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Identification of probability measures via distribution of quotients

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Abstract

Straightforward generalizations of the classical Kotlarski characterization of normality using bivariate Cauchy distribution of quotients of independent r.v.'s are given. The symmetry assumption in Kotlarski's result is omitted. Two larger families of bivariate distributions are considered: symmetric second kind beta and elliptically contoured measures. © 1997 Elsevier Science B.V.

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1. Introduction

It is well known that for independent normal zero-mean r.v.'s X and Y the quotient X/Y has a symmetric Cauchy distribution. Beginning with late 1950s, many efforts were devoted to study different versions of the converse problem. Some examples of non-normal r.v.'s with Cauchy quotient were given, for instance, in Laha (1958, 1959a) and Kotlarski (1960). In Laha (1959b) also some additional analytical conditions forcing normality were proposed. In order to identify the normal distribution, Seshadri (1969) assumed additional independence of the quotient and the euclidean norm. These assumptions have been weakened in Wesolowski (1991). A bivariate version of Seshadri's result, featuring elliptically contoured distributions, has been given recently in Wesolowski (1992).

A remarkable contribution to this field establishing normality via bivariate Cauchy quotients was given in Kotlarski (1967).

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Theorem 1 (Kotlarski, 1967). *Let X_1, X_2, X_3 be independent symmetric r.v.'s with $X_3 \neq 0$ a.s. The random vector $(X_1/X_3, X_2/X_3)$ has the bivariate Cauchy distribution, i.e. it has a density f of the form*

$$f(x,y) = \frac{1}{\pi(1+x^2+y^2)^{3/2}}, \quad x,y \in \mathbb{R}. \tag{1}$$

iff X's are normal.

This was a consequence of Kotlarski's interest in problems of identifiability of distributions of independent r.v.'s by the joint distribution of linear forms. A considerable development here was given in Rao (1973).

A starting point of our interest in this characterization was a feeling that the symmetry assumption in Theorem 1 is somewhat technical. In the proof, as given by Kotlarski, the distribution of squares of r.v.'s was identified as chi-square with one degree of freedom. If one does not assume symmetry then the family of distributions of r.v.'s with squares having the chi-square distribution is much wider. For details see, for instance, Roberts (1971). However, condition (1) is much more informative as it will be shown in the sequel.

Notice that the symmetry assumption may be replaced by identity of distributions. This is an immediate consequence of the following observation due to Laha (1959b): for i.i.d. r.v.'s X and Y, a distribution of the quotient X/Y is symmetric iff X has a symmetric distribution.

The following characterization of the normal law is given in Letac (1981).

Theorem 2 (Letac, 1981). *Let $\mathbf{X} = (X_1,X_2,X_3)$ be an a.s. non-zero random vector with independent components. If $\mathbf{X}/||\mathbf{X}||$ has a uniform distribution on the unit sphere in $\mathbb{R}^3$, where $||\cdot||$ denotes the euclidean norm, then $\mathbf{X}$ is Gaussian.*

It may be treated as another version of Kotlarski's theorem without explicitly giving a symmetry assumption. However, once again, it is hidden among other assumptions.

While relaxing symmetry, we consider at the same time wider families of bivariate measures instead of the Cauchy law. In Section 2, the bivariate symmetric beta distribution of the second kind is investigated. Letac's theorem is obtained here as a special case of a more general result. In Section 3, elliptically contoured measures are studied.

Let us point out that in both the cases only partial identification is possible, i.e. not all distributions of the r.v.'s involved are uniquely identified by conditions like (1). However, all the results are straightforward extensions of Kotlarski's theorem.

2. Quotients with the bivariate symmetric beta distribution of the second kind

Denote by $G_r(\alpha,\beta,\gamma)$ the reflected generalized gamma distribution defined by the density

$$f(x) = \frac{\gamma\beta^\alpha}{2\Gamma(\alpha)}|x|^{\alpha\gamma-1}\exp^{-\beta|x|^\gamma}, \quad x \in \mathbb{R}$$

for positive α, β, γ. This distribution was introduced in Plucinska (1966) for some models concerning rheostat resistance (see also Johnson et al., 1994). For $\alpha = 1/\gamma = \frac{1}{2}$, it is the normal distribution and for $\alpha = \gamma = 1$, it is the Laplace distribution.

By SB2(α, γ) denote a bivariate symmetric beta distribution of the second kind with the density

$$f(x,y) = \frac{\gamma^2 \Gamma(3\alpha)}{4\Gamma^3(\alpha)} \frac{|xy|^{\alpha\gamma-1}}{(1+|x|^\gamma+|y|^\gamma)^{3\alpha}}, \quad x,y \in \mathbb{R} \tag{2}$$

for positive α and γ. It is a symmetric version of the bivariate Burr-type XII distribution given in Takahasi (1965). For $\alpha = 1/\gamma = \frac{1}{2}$, it is the bivariate Cauchy distribution.

The sufficiency part of Theorem 1 can be easily extended to

Proposition 1. *Let X_1, X_2, X_3 be i.i.d. r.v.'s having a $G_r(\alpha,\beta,\gamma)$ distribution. Then $(X_1/X_3,\ X_2/X_3)$ has an* SB2(α, γ) *distribution.*

Proof. It follows immediately from the formula for the joint density g of the quotients

$$g(x,y) = \int_{\mathbb{R}} z^2 f_1(xz) f_2(yz) f_3(z)\,\mathrm{d}z\,, \tag{3}$$

where f_i is the density of X_i, $i = 1,2,3$. □

The main result of this section is the following:

Theorem 3. *Let X_1, X_2, X_3 be independent r.v.'s and $X_3 \neq 0$ a.s. The random vector $(X_1/X_3,\ X_2/X_3)$ has an* SB2(α,γ) *distribution iff two of the X's have $G_r(\alpha,\beta,\gamma)$ distributions and γth power of the absolute value of the third of the X's has a gamma distribution with the shape parameter α and the scale β.*

Obviously, Kotlarski's theorem is an immediate consequence of Theorem 3 since for the Cauchy case we have

Corollary 1. *Let X_1, X_2, X_3 be independent r.v.'s with $X_3 \neq 0$ a.s. Assume that $(X_1/X_3,\ X_2/X_3)$ has the bivariate Cauchy distribution with the density* (1). *Then two of the X's are equi-distributed normal and square of the third one of the X's has $\chi^2(1)$ distribution (chi-square with one degree of freedom).*

Before proving the result, let us formulate two other immediate consequences of Theorem 3.

Corollary 2. *Let X_1, X_2, X_3 be independent r.v.'s and suppose that X_3 is positive a.s. If $(X_1/X_3,\ X_2/X_3)$ has the bivariate* SB2(α, γ) *distribution then X_1 and X_2 have a common $G_r(\alpha,\ \beta,\ \gamma)$ distribution for some $\beta > 0$.*

Corollary 3. *Let X_1, X_2, X_3 be independent r.v.'s and suppose that X_3 is positive a.s. If $(X_1/X_3,\ X_2/X_3)$ has the bivariate Cauchy distribution with the density* (1) *then X_1 and X_2 are normal $\mathcal{N}(0,\ \sigma^2)$ for some $\sigma > 0$.*

Remark. Let us now quickly show how Letac's theorem (Theorem 2) follows from Corollary 3. Since the $\boldsymbol{X}/||\boldsymbol{X}||$ is uniform on the unit sphere then the distribution of $(X_1/|X_3|, X_2/|X_3|)$ is uniquely determined to be Cauchy with the density (1) (we know that it is the case if X's are normal). Hence, by Corollary 3, X_1 and X_2 are normal. That X_3 is normal, too, follows by symmetry of the problem with respect to permutation of X's. Yet another proof of the Letac theorem, based on the Deny theorem, was given in Rao and Shanbhag (1989).

Proof of Theorem 3 Necessity. Define $\boldsymbol{Y} = (Y_1, Y_2) = (\ln|X_1| - \ln|X_3|, \ln|X_2| - \ln|X_3|)$. Then by Proposition 1 $(Y_1, Y_2) \stackrel{d}{=} (Z_1, Z_2)$, where $Z_1 = \ln|U_1| - \ln|U_3|$, $Z_2 = \ln|U_2| - \ln|U_3|$ and U_i, $i = 1,2,3$, are i.i.d. $G_r(\alpha,\beta,\gamma)$ r.v.'s. The ch.f. ψ of $\ln|U_1|$ has the form

$$\psi(t) = \frac{\Gamma(\alpha + it/\gamma)}{\Gamma(\alpha)\beta^{it/\gamma}}, \quad t \in \mathbb{R}.$$

Consequently, for the ch.f. ϕ of $\boldsymbol{Y}$ we have

$$\phi(s,t) = \frac{\Gamma(\alpha + is/\gamma)\Gamma(\alpha + it/\gamma)\Gamma(\alpha - i(s+t)/\gamma)}{\Gamma^3(\alpha)}, \quad s,t \in \mathbb{R}.$$

So it does not vanish. Now by Lemma 2 in Kotlarski (1967) it follows that the $|X|$'s have the same distribution as rescaled $|U|$'s. Further (2) implies that a scale is common for all $|X|$'s. Thus $|X_i|^\gamma$, $i = 1,2,3$, have the same gamma distribution with parameters β and α, where β is a positive number. Without any loss of generality we can assume $\beta = 1$.

Consequently, by Theorem 1 from Roberts (1971), X_i has a density

$$f_i(x) = \frac{\gamma}{\Gamma(\alpha)} h_i(x)|x|^{\alpha\gamma - 1} \exp(-|x|^\gamma), \quad x \in \mathbb{R},$$

where $h_i(x) + h_i(-x) = 1, \quad x \in \mathbb{R}, \ i = 1,2,3.$

Denote

$$b(x,y) = \frac{\Gamma(3\alpha)}{4\gamma(1 + |x|^\gamma + |y|^\gamma)^{3\alpha}}, \quad x,y \in \mathbb{R}.$$

Then by (3) we have

$$\int_{\mathbb{R}} h_1(xz)h_2(yz)h_3(z)|z|^{3\alpha\gamma - 1}e^{-|z|^\gamma(1+|x|^\gamma+|y|^\gamma)}\,dz = b(x,y). \tag{4}$$

Observe that $b(x,y)$ does not depend on the sign of x and y. Hence (4), upon changing z to $-z$ and then x to $-x$ and y to $-y$, takes the form

$$\int_{\mathbb{R}} h_1(xz)h_2(yz)h_3(-z)|z|^{3\alpha\gamma - 1}e^{-|z|^\gamma(1+|x|^\gamma+|y|^\gamma)}\,dz = b(x,y). \tag{5}$$

Now we add (4) and (5) to obtain

$$\int_{\mathbb{R}} h_1(xz)h_2(yz)|z|^{3\alpha\gamma - 1}e^{-|z|^\gamma(1+|x|^\gamma+|y|^\gamma)}\,dz = 2b(x,y). \tag{6}$$

Notice that (6) is satisfied for $h_i(x) = \frac{1}{2}$, $i = 1, 2, 3$. Next define $g_i(x) = h_i(x) - \frac{1}{2}$, $x \in \mathbb{R}$, $i = 1, 2$, and notice that they are odd functions. Consequently, (6) implies

$$\int_{\mathbb{R}} g_1(xz)g_2(yz)|z|^{3\alpha\gamma-1}\mathrm{e}^{-|z|^\gamma(1+|x|^\gamma+|y|^\gamma)}\,\mathrm{d}z = 0\,.$$

For an arbitrary (but fixed) w put $y = wx$ in the above equation and consider any positive x. Since $g_1(xz)g_2(yz)$ is even in z then the above equation yields

$$\int_0^\infty g_1(xz)g_2(wxz)z^{3\alpha\gamma-1}\mathrm{e}^{-z^\gamma(1+x^\gamma(1+|w|^\gamma))}\,\mathrm{d}z = 0\,.$$

Upon changing the variable $u = zx$ and denoting $s = x^{-\gamma}$ we get

$$\int_0^\infty g_1(u)g_2(wu)u^{3\alpha\gamma-1}\mathrm{e}^{-u^\gamma(1+|w|^\gamma)}\mathrm{e}^{-su^\gamma}\,\mathrm{d}u = 0$$

for any $s > 0$. This indicates that a Laplace transform is equal to zero. Hence, $g_1(x)$ $g_2(wx) = 0$ a.e. in $\mathbb{R}$. Since w is arbitrary then it follows that for some versions of g_1 and g_2, $g_1(x)g_2(y) = 0$ for any real x and y. Consequently, one of these functions is identically zero.

Now observe that we can renumber the X's and still the joint distribution remains the same, i.e. SB2(α,γ). Consider, for example, $(X_1/X_2,\ X_3/X_2)$. Denote its density by $\hat{f}$. Then for f given in (2) it follows by (3) that

$$\begin{aligned}\hat{f}(x,y) &= \int_{\mathbb{R}} z^2 f_1(xz)f_3(yz)f_2(z)\,\mathrm{d}z = |y|^{-3}\int_{\mathbb{R}} u^2 f_1\left(\frac{x}{y}u\right) f_2\left(\frac{1}{y}u\right) f_3(u)\,\mathrm{d}u \\ &= |y|^{-3} f\left(\frac{x}{y},\frac{1}{y}\right) = f(x,y)\end{aligned}$$

for any $(x,\ y) \in \mathbb{R}^2$.

Consequently, we can repeat the above argument to get $g_i(x)g_j(y) = 0$ for any real x and y and for any $i,j = 1,2,3$, $i \neq j$. Hence, two of g's must be zero.

Sufficiency. Assume that X_3 is non-symmetric. Then it suffices to prove that

$$\int_{\mathbb{R}} |z|^{3\alpha\gamma-1}h_3(z)\mathrm{e}^{-|z|^\gamma(1+|x|^\gamma+|y|^\gamma)}\,\mathrm{d}z = 4b\,(x,y)\,,\quad x,y \in \mathbb{R}. \tag{7}$$

Observe that

$$\int_{\mathbb{R}} |z|^{3\alpha\gamma-1}h_3(z)\mathrm{e}^{-|z|^\gamma(1+|x|^\gamma+|y|^\gamma)}\,\mathrm{d}z = \int_{\mathbb{R}} |z|^{3\alpha\gamma-1}h_3(-z)\mathrm{e}^{-|z|^\gamma(1+|x|^\gamma+|y|^\gamma)}\,\mathrm{d}z\,. \tag{8}$$

By the definition of h_3 upon adding both sides of (8) we obtain easily (7). Similar argument holds if X_1 or X_2 is non-symmetric. □

3. Quotients with bivariate elliptically contoured distributions

Let us recall that a real random vector (X_1, X_2) has the central elliptically contoured (c.e.c) EC$(\sigma_1^2,\ \sigma_2^2,\ \rho)$ ($\sigma_1 > 0$, $\sigma_2 > 0$, $|\rho| \leqslant 1$) distribution iff its ch.f. at any point

$(s,t) \in \mathbb{R}^2$ is a function of the quadratic form $\sigma_1^2 s^2 - 2\rho\sigma_1\sigma_2 st + \sigma_2^2 t^2$ (for $\sigma_1 = \sigma_2 = 1$, $\rho = 0$ it is called spherically invariant, s.i.). If only the distribution has no atom in the origin then the quotient follows the symmetric Cauchy law as it was observed in Szablowski (1986). See also Philips (1989) for a more general result involving spherical matrix distribution. Some versions of the converse result were given in Letac (1981) and Wesolowski (1992).

On the other hand, the bivariate Cauchy is c.e.c. All these facts suggest that an extension of Kotlarski's theorem in this direction might hold. This is confirmed below.

Let $\boldsymbol{X} = (X_1, X_2)$ have a non-degenerate c.e.c. distribution. Then it is well known (see, for example, Fang et al., 1990) that there exist: a 2×2 matrix $\mathbf{A}$, a non-negative r.v. R and independent of it bivariate random vector $\boldsymbol{Z} = (Z_1, Z_2)$ distributed uniformly on the unit sphere, such that $\boldsymbol{X} \stackrel{d}{=} R\mathbf{A}\boldsymbol{Z}^{\mathrm{T}}$. All these quantities are uniquely determined by the distribution of $\boldsymbol{X}$. On the other hand, if $\boldsymbol{Y} = (Y_1, Y_2)$ has an c.e.c. distribution and R is a r.v. independent of $\boldsymbol{Y}$ then $\boldsymbol{X} = (X_1, X_2) \stackrel{d}{=} R\boldsymbol{Y}$ is also c.e.c. In this section, first, we consider a converse problem.

Suppose that $\boldsymbol{X} = (X_1, X_2)$ is c.e.c. Let us also assume that $\boldsymbol{X} \stackrel{d}{=} R\boldsymbol{Y}$, where R is a non-negative r.v. independent of $\boldsymbol{Y} = (Y_1, Y_2)$. Is $\boldsymbol{Y}$ c.e.c.?

As an answer, we have the following theorem.

Theorem 4. *Let $\boldsymbol{X} = (X_1, X_2) \stackrel{d}{=} R\boldsymbol{Y}$, where R is a positive r.v. independent of $\boldsymbol{Y} = (Y_1, Y_2)$, be c.e.c. Then $\boldsymbol{Y}$ is c.e.c. if either of the following conditions hold*:
(i) $ER^{it} \neq 0$ *for all* $t \in \mathbb{R}$;
(ii) $E\|\boldsymbol{X}\|^{\alpha} < \infty$ *for some* $\alpha \neq 0$ *(G. Letac – private communication).*

As an immediate consequence of Theorem 4 we have the following characterization of normality:

Corollary 4. *Let X_1, X_2, X_3 be independent r.v.'s with X_3 positive a.s. Suppose that $(X_1/X_3, X_2/X_3)$ is s.i. and* (i) $EX_3^{it} \neq 0$ *for all* $t \in \mathbb{R}$ *or* (ii) $EX_3^{\alpha} < \infty$ *for some* $\alpha \neq 0$. *Then X_1 and X_2 are normal $\mathcal{N}(0, \sigma^2)$ for some $\sigma > 0$.*

Proof of Theorem 4. It is easy to notice that, upon taking a suitable linear transformation, instead of c.e.c. measures it suffices to consider s.i. distributions. Take now any $\mathbf{s} \in \mathbb{R}^2$ and some complex z ($z = it$ for any $t \in \mathbb{R}$ under (i) or $z = t$ for any $t \in (0, \alpha)$ ($(\alpha, 0)$) under (ii). Then by independence of R and $\boldsymbol{Y}$ and the definition of $\boldsymbol{X}$ we have

$$h(\mathbf{s},t) = E\left(\exp\left[\left(\mathbf{s}, \frac{\boldsymbol{X}}{\|\boldsymbol{X}\|}\right)\right] \|\boldsymbol{X}\|^z\right) = E\left(\exp\left[\left(\mathbf{s}, \frac{\boldsymbol{Y}}{\|\boldsymbol{Y}\|}\right)\right] R^z \|\boldsymbol{Y}\|^z\right)$$

$$= E\left(\exp\left[\left(\mathbf{s}, \frac{\boldsymbol{Y}}{\|\boldsymbol{Y}\|}\right)\right] \|\boldsymbol{Y}\|^z\right) ER^z .$$

On the other hand, additionally using the fact that $\boldsymbol{X}/\|\boldsymbol{X}\|$ and $\|\boldsymbol{X}\|$ are independent, we obtain

$$h(\mathbf{s},z)=E\left(\exp\left[\left(\mathbf{s},\frac{\boldsymbol{X}}{\|\boldsymbol{X}\|}\right)\right]\right)E(\|\boldsymbol{X}\|^z)$$
$$=E\left(\exp\left[\left(\mathbf{s},\frac{\boldsymbol{Y}}{\|\boldsymbol{Y}\|}\right)\right]\right)E(\|\boldsymbol{Y}\|^z)ER^z .$$

Finally, since in both cases (i) and (ii), we can cancel ER^z, it follows that

$$E\left(\exp\left[\left(\mathbf{s},\frac{\boldsymbol{Y}}{\|\boldsymbol{Y}\|}\right)\right]\|\boldsymbol{Y}\|^z\right)=E\left(\exp\left[\left(\mathbf{s},\frac{\boldsymbol{Y}}{\|\boldsymbol{Y}\|}\right)\right]\right)E(\|\boldsymbol{Y}\|^z)$$

for all $z=it$, $t\in\mathbb{R}$ in the case (i) and for all $z=t$, $t\in(0,\alpha)$ or $(\alpha,0)$ in the case (ii). Hence, $\boldsymbol{Y}/\|\boldsymbol{Y}\|$ and $\|\boldsymbol{Y}\|$ are independent. Consequently, $\boldsymbol{Y}$ is s.i. □

Now we use the above result to give another straightforward extension of Kotlarski's theorem in the case of s.i. quotients for independent parent r.v.'s.

Theorem 5. *Let X_1, X_2, Z be independent r.v.'s and $Z\neq 0$ a.s. Suppose that the distribution of $(X_1/Z, X_2/Z)$ is s.i. and* (i) $E|Z|^{it}\neq 0$ *for any $t\in\mathbb{R}$ or* (ii) $E|Z|^{\alpha}<\infty$ *for some $\alpha\neq 0$. Then the X^2's have a $\chi^2(1)$ distribution, and one of the X's is normal.*

Proof. One can assume that Z is symmetric. To understand this denote the value of the ch.f. of $(X_1/Z, X_2/Z)$ at the point $(s,t)\in\mathbb{R}^2$ by $\phi(s^2+t^2)$, i.e.

$$\phi(s^2+t^2)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\left(is\frac{x}{z}+it\frac{y}{z}\right)\mathrm{d}F_Z(z)\,\mathrm{d}F_{X_1}(x)\,\mathrm{d}F_{X_2}(y)\,, \tag{9}$$

where F_Z, F_{X_1}, F_{X_2} are d.f.'s of Z, X_1, X_2, respectively. Now let us change the z to $-z$ in (9). Then

$$\phi(s^2+t^2)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\left(is\frac{x}{z}+it\frac{y}{z}\right)\mathrm{d}(1-F_Z(-z))\,\mathrm{d}F_{X_1}(x)\,\mathrm{d}F_{X_2}(y)\,, \tag{10}$$

since one can substitute $-t$ for t and $-s$ for s (ϕ is even with respect to s and t). Now, upon adding (9) and (10) divided by 2, we conclude that Z can be replaced by some r.v. with a distribution function $\frac{1}{2}(F_Z(z)+1-F_Z(-z))$, i.e. by a symmetric one.

Hence, assume now that Z is symmetric. Let us notice that $\frac{1}{|Z|}(\mathrm{sgn}(Z)X_1, \mathrm{sgn}(Z)X_2)$ is s.i. and that $\mathrm{sgn}(Z)$ and $|Z|$ are independent since Z is symmetric. Hence, Theorem 4 ensures spherical invariance of $(\mathrm{sgn}(Z)X_1, \mathrm{sgn}(Z)X_2)$. Let $f(s^2+t^2)$ be its ch.f. and denote the ch.f. of X_i by ϕ_i, $i=1,2$. Then

$$f(s^2+t^2)=\frac{\phi_1(s)\phi_2(t)+\phi_1(-s)\phi_2(-t)}{2}\,,\quad s,t\in\mathbb{R}, \tag{11}$$

since $P(\mathrm{sgn}(Z)=\pm 1)=\frac{1}{2}$.

Now (11) implies $\operatorname{Re}\phi_i(t) = f(t^2)$, $i = 1, 2$. Hence the relation (11) takes the form

$$f(s^2 + t^2) = f(s^2)f(t^2) - \eta_1(s)\eta_2(t), \quad s, t \in \mathbb{R},$$

where $\eta_i = \operatorname{Im}\phi_i$, $i = 1, 2$. Since $f(s^2 + t^2)$ and $f(s^2)$ are even with respect to s then we deduce that $\eta_1(s)\eta_2(t) = 0$ for all $s, t \in \mathbb{R}$. Consequently one of the η's is a zero function and we have $f(s^2 + t^2) = f(s^2)f(t^2)$ for all $s, t \in \mathbb{R}$. This is a version of the Cauchy equation with $f(0) = 1$. It is well known that in such a case $f(s^2) = \exp(-\sigma^2 s^2)$, $s \in \mathbb{R}$. Notice that the condition $\operatorname{Re}\phi_i(t) = \exp(-\sigma^2 t^2)$ is equivalent to $\sigma^{-2}X_i^2 \overset{d}{=} \chi^2(1)$ for some $\sigma > 0$, $i = 1, 2$ (see, for example, Roberts, 1971). □

Observe that Corollary 1 is an immediate consequence of Theorem 5 since the Cauchy distribution is s.i. and it allows to renumber the r.v's involved in the quotients.

To have a kind of a converse of Theorem 5 symmetry of Z is needed.

Theorem 6. *Let X_1, X_2, Z be independent r.v.'s such that both the X^2's are $\chi^2(1)$ with a scale σ^2, one of the X's is normal $\mathcal{N}(0, \sigma^2)$ and $Z \neq 0$ a.s. is symmetric. Then $(X_1/Z, X_2/Z)$ is s.i.*

Proof. Without any loss of generality, we can assume that $\sigma = 1$ and X_1 is normal. Then the ch.f. ϕ of $(X_1/Z, X_2/Z)$ has the form

$$\phi(s, t) = \sqrt{\frac{\pi}{2}} \int_{\mathbb{R}} \exp\left(-\frac{s^2}{2z^2}\right) \int_{\mathbb{R}} h(y) \exp\left(i\frac{t}{z}y\right) \exp\left(-\frac{y^2}{2}\right) \mathrm{d}y \, \mathrm{d}F_Z(z),$$

where $h(y) + h(-y) = 1$. Now the result follows once again by changing variables y to $-y$ and then z to $-z$ (since Z is symmetric) and further on adding both sides of the original and the resulting formulas. □

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