s^Y$  is integrable, for any  $s < 1$ . Thus again by (3)

$$E(s^Y) = E(\exp[\lambda_2 \lambda_3^X (s - 1)]), \quad s < 1$$

By (2) and (3)  $Xs^Y$ ,  $s < 1$ , is also integrable. Hence (2) implies

$$E(Xs^Y) = \lambda_1 E[(\lambda_3 s)^Y], \quad s < 1$$

The last three equations yield

$$E(X \exp[\lambda_2 \lambda_3^X (s - 1)]) = \lambda_1 E(\exp[\lambda_2 \lambda_3^X (\lambda_3 s - 1)])$$

for any  $s < 1$ . Put in the last equation  $H(x) = \exp(-\lambda_2 \lambda_3^x) x! P(X = x)$ ,  $x \in \mathbb{N}$ . Then

$$\sum_{x=0}^{\infty} x \exp(\lambda_2 \lambda_3^x s) \frac{H(x)}{x!} = \lambda_1 \sum_{x=0}^{\infty} \exp(\lambda_2 \lambda_3^{x+1} s) \frac{H(x)}{x!}, \quad s \leq 0$$

implies

$$\sum_{x=0}^{\infty} \exp[(\lambda_2 \lambda_3 s) \lambda_3^x] \frac{H(x+1)}{x!} = \lambda_1 \sum_{x=0}^{\infty} \exp[(\lambda_2 \lambda_3 s) \lambda_3^x] \frac{H(x)}{x!}, \quad s \leq 0 \tag{4}$$

Consider now new r.v.'s  $U$  and  $V$  with distributions

$$P(U = \lambda_3^x) = kH(x+1)/x!, \quad P(V = \lambda_3^x) = k\lambda_1 H(x)/x!$$

$x \in \mathbb{N}$ . Then by (4) for any  $z \geq 0$

$$E[\exp(-zU)] = E[\exp(-zV)]$$

Consequently for all  $x \in \mathbb{N}$

$$H(x+1) = \lambda_1 H(x)$$

and hence

$$H(x) = \lambda_1^x H(0)$$

Finally by the definition of  $H$

$$P(X = x) = c(\lambda_1, \lambda_2, \lambda_3) \exp(\lambda_2 \lambda_3^x) \frac{\lambda_1^x}{x!}$$

for any  $x \in \mathbb{N}$  and some constant  $c(\lambda_1, \lambda_2, \lambda_3) > 0$ . □

Now we show that under (3) constancy of regression of  $X$  given  $Y$  implies independence of  $X$  and  $Y$ . Consequently  $Y$  is a Poisson r.v. with the parameter  $\lambda_2$  and  $X$  is optional (integrable).

PROPOSITION 1. Assume that  $(X, Y)$  is a random vector with  $\text{supp}(X) = \text{supp}(Y) = \mathbb{N}$ . If for all  $x, y \in \mathbb{N}$  the condition (3) holds and

$$E(X|Y) = c \quad (5)$$

where  $\lambda_2 > 0$ ,  $0 < \lambda_3 \leq 1$  and  $c > 0$  are some constants then  $\lambda_3 = 1$  and consequently r.v's  $X$  and  $Y$  are independent.

PROOF: Similarly as in the proof of Theorem 1 by (5) we obtain an analogue of (4)

$$\sum_{x=0}^{\infty} \exp[\lambda_3(\lambda_2 s)\lambda_3^s] \frac{H(x+1)}{x!} = c \sum_{x=0}^{\infty} \exp[(\lambda_2 s)\lambda_3^s] \frac{H(x)}{x!}, \quad s \leq 0$$

Hence the r.v.  $\lambda_3 U$  and  $V$ , where  $U, V$  are defined above, have the same distribution. Consequently we should have  $\text{supp}(\lambda_3 U) = \text{supp}(V)$ . But this is possible only in the case  $\lambda_3 = 1$ .  $\square$

Finally I point out that unique determination of the joint distribution by (3) and any consistent regression function (not only of power type) remains open.

## References

- AHSANULLAH, M. and J. WESOLOWSKI (1993), Bivariate normality via Gaussian conditional structure, *preprint*.
- ARNOLD, B. C., E. CASTILLO and J. M. SARABIA (1992), *Conditionally specified distributions*, Lecture Notes in Statistics 73, Springer, Berlin.
- ARNOLD, B. C., E. CASTILLO and J. M. SARABIA (1993), Conditionally specified models: structure and inference, in: C. M. Coudras and C. R. Rao (eds), *Multivariate analysis: Future Directions 2*, Elsevier Science B.V., Amsterdam, 441–450.
- ARNOLD, B. C. and D. J. STRAUSS (1991), Bivariate distributions with conditionals in prescribed exponential families, *Journal of the Royal Statistical Society B* 53, 365–375.
- CACOULLOS, T. and H. PAPAGEORGIU (1983), Characterizations of discrete distributions by a conditional distribution and a regression function, *Annals of the Institute of Statistical Mathematics* 35, 95–103.
- JOHNSON, N. L. and S. KOTZ (1992), Non-existence of certain polynomial regression in random sum theory, *Sankhyā, Series A* 54, 455–459.
- KORWAR, R. M. (1975), On characterizing some discrete distributions by linear regression, *Communications in Statistics* 4, 1133–1147.
- KYRIAKOUSSIS, A. (1988), Characterization of bivariate discrete distributions, *Sankhyā, Series A* 50, 286–287.
- KYRIAKOUSSIS, A. and H. PAPAGEORGIU (1991), Characterizations of logarithmic series distributions, *Statistica Neerlandica* 45, 1–8.
- WESOLOWSKI, J. (1995a), Bivariate distributions via a Pareto conditional distribution and a regression function, *Annals of the Institute of Statistical Mathematics* 47, 177–183.
- WESOLOWSKI, J. (1995b), Bivariate discrete measures via a power series conditional distribution and a regression function, *Journal of Multivariate Analysis* 55, 219–229.

Received: October 1993. Revised: September 1994.