Austral. J. Statist. **39**(1), 1997, 69–78

ON CHARACTERIZING DISTRIBUTIONS VIA LINEARITY OF REGRESSION FOR ORDER STATISTICS

JACEK WESOLOWSKI[1] AND MOHAMMAD AHSANULLAH[2]

Warsaw University of Technology and *Rider University*

Summary

Let $X_1, \ldots, X_n$ be a random sample from an absolutely continuous distribution with the corresponding order statistics $X_{1:n} \leq X_{2:n} \leq X_{n:n}$. A complete solution of the problem, posed in 1967 by T. Ferguson, of determining the distribution by linearity of regression of $X_{k+2:n}$ with respect to $X_{k:n}$ is given. The only possible distributions are of the exponential, power and Pareto type. A linear regression relation for exponents of order statistics is also considered.

Key words: Order statistics; linearity of regression; exponential distribution; power distribution; Pareto distribution.

1. Introduction

Let $X, X_1, \ldots, X_n$ be independent and identically distributed (i.i.d.) random variables (r.v.s) with distribution function (d.f.) F. Denote by $X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{n:n}$ the respective order statistics. If we want to predict $X_{k+m:n}$ knowing $X_{k:n}$ then the best unbiased predictor with respect to the squared-error loss is $\mathrm{E}(X_{k+m:n} \mid X_{k:n})$. It is not difficult to check that it is linear if X is of exponential, power function or Pareto type.

Extending earlier work, Ferguson (1967) first gave a complete solution to the problem of determining all d.f.s F for which

$$\mathrm{E}(X_{k+m:n} \mid X_{k:n}) = aX_{k:n} + b, \qquad \text{a.s.} \tag{1}$$

for $m = 1$ under continuity assumption, and posed the question for $m = 2$ (essentially he dealt with $m = -1$, but there is an obvious duality in taking positive and negative ms in (1)). Nagaraja (1988a) considered the problem again for $m = 1$ but within the class of discrete distributions. For the continuous case,

Received August 1995; revised February 1996; accepted October 1996.

[1]Mathematical Institute, Warsaw University of Technology, Plac Politechniki 1, 00-661 Warsaw, Poland. email: wesolo@alpha.im.pw.edu.pl

[2]Dept of Management Sciences, Rider University, Lawrenceville, NJ 08648-3099, USA. email: ahsan@enigma.rider.edu

Acknowledgments. The authors thank the referee for a thorough review and stimulating comments that considerably improved the presentation of the results. The referee's remarks concerning unnecessary integrability assumptions are particularly appreciated. Anna Dembińska's careful reading also helped to clarify several points.

Nagaraja (1988b) showed that if $\mathrm{E}(X_{m+1:n} \mid X_{m:n})$ and $\mathrm{E}(X_{m:n} \mid X_{m+1:n})$ are both linear then the parent distribution is exponential. Arnold *et al.* (1992) observed that the cases where $m > 1$ remained open. The present paper considers the case $m = 2$, giving a complete solution when F is absolutely continuous.

Ferguson's (1967) result led to other developments. Assume that X is of the continuous type and determine its distribution by the condition

$$\mathrm{E}[G(X_{1:n}, X_{2:n}, \ldots, X_{n:n}) \mid X_{k:n}] = H(X_{k:n}), \tag{2}$$

where $G\colon \mathbb{R}^n \to \mathbb{R}$ and $H\colon \mathbb{R} \to \mathbb{R}$ are known functions and $k \in \{1, \ldots, n\}$ is fixed. The general solution of this question is not known. However many special cases have been treated in the literature. To give a wider perspective on our results we recall them now briefly.

Fisz (1958) considered the case $n = 2$, $G(x,y) = y - x$, $k = 1$ and $H(x) =$ const. (essentially he was interested in independence of $X_{2:2} - X_{1:2}$ and $X_{1:2}$) for absolutely continuous F. Rogers (1963) extended the result to any n by taking $G(x_1, \ldots, x_n) = x_{k+1} - x_k$. Then, as mentioned earlier, Ferguson (1967) studied the case $G(x_1, \ldots, x_n) = x_{k-1}$ and $H(x) = ax + b$ for continuous F. Dallas (1973) considered (2) for $G(x_1, \ldots, x_n) = (n-1)^{-1} \sum_{i=2}^{n} (x_i - x_1)$ and linear H. The results given in Beg & Kirmani (1974) also fit the scheme with $G(x_1, \ldots, x_n) = (x_1 + \cdots + x_n)/n$ and linear H. Dallas (1976) treated the case $G(x_1, \ldots, x_n) = x_1^r + \cdots + x_n^r$ and $H(x) = cx^r$. Wang & Srivastava (1980) gave characterization theorems by linearity of regression in (2) with $G(x_1, \ldots, x_n) = (n-k)^{-1} \sum_{i=k+1}^{n} (x_i - x_k)$ and $G(x_1, \ldots, x_n) = (k-1)^{-1} \sum_{i=1}^{k-1} (x_k - x_i)$. Khan & Khan (1987) dealt with the problem for $G(x_1, \ldots, x_n) = x_{k+1}^r$ and $H(x) = cx^r$. Exactly the same G with $H(x) = cx^r + d$ was considered in Khan & Ali (1987) (see also El-Din *et al.*, 1991). A recent development has been given in Beg & Balasubramanian (1990) with $G(x_1, \ldots, x_n) = (s-1)^{-1} \sum_{i=1}^{s-1} g(X_{i:n})$ and $H(x) = \frac{1}{2}[g(x) + g(a^+)]/2$, where g is a continuous function on the interval (a, b) which is the support of F. Beg & Kirmani (1978) obtained related results involving relations between conditional moments, as also did Mukherjee & Roy (1986), Khan & Beg (1987), Roy & Mukherjee (1991), Swanepoel (1991) and Pakes *et al.* (1996). For recent surveys see Arnold *et al.* (1992) or Johnson *et al.* (1994). Many important statistical distributions, including exponential, Weibull, Pareto and power, are involved in these investigations.

Observe that the following two specializations of (2) allow an immediate answer.

(i) Consider (2) with $G(x_1, \ldots, x_n) = x_{k+1}$ for some $k \in \{1, \ldots, n-1\}$ and assume that F has a density f. Then, under an integrability assumption and writing $\bar{F} = 1 - F$, (2) implies that

$$(n-k) \int_x^{r_X} y[\bar{F}(y)]^{n-k-1} f(y)\, dy = [\bar{F}(x)]^{n-k} H(x) \quad (\text{all } x \in (\ell_X, r_X)),$$

where $[\ell_X, r_X]$ is the support of the X (this notation is kept throughout the paper), $-\infty \le \ell_X < r_X \le \infty$ if only H is continuous. Hence it follows easily that for any $x \in (\ell_X, r_X)$

$$\bar{F}(x) = \bar{F}(\ell_X)\exp\left[-\frac{1}{n-k}\int_{\ell_X}^{x}\frac{H'(u)}{H(u)-u}\,du\right].$$

(ii) Similarly for $G(x_1,\ldots,x_n) = x_{k-1}$ for some $k \in \{2,\ldots,n\}$ and continuous H it is not difficult to observe, under an integrability and absolute continuity assumption, that for any $x \in (\ell_X, r_X)$,

$$F(x) = F(r_X)\exp\left[\frac{1}{k-1}\int_{x}^{r_X}\frac{H'(u)}{u-H(u)}\,du\right].$$

In this paper we are interested essentially in (2) with $G(x_1,\ldots,x_n) = x_{k+2}$ for some $k \in \{1,\ldots,n-2\}$. Linearity of regression is considered, i.e. $H(x) = ax + b$. It leads to new characterizations of the exponential, power and Pareto distributions: these are the main results of the paper and are given in Section 3. However Section 2 is devoted to a new characterization of the exponential distribution by considering $G(x_1,\ldots,x_n) = \exp(cx_{k+2})$ and $H(x) = a\exp(cx)$ in (2); that result, while being of independent interest, is used in the proof of our main result.

Throughout the paper $\mathcal{L}(X)$ denotes the probability distribution of the r.v. X; all the equations between r.v.s are understood in the a.s. sense.

2. Regression of Exponents

Let $\mathcal{E}(\lambda,\gamma)$ denote the shifted exponential distribution with the tail of d.f. $\bar{F}(x) = \exp[-\lambda(x-\gamma)^+]$, where $\lambda > 0$, γ are some constants. For $\mathcal{L}(X) = \mathcal{E}(\lambda,\gamma)$, if $c < (n-k-1)\lambda$, then $\mathrm{E}[\exp(cX_{k+2:n})] < \infty$ and it can be easily checked that

$$\mathrm{E}(e^{cX_{k+2:n}} \mid X_{k:n}) = \frac{\lambda^2(n-k)(n-k-1)}{[\lambda(n-k)-c][\lambda(n-k-1)-c]}e^{cX_{k:n}}. \tag{3}$$

Here we are interested in a converse result.

Theorem 1. *Let* $\mathrm{E}[\exp(cX_{k+2:n})] < \infty$. *Assume that* X *is absolutely continuous. If*

$$\mathrm{E}[\exp(cX_{k+2:n}) \mid X_{k:n}] = a\exp(cX_{k:n}), \tag{4}$$

where $k \le n-2$, $c \ne 0$ *and* $a > 0$, *then* $\mathcal{L}(X) = \mathcal{E}(\lambda,\gamma)$ *for some real* γ *and*

$$\lambda = c\frac{a(2n-2k-1)+\sqrt{a^2+4a(n-k)(n-k-1)}}{2(a-1)(n-k)(n-k-1)};$$

further, $a \in (0,1)$ *if* $c < 0$ *and* $a > 1$ *if* $c > 0$.

Proof. Obviously $a \leq 0$ is impossible since the left hand side of (4) is positive. Also by (4), $a = 1$ is impossible since X is not degenerate at a point.

By (4) and continuity of F it follows that one can assume that the equation

$$\int_x^{r_X} e^{cy}[\bar{F}(x) - \bar{F}(y)][\bar{F}(y)]^{n-k-2} f(y)\,dy = \frac{a[\bar{F}(x)]^{n-k}\,e^{cx}}{(n-k)(n-k-1)}, \tag{5}$$

holds for any $x \in (\ell_X, r_X)$, where f is the density of F. Consequently (5) yields that for any $x \in (\ell_X, r_X)$, $f(x) = F'(x)$ and

$$\begin{aligned} -f(x)\int_x^{r_X} & e^{cy}[\bar{F}(y)]^{n-k-2} f(y)\,dy \\ &= -\frac{a[\bar{F}(x)]^{n-k-1} f(x) e^{cx}}{n-k-1} + \frac{ac[\bar{F}(x)]^{n-k} e^{cx}}{(n-k)(n-k-1)}. \end{aligned} \tag{6}$$

By (6) we conclude that $f > 0$ in (ℓ_X, r_X). Thus we can divide both sides of (6) by $-f$ obtaining

$$\int_x^{r_X} e^{cy}[\bar{F}(y)]^{n-k-2} f(y)\,dy = \frac{a[\bar{F}(x)]^{n-k-1} e^{cx}}{n-k-1} + \frac{ac\{[\bar{F}(x)]^{n-k} e^{cx}\}}{(n-k)(n-k-1)f(x)}. \tag{7}$$

Now again the left hand side of (7) is differentiable and consequently f' exists in (ℓ_X, r_X). Hence, upon differentiation of (7) and some elementary algebra it follows that

$$\begin{aligned} acf'(x)\bar{F}^2(x) - ac^2 f(x)\bar{F}^2(x) &+ 2ac(n-k)f^2(x)\bar{F}(x) \\ &+ (1-a)(n-k)(n-k-1)f^3(x) = 0, \end{aligned} \tag{8}$$

for any $x \in (\ell_X, r_X)$. Denoting $y = \bar{F}$ (i.e. $f = -y'$, $f' = -y''$) we get by (8) a second order differential equation in (ℓ_X, r_X),

$$-y''y^2 + cy'y^2 + 2(n-k)y'^2 y - \frac{1-a}{ac}(n-k)(n-k-1)y'^3 = 0. \tag{9}$$

Substituting $u(y) = y'$ in (9) we have

$$-u'y^2 + cy^2 + 2(n-k)uy - \frac{1-a}{ac}(n-k)(n-k-1)u^2 = 0, \tag{10}$$

since $y' \boxminus 0$ is impossible (it yields $f \equiv 0$). In (10), substitute $u(y) = v(y) - \beta y$, where β is a real constant such that

$$\frac{1-a}{ac}(n-k)(n-k-1)\beta^2 + (2n-2k-1)\beta - c = 0.$$

Hence β has to be equal to one of the numbers

$$\beta_{1,2} = -c\frac{a(2n-2k-1) \pm \sqrt{a^2+4a(n-k)(n-k-1)}}{2(1-a)(n-k)(n-k-1)}.$$

Then (10) takes the form

$$-y^2v' + 2(n-k)\left(1+\frac{1-a}{ac}(n-k-1)\beta\right)yv - \frac{1-a}{ac}(n-k)(n-k-1)v^2 = 0, \quad (11)$$

which is the Bernoulli equation.

Take $\beta = \beta_1$. Consider first the trivial solution $v \equiv 0$. Then we have as a solution of (9),

$$y = Ke^{-\beta_1 x} \qquad (x \in (\ell_X, r_X)), \quad (12)$$

where K is a constant. By properties of a d.f. it follows from (12) that $\gamma = \ell_X > -\infty$, $r_X = \infty$ and

$$\bar{F}(x) = \exp[-\beta_1(x-\gamma)] \qquad (x \in (\gamma, \infty)). \quad (13)$$

Now consider the case that v is not identically zero. Then applying standard techniques for a Bernoulli equation to (11), we obtain

$$-y' = y\frac{\beta_1 Dy^B + C}{Dy^B + A}, \quad (14)$$

where

$$A = \frac{1}{\beta_1 - \beta_2}, \quad B = \sqrt{1+\frac{4(n-k)(n-k-1)}{a}}, \quad C = \frac{\beta_2}{\beta_1 - \beta_2},$$

and D is a constant. Observe that (14) implies

$$f(x) = \bar{F}(x)\frac{\beta_1 D\bar{F}^B(x) + C}{D\bar{F}^B(x) + A} \qquad (x \in (\ell_X, r_X)). \quad (15)$$

Consider two possible cases. For the first, observe that for $c < 0$ we have $a \in (0,1)$. Consequently $\beta_2 < 0 < \beta_1$. Then $A > 0$, $B > 0$ and $C < 0$. Observe that by taking x less than but close to r_X in (15) its right hand side becomes negative, which is impossible since on the left hand side we have a density function. Consequently in this case the solution of (9) is not a tail of a distribution function.

In the second case, $c > 0$ and so $a > 1$. Consequently $\beta_1 > \beta_2 > 0$. This time $A > 0$, $B > 0$ and $C > 0$. Hence solving (14) we have

$$y = \left(\frac{\beta_1 Dy^B + C}{\beta_1 D + C}\right)^{\beta_2/c} e^{-\beta_2 x}, \quad (16)$$

for $x \in (\ell_X, r_X)$. Observe that (16), by taking limits for $x \uparrow r_X$, implies $r_X = \infty$. Additionally it follows from (16) that

$$\bar{F}(x) \geq K e^{-\beta_2 x} \qquad (x \geq \ell_X),$$

where $K = 1$ if $D \leq 0$ and

$$K = \left(\frac{C}{\beta_1 D + C}\right)^{\beta_2/C}$$

if $D > 0$. On the other hand since $\mathrm{E}[\exp(cX_{k+2:n})] < \infty$ then for sufficiently large x

$$\bar{F}(x)^{n-k-1} \leq m e^{-cx},$$

where m is a constant. If both the above inequalties are to be satisfied then we must have $\beta_2(n-k-1) > c$. Hence

$$1 < \frac{a}{2(n-k)(a-1)}\left[2(n-k) - 1 - \sqrt{\frac{4(n-k)(n-k-1)}{a} + 1}\right],$$

yielding $a < 1$, a contradiction.

Hence the only solution is given in (13).

3. Linearity of Regression

In this section we are interested in the conditional moment $\mathrm{E}(X_{k+2:n} \mid X_{k:n})$, not only in the exponential case, but also for the power and Pareto distributions. Denote by $\mathrm{POW}(\theta; \mu, \nu)$ a power distribution defined by the density

$$f(x) = \frac{\theta(\nu - x)^{\theta-1}}{(\nu-\mu)^\theta} I_{(\mu,\nu)}(x),$$

where $\theta > 0$, $-\infty < \mu < \nu < \infty$ are some constants. By $\mathrm{PAR}(\theta; \mu, \delta)$ denote the Pareto distribution with the probability density function (p.d.f.)

$$f(x) = \frac{\theta(\mu+\delta)^\theta}{(x+\delta)^{\theta+1}} I_{(\mu,\infty)}(x),$$

where $\theta > 0$, and μ, δ are some real constants such that $\mu + \delta > 0$.

Observe that if X has the d.f. F and the p.d.f. f then for any $x \in (\ell_X, r_X)$,

$$\begin{aligned} &\mathrm{E}(X_{k+2:n} \mid X_{k:n} = x) \\ &\quad = \frac{(n-k)(n-k-1)}{(\bar{F}(x))^{n-k}} \int_x^{r_X} y[\bar{F}(x) - \bar{F}(y)][\bar{F}(y)]^{n-k-2} f(y)\, dy. \end{aligned}$$

Consequently it can be easily verified that in all three cases of the exponential, power and Pareto distributions, the regression relation we are interested in is linear, i.e.

$$\mathrm{E}(X_{k+2:n} \mid X_{k:n}) = aX_{k:n} + b, \tag{17}$$

where the constants a and b have the following forms:

1. for the POW$(\theta;\mu,\nu)$ distribution,

$$a = \frac{\theta^2(n-k)(n-k-1)}{[\theta(n-k-1)+1][\theta(n-k)+1]} < 1, \qquad b = \nu(1-a);$$

2. for the PAR$(\theta;\mu,\delta)$ distribution with $\theta > 1$,

$$a = \frac{\theta^2(n-k)(n-k-1)}{[\theta(n-k-1)-1][\theta(n-k)-1]} > 1, \qquad b = \delta(a-1);$$

3. for the $\mathcal{E}(\lambda,\gamma)$ distribution,

$$a = 1, \qquad b = \frac{2n-2k-1}{\lambda(n-k)(n-k-1)}.$$

The question we address here is the following: are the distributions given above the only cases for which linearity of regression (17) holds? The affirmative answer given in Theorem 2 is the main result of the paper.

Theorem 2. *Let* $\mathrm{E}(|X_{k+2:n}|) < \infty$. *Assume that* X *is absolutely continuous. If for some* $k \le n-2$ *and real* a *and* b *the linearity of regression (17) holds, then only the following three cases are possible:*

1. *if* $a < 1$ *then* $\mathcal{L}(X) = \mathrm{POW}(\theta;\mu,\nu)$ *where*

$$\theta = \frac{a(2n-2k-1)+\sqrt{a^2+4a(n-k)(n-k-1)}}{2(1-a)(n-k)(n-k-1)}, \qquad \nu = \frac{b}{1-a},$$

and $\mu < \nu$ *is a real number;*

2. *if* $a > 1$ *then* $\mathcal{L}(X) = \mathrm{PAR}(\theta;\mu,\delta)$ *where* μ *is a real number and*

$$\theta = \frac{a(2n-2k-1)+\sqrt{a^2+4a(n-k)(n-k-1)}}{2(a-1)(n-k)(n-k-1)} > 1, \qquad \delta = \frac{b}{a-1};$$

3. *if* $a = 1$ *then* $b > 0$ *and* $\mathcal{L}(X) = \mathcal{E}(\lambda,\gamma)$ *where* γ *is a real number and*

$$\lambda = \frac{2n-2k-1}{b(n-k)(n-k-1)}.$$

Proof. The proofs in cases 1 and 2 follow easily from Theorem 1 and are considered first; case 3 needs a separate argument.

Case 1. First consider the case $a < 1$ and denote $\nu = b/(1-a)$. Assume that $r_X = \infty$; then by (17), $x \leq ax + b$ for any $x \in \mathbb{R}$, which is impossible since $a < 1$. Consequently $r_X = \nu < \infty$. Define a new r.v. Z by the relation $Z = -\ln(\nu - X)$. Observe that by (17) for the order statistics from the sample $Z_1, \ldots, Z_n$,

$$\mathrm{E}[\exp(-Z_{k+2:n}) \mid Z_{k:n}] = a\exp(-Z_{k:n}).$$

Hence by Theorem 1, $\mathcal{L}(Z) = \mathcal{E}(\lambda, \gamma)$. Consequently $\mathcal{L}(X) = \mathrm{POW}(\theta; \mu, \nu)$ with $\theta = \lambda$, ν given above and $\mu = \nu - \exp(-\gamma)$.

Case 2. Consider now $a > 1$ in (17). For $\delta = b/(a-1)$ define $Z = \ln(\delta + X)$ and consider order statistics for the sample of Zs . Then by (17)

$$\mathrm{E}[\exp(Z_{k+2:n}) \mid Z_{k:n}] = a\exp(Z_{k:n}).$$

Consequently by Theorem 1, $\mathcal{L}(Z) = \mathcal{E}(\lambda, \gamma)$. Thus $\mathcal{L}(X) = \mathrm{PAR}(\theta; \mu, \delta)$ with $\theta = \lambda$, $\mu = \exp(\lambda\gamma) - \delta = \ell_X$ and δ defined above. Observe that (17) implies that $\ell_X > b/(1-a)$ and consequently $\mu + \delta > 0$.

Case 3. Now take $a = 1$ in (17). Then due to (17) and continuity of F it can be assumed that the equation

$$\int_x^{r_X} y[\bar{F}(x) - \bar{F}(y)][\bar{F}(y)]^{n-k-2} f(y)\,dy = \frac{[\bar{F}(x)]^{n-k}(x+b)}{(n-k)(n-k-1)}, \tag{18}$$

holds for any $x \in (\ell_X, r_X)$, where $\bar{F} = 1 - F$ and f is the density of F. Now repeating the argument from the proof of Theorem 1 we derive from (18) the differential equation

$$-u'y^2 + 2(n-k)uy + cu^2 = 0, \tag{19}$$

where $y = \bar{F}$ (i.e. $f = -y'$, $f' = -y''$), $u(y) = y'$ and $c = b(n-k)(n-k-1)$.

Observe that (19) is a kind of Bernoulli differential equation, which can be easily solved with a help of the routine technique. Consequently

$$y' = \frac{y^{2(n-k)}}{D - cy^{2(n-k)-1}/[2(n-k)-1]}. \tag{20}$$

Rewrite (20) as

$$f(x) = \frac{[\bar{F}(x)]^{2(n-k)}}{c[\bar{F}(x)]^{2(n-k)-1}/[2(n-k)-1] - D} \qquad (x \in (\ell_X, r_X)). \tag{21}$$

Observe that the left hand side of (21) is always non-negative in (ℓ_X, r_X). However if $x \uparrow r_X$ then the right hand side becomes negative (since $\bar{F}(x) \to 0$) unless

$D \leq 0$. Obviously $c \leq 0$ is impossible due to (17) and the absolute continuity assumption.

Now we solve (20) obtaining

$$-\frac{D}{[2(n-k)-1]y^{2(n-k)-1}} - \frac{c}{2(n-k)-1}\ln(y) = x+d \qquad (x \in (\ell_X, r_X)), \quad (22)$$

(since F is continuous), where d is a real constant. Since $\lim_{x \downarrow \ell_X} y(x) = 1$ then by (22) it follows that $\gamma = \ell_X > -\infty$ and

$$-\frac{D}{2(n-k)-1} = \gamma + d \geq 0. \qquad (23)$$

Then

$$\bar{F}(x) = \exp\left[-\frac{2(n-k)-1}{c}\left(x+d-\frac{\gamma+d}{[\bar{F}(x)]^{2(n-k)-1}}\right)\right] \qquad (24)$$

for $x \in (\gamma, r_X)$. Take first $r_X < \infty$. Then upon allowing $x \uparrow r_X$ in (22) we observe that its left hand side tends to $+\infty$, while the right hand side remains finite positive. Hence $r_X = \infty$.

Observe now that by the integrability assumption $\mathrm{E}(|X_{k+2:n}|) < \infty$ it follows that

$$\bar{F}(x)^{n-k-1} \leq \frac{m}{x}$$

for sufficiently large positive x, where m is a constant. Consequently (24) implies

$$\bar{F}(x) \geq \exp\left[-\frac{2(n-k)-1}{c}\left(x+d-\frac{(\gamma+d)x^{n-k}}{m^{2(n-k)-1}}\right)\right] \qquad (25)$$

for $x \geq \gamma$. Observe that for $\gamma + d > 0$ the right hand side of (25) is unbounded. Hence by (23) we conclude that $d = -\gamma$ and then the result follows by (24).

References

ARNOLD, B.C., BALAKRISHNAN, N. & NAGARAJA, H.N. (1992). *A First Course in Order Statistics*. New York: Wiley.

BEG, M.I. & BALASUBRAMANIAN, K. (1990). Distributions determined by conditioning on a single order statistic. *Metrika* **37**, 37–43.

— & KIRMANI, S.N.U.A. (1974). On a characterization of exponential and related distributions. *Austral. J. Statist.* **16**, 163–166; Correction (1976) *ibid.* **18**, 85.

— & — (1978). Characterization of the exponential distribution by a weak homoscedasticity. *Comm. Statist. A — Theory Methods* **7**, 307–310.

DALLAS, A.C. (1973). A characterization of the exponential distribution. *Bull. Soc. Math. Grèce* **14**, 172–175.

— (1976). Characterizing the Pareto and power distributions. *Ann. Inst. Statist. Math.* **28**, 491–497.

EL-DIN, M.M.M., MAHMOUD, M.A.W. & YOUSSEF, S.E.A. (1991). Moments of order statistics from parabolic and skewed distributions and a characterization of Weibull distribution. *Comm. Statist. B — Simulation Comput.* **20**, 639–645.

FERGUSON, T.S. (1967). On characterizing distributions by properties of order statistics. *Sankhyā Ser. A* **29**, 265–278.

FISZ, M. (1958). Characterization of some probability distributions. *Skand. Aktuarietidskr.* **41**, 65–70.

JOHNSON, N.L., KOTZ, S. & BALAKRISHNAN, N. (1994). *Continuous Univariate Distributions*, Vol. 1. New York: Wiley.

KHAN, A.H. & ALI, M.M. (1987). Characterization of probability distributions through higher order gap. *Comm. Statist. A — Theory Methods* **16**, 1281–1287.

— & BEG, M.I. (1987). Characterization of the Weibull distribution by conditional variance. *Sankhyā Ser. A* **49**, 268–271.

— & KHAN, S.A. (1987). Moments of order statistics from Burr distribution and its characterizations. *Metron* **45**, 21–29.

MUKHERJEE, S.P. & ROY, D. (1986). Some characterizations of the exponential and related life distributions. *Calcutta Statist. Assoc. Bull.* **35**, 189–197.

NAGARAJA, H.N. (1988a). Some characterizations of discrete distributions based on linear regressions of adjacent order statistics. *J. Statist. Plann. Inference* **20**, 65–75.

— (1988b). Some characterizations of continuous distributions based on regressions of adjacent order statistics and record values. *Sankhyā Ser. A* **50**, 70–73.

PAKES, A.G., FAKHRY, M.E., MAHMOUD, M.R. & AHMAD, A.A. (1996). Characterizations by regressions of order statistics. *J. Appl. Statist. Sci.* **3**, 11–23.

ROGERS, G.S. (1963). An alternative proof of the characterization of the density Ax^B. *Amer. Math. Monthly* **70**, 857–858.

ROY, D. & MUKHERJEE, S.P. (1991). Revival of a life-time distribution. *Econ. Qual. Control* **6**, 164–173.

SWANEPOEL, J.W.H. (1991). A note on the characterization of the exponential distribution by the covariance of spacings. *South African Statist. J.* **25**, 73–78.

WANG, Y.H. & SRIVASTAVA, R.C. (1980). A characterization of the exponential and related distributions by linear regression. *Ann. Statist.* **8**, 217–220.