

On some distributional and limit properties of factorizable distributions

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Abstract

Factorizable distributions are investigated in the context of decomposibility and sphericity. Also a version of the Lindeberg limit theorem for 2-factorizable arrays is obtained.

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1. Introduction

One of the most prominent results in characterization of probability distributions belongs to Darmois (1953) and Skitovitch (1953): *If $X = (X_1, \dots, X_n)$ is a random vector with independent components such that linear forms $L_1 = a_1X_1 + \dots + a_nX_n$ and $L_2 = b_1X_1 + \dots + b_nX_n$ are independent then the X 's present in both linear forms are Gaussian.* This result gave rise to intensive studies in the field of analytical theory of linear statistics. This is essentially the main subject of the monograph of Kagan et al. (1973). One direction was connected with dimension of the problem – see Ghurye and Olkin (1962) for multivariate version, Krakowiak (1985) for a Banach space version and Feldman (1988) for a group version.

In Linnik (1956), strong relations between the Darmois–Skitovitch theorem and the Cramér theorem on decomposition of the normal law were revealed. This line in multidimensional case was continued in Kagan (1987, 1988a, b) leading to some analytic weakenings of the concept of independence. A culmination of these investigations is a notion of factorizable measures, originally named $\mathcal{L}_{n,k}$ distributions, introduced in Kagan (1988b).

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Definition 1. A random vector $X = (X_1, \dots, X_n)$ (or its distribution) is k -factorizable (belongs to the class $\mathcal{L}_{n,k}$) if its characteristic function ϕ has the form

$$\phi(t_1, \dots, t_n) = \prod_{1 \leq i_1 < \dots < i_k \leq n} R_{i_1, \dots, i_k}(t_{i_1}, \dots, t_{i_k}) \quad (1)$$

for any $(t_1, \dots, t_n) \in \mathbb{R}^n$, where R_{i_1, \dots, i_k} is a continuous complex-valued function such that $R_{i_1, \dots, i_k}(0, \dots, 0) = 1$ for any $1 \leq i_1 < \dots < i_k \leq n$.

The random vector X (or its distribution) is locally k -factorizable if the representation (1) holds in some neighbourhood of the origin.

In Kagan (1988b) some nice properties of $\mathcal{L}_{n,k}$ families were discovered and interesting examples were provided. One of the most intriguing observations, given there, is contained in the following.

Proposition 1. *If X is a k -factorizable random vector with all k -variate marginals Gaussian then X is also Gaussian.*

The main result of Kagan (1988b) was a new version of the Darmois–Skitovitch theorem for factorizable linear forms. Its proof was based on a solution of a generalized Cauchy equation (see Lemma 1 below). An abstract analogue of this result has been given in Lisyanyoy (1995), recently.

The investigations were continued in Wesolowski (1991a), where a formula expressing the characteristic function (ch.f.) of a k -factorizable measure in terms of ch.f.'s of all its k -variate marginals, was given. We recall it here since it will be applied in the sequel.

Proposition 2. *If a random vector (X_1, \dots, X_n) is (locally) k -factorizable then its ch.f. ϕ has the form*

$$\phi(t_1, \dots, t_n) = \prod_{r=1}^k \left\{ \prod_{1 \leq i_1 < \dots < i_r \leq n} \phi_{i_1, \dots, i_r}(t_{i_1}, \dots, t_{i_r}) \right\}^{a_{n,k,r}} \quad (2)$$

in some neighbourhood of the origin, where

$$a_{n,k,r} = \sum_{i=0}^{k-r} \binom{n-r}{i} (-1)^i$$

and ϕ_{i_1, \dots, i_k} denotes the ch.f. of the marginal $(X_{i_1}, \dots, X_{i_k})$, $1 \leq i_1 < \dots < i_k \leq n$, $r = 1, \dots, k$.

It can be easily seen that Proposition 2 is a straightforward extension of Proposition 1.

The formula (2) was then applied in investigating relations between the $\mathcal{L}_{n,2}$ classes and the Gaussian conditional structure of the second order in Wesolowski (1991b). Central limit problem for factorizable arrays was studied in Wesolowski (1994a, b). A comprehend review of all those results is given in Wesolowski (1993). A related concept of (n, k) -equivalence, being an analytic weakening of equidistribution, was introduced and studied in Kagan (1989).

This paper is a further contribution towards understanding the analytic notion of factorizability. In Section 2 we study first some problems connected with decomposition of linear forms in factorizable r.v.'s. We are interested in the degenerate and Gaussian cases. An example provided there gives a lower bound for the dimension of the problem. The section is closed with a characterization of the Gaussian distribution by sphericity and factorizability. The main tool used in that section is the solution of an extension of the Cauchy functional equation given in Kagan (1988b). Section 3 is devoted to a central limit theorem (clt) of the Lindeberg type for rowwise 2-factorizable arrays. This is an analogue of the Lyapounov clt for factorizable distributions, obtained in Wesolowski (1994a), and considerably refined in Wesolowski (1994b). Here our arguments are based on the formula (2).

2. Distributional properties

We begin with a technical result, mentioned in the Introduction, being the core of most of the arguments used in this section. It deals with a kind of the Cauchy equation and was obtained by Kagan (1988b).

Lemma 1. *Let $m > 1$ be an integer. If a continuous (in a neighbourhood of the origin) complex function ψ defined on \mathbb{R} satisfies for some $\varepsilon > 0$ and $\|t\| < \varepsilon$ the equation*

$$\psi(a_1 t_1 + \dots + a_m t_m) = \sum_{j=1}^m \rho_j(t_1, \dots, t_{j-1}, \dots, t_{j+1}, \dots, t_m), \tag{3}$$

where $a_1 \dots a_m \neq 0$, are some real constants and ρ 's are some complex functions on \mathbb{R}^{m-1} . Then for some $\delta > 0$, $\psi(x)$ is a polynomial of the order not exceeding $(m - 1)$ for $|x| < \delta$.

This result was the main tool of the proof of the version of the Darmois–Skitovitch theorem given in Kagan (1988b). It will be shown in the sequel that other interesting properties of factorizable distributions can be deduced by means of Lemma 1.

2.1. Decomposition problems

We begin with some decomposition problems, i.e. questions of the following type: Take an n -dimensional random vector X and consider its m -dimensional linear transformation $L = AX^T$, where A is an $m \times n$ matrix with real entries. Assume that the distribution of L is known. What is the distribution of X ? It is trivial, that if $\text{rank}(A) = n$ the distribution of L determines that of X . Otherwise, even with a strong assumption of independence of components of X , it is usually a difficult question and its solution is known only for some special distributions of L . An example of a non-trivial result is the Cramér theorem on decomposition of the normal law (the case $n > m = 1$). Other results on reconstructing the distribution of X from that of L can be found, for instance, in Reiersol (1950) or Rao (1966, 1973). On the other hand, it is quite easy to observe that any degenerate distribution has only degenerate independent components, i.e. if $X \in \mathcal{L}_{n,1}$ and $L = L_1 = a$ a.s. (a constant) then X is degenerate. If we do not assume independence of components, then it can easily be seen that the result does not hold. Now we will study the same question for factorizable X .

Let $X \in \mathcal{L}_{n,k}$ and assume that $L = a$ a.s., where $a \in \mathbb{R}^m$ is a given point. We want to determine the distribution of X . We begin with the following, more general, negative result.

Proposition 3. *Let A be an $(m - 1) \times (m + 1)$ ($m > 1$) matrix. Then there exist random vectors, $X, Y \in \mathcal{L}_{m+1,m}$ such that AX^T and AY^T have the same distribution, while X_i and Y_i have different distributions, $i = 1, \dots, m + 1$.*

Proof. Only the case $m = 2$ is considered since a similar argument holds for any $m > 1$. Take some $X = (X_1, X_2, X_3) \in \mathcal{L}_{3,2}$. Let $U = (U_1, U_2, U_3)$ be a random vector with independent components such that X and U are independent. Let $A = (a_1, a_2, a_3)$. Define $Y = (Y_1, Y_2, Y_3)$ by $Y_1 = X_1 + a_2 U_3 - a_3 U_2$, $Y_2 = X_2 + a_3 U_1 - a_1 U_3$, $Y_3 = X_3 + a_1 U_2 - a_2 U_1$. Then, obviously, $AX^T = AY^T$.

Observe that the ch.f. ϕ_Y of Y has the form

$$\begin{aligned} \phi_Y(t) &= Ee^{i(t_1 Y_1 + t_2 Y_2 + t_3 Y_3)} = Ee^{i(t_1(X_1 + a_2 U_3 - a_3 U_2) + t_2(X_2 + a_3 U_1 - a_1 U_3) + t_3(X_3 + a_1 U_2 - a_2 U_1))} \\ &= \phi_X(t) \phi_U(a_3 t_2 - a_2 t_3, a_1 t_3 - a_3 t_1, a_2 t_1 - a_1 t_2) \end{aligned}$$

for any $t = (t_1, t_2, t_3) \in \mathbb{R}^3$, where ϕ_X, ϕ_U are ch.f.'s of X, U , respectively. Now, by independence of components of U and 2-factorizability of X , it follows that $Y \in \mathcal{L}_{3,2}$. Observe that $V = (a_2 U_3 - a_3 U_2, a_3 U_1 - a_1 U_3, a_1 U_2 - a_2 U_1)$ is a 2-factorizable, non-degenerate, random vector such that $AV^T = 0$. \square

However if the number of linear forms is increased then a decomposition problem for degenerate distribution may have a unique solution.

Theorem 1. *Assume that $X = (X_1, \dots, X_m) \in \mathcal{L}_{m, m-1}$. If, for an $(m-1) \times m$ matrix A of the rank $(m-1)$ the random vector $L = AX^T$ has a degenerate distribution, then X is also concentrated in a point.*

Proof. Assume that $L = b$ a.s. for some point $b = (b_1, \dots, b_{m-1}) \in \mathbb{R}^{m-1}$. Then, since $\text{rank}(A) = m-1$, we can solve this systems of linear equations to get

$$X_i = \alpha_i + \beta_i X_m, \quad i = 1, \dots, m-1,$$

where $\alpha_j, \beta_j, j = 1, \dots, m$, are some real numbers. Hence, the ch.f. ϕ_X of the X has the form

$$\phi_X(t) = E\left(\exp\left(i \sum_{j=1}^m t_j(\alpha_j + \beta_j X_m)\right)\right) = \phi_m\left(\sum_{j=1}^m \beta_j t_j\right) \prod_{j=1}^m e^{i\alpha_j t_j},$$

for any $t = (t_1, \dots, t_m) \in \mathbb{R}^m$, where ϕ_m is a ch.f. of X_m . On the other hand, by (1),

$$\phi_X(t) = \prod_{j=1}^m \psi_j(t_1, \dots, t_{j-1}, t_{j+1}, \dots, t_m),$$

where $\psi_j = R_{1, \dots, j-1, j+1, \dots, m}, j = 1, \dots, m$. Hence, the above two equations, by Lemma 1, imply that X_m is a normal or degenerate r.v. (It follows immediately from the Marcinkiewicz theorem.) Consequently, the joint distribution of X is Gaussian or degenerate. However in the Gaussian case AX^T has to be non-degenerate Gaussian, too. Hence, the only possibility is the degeneracy of the distribution of X . \square

Observe that Theorem 1 is a straightforward and generalization of the trivial independent case, which follows by taking $m = 2$. However, the decomposition problem for $m-1$ degenerate linear forms in n -dimensional, $n > m$, $(m-1)$ -factorizable random vector remains open.

Gaussian decomposition problems are much more complicated. Here we present a contribution involving factorizable measures with some marginals being Gaussian. The result may be also treated as a complement to Proposition 1.

Theorem 2. *Assume that $X = (X_1, \dots, X_n)$ is a k -factorizable random vector for some $k < n$ with all $(k-1)$ -dimensional marginals being Gaussian. Assume that A is a $k \times n$ real matrix having all its $k \times k$ submatrices of full rank. If the linear form $L = (L_1, \dots, L_k) = AX^T$ has a Gaussian distribution then X is Gaussian.*

Proof. Consider the ch.f. ϕ_L of L . Obviously,

$$\phi_L(t) = Ee^{i(t, AX^T)} = Ee^{i(A^T t, X)} = \phi_X((a_1, t), \dots, (a_n, t))$$

for any $t = (t_1, \dots, t_k) \in \mathbb{R}^k$, where ϕ_X is the ch.f. of X and a_i is the i th column of the matrix $A, i = 1, \dots, n$. From (2) we have

$$\begin{aligned} & \prod_{1 \leq i_1 < \dots < i_k \leq n} \phi_{i_1, \dots, i_k}((a_{i_1}, t), \dots, (a_{i_k}, t)) \\ &= \phi_X((a_1, t), \dots, (a_n, t)) \prod_{r=1}^{k-1} \left\{ \prod_{1 \leq i_1 < \dots < i_r \leq n} \phi_{i_1, \dots, i_r}((a_{i_1}, t), \dots, (a_{i_r}, t)) \right\}^{-a_{n, k, r}} \end{aligned}$$

in a neighbourhood of the origin, where ϕ_{i_1, \dots, i_r} is the ch.f. of the marginal $(X_{i_1}, \dots, X_{i_r})$, $1 \leq i_1 < \dots < i_r \leq n$, $r = 1, \dots, k$. Observe that the first factor on the right-hand side of the above equation is a ch.f. of some Gaussian measure (since it is the ch.f. of L). Since all $(k - 1)$ -dimensional marginals are Gaussian then the right-hand side has the form $\exp(Q(t_1, \dots, t_k))$ and it is a ch.f. since the left-hand side is a ch.f. Consequently, the property of the ranks of the submatrices of A implies that the product of ch.f.'s of all k -dimensional marginals of the X is a Gaussian k -dimensional ch.f. Now by the multivariate Cramér decomposition theorem it follows that all k -variate marginals are Gaussian. Hence, the result is a consequence of Proposition 1. \square

Obviously, general Gaussian decomposition problem for factorizable measures: is it true for $X \in \mathcal{L}_{n,k}$ that the normality of AX^T for some $k \times n$ matrix, implies normality of X (possibly with some assumption on the matrix and with no prior conditions on the marginals); remains open.

2.2. Sphericity

Now we consider spherical measures. An n -dimensional random vector X (or its distribution) is said to be spherical (or spherically invariant) if its ch.f. ϕ_X has the form

$$\phi_X(t) = \psi(t_1^2 + \dots + t_n^2)$$

for any $t = (t_1, \dots, t_n) \in \mathbb{R}^n$, where ψ is some function. In this case we write $X \in SI(n)$. Spherical, more generally, elliptically contoured, measures are a subject of intensive studies in recent years as a useful and non-trivial generalization of Gaussian multivariate distributions – see the monograph of Fang et al. (1990). It is well known that a spherical random vector with independent components has to be Gaussian – see Theorem 4.11 in the monograph. This can be stated as: If $X \in SI(n) \cap \mathcal{L}_{n,1}$ then it is Gaussian. We want to extend this result to factorizable measures of any order k . Since any k -factorizable measure is $(k + 1)$ -factorizable, $k < n$, then it suffices to consider the case $k = n - 1$.

Theorem 3. *If an $(n - 1)$ -factorizable random vector X belong to $SI(n)$ then it is Gaussian.*

Proof. Both the assumption imply that

$$\psi(t_1^2 + \dots + t_n^2) = \prod_{j=1}^n R_j(t_1, \dots, t_{j-1}, t_{j+1}, \dots, t_n), \tag{4}$$

where the ch.f. of X has the form $\phi_X(t) = \psi(t_1^2 + \dots + t_n^2)$, $(t_1, \dots, t_n) \in \mathbb{R}^n$. It is not difficult to observe, applying exactly the same argument as in the original proof, that Lemma 1 holds true also if Eq. (3) holds only for $\|t\| < \varepsilon$ with positive components. Then ψ is a polynomial for $0 \leq t < \delta$. Hence, considering only positive t 's in (4) upon taking logarithms for t 's close to zero, we conclude by Lemma 1, that ϕ_X has a form $\exp(Q(t_1, \dots, t_n))$ in a neighbourhood of the origin. Consequently, it can be extended to the whole space and by the Marcinkiewicz theorem it follows that X is Gaussian. \square

3. Limit theorem

In Wesolowski (1994a) the following version of the Lyapounov central limit theorem for factorizable measures was obtained.

Theorem 4. Let $X_n = (X_{n,1}, \dots, X_{n,k_n})$ be a zero mean 2-factorizable random vector, $n = 1, 2, \dots$, and $k_n \rightarrow \infty$ while $n \rightarrow \infty$. If

$$\lim_{n \rightarrow \infty} \sum_{i,j=1}^{k_n} EX_{n,i}X_{n,j} = \sigma^2 > 0, \quad (5)$$

$$\lim_{n \rightarrow \infty} k_n \sum_{i=1}^{k_n} EX_{n,i}^2 = \tau^2, \quad (6)$$

$$\lim_{n \rightarrow \infty} k_n \sum_{i=1}^{k_n} E|X_{n,i}|^3 = 0, \quad (7)$$

then $S_n = X_{n,1} + \dots + X_{n,k_n}$ converges in distribution to the normal law with the mean zero and the variance $\sigma^2(S_n \xrightarrow{d} \mathcal{N}(0, \sigma^2))$ as $n \rightarrow \infty$.

Then the result was considerably strengthened in Wesolowski (1994b), where rowwise k -factorizable arrays were considered and no version of the technical condition (6) was needed. Here we are interested in the Lindeberg version of the above result, i.e. we replace the Lyapounov condition (7) by its Lindeberg analogue

$$\lim_{n \rightarrow \infty} k_n \sum_{i=1}^{k_n} EX_{n,i}^2 I(|X_{n,i}| > \varepsilon) = 0, \quad (8)$$

where ε is a positive number.

Theorem 5. Let $X_n = (X_{n,1}, \dots, X_{n,k_n})$ be a zero mean 2-factorizable random vector, $n = 1, 2, \dots$, and $k_n \rightarrow \infty$ while $n \rightarrow \infty$. If the conditions (5), (6) and (8) for any $\varepsilon > 0$ hold then $S_n \xrightarrow{d} \mathcal{N}(0, \sigma^2)$ as $n \rightarrow \infty$.

Before we give the proof of Theorem 5 let us formulate and prove the following technical, but general inequality, which will be used in that proof.

Lemma 2. Consider r integrable r.v.'s X and Y . Then for any $\varepsilon > 0$

$$E(|X + Y|^r I(|X + Y| > \varepsilon)) \leq 2^r [E(|X|^r I(|X| > \varepsilon/2)) + E(|Y|^r I(|Y| > \varepsilon/2))]. \quad (9)$$

Proof of Lemma 2. It is well known that for any real numbers a, b and any natural r the following inequality holds: $|a + b|^r \leq 2^{r-1}(|a|^r + |b|^r)$. Additionally, $I(|X + Y| > \varepsilon) \leq I(|X| > \varepsilon/2) + I(|Y| > \varepsilon/2)$ a.s. Then

$$\begin{aligned} E(|X + Y|^r I(|X + Y| > \varepsilon)) &\leq 2^{r-1} [E(|X|^r I(|X| > \varepsilon/2)) + E(|X|^r I(|Y| > \varepsilon/2)) \\ &\quad + E(|Y|^r I(|X| > \varepsilon/2)) + E(|Y|^r I(|Y| > \varepsilon/2))]. \end{aligned} \quad (10)$$

But

$$\begin{aligned} E(|X|^r I(|Y| > \varepsilon/2)) &= E(|X|^r I(|X| > \varepsilon/2, |Y| > \varepsilon/2)) + E(|X|^r I(|Y| > \varepsilon/2 \geq |X|)) \\ &\leq E(|X|^r I(|X| > \varepsilon/2)) + E(|Y|^r I(|Y| > \varepsilon/2)) \end{aligned}$$

since

$$E(|X|^r I(|Y| > \varepsilon/2 \geq |X|)) \leq E(|Y|^r I(|Y| > \varepsilon/2)).$$

Similarly,

$$E(|X|^r I(|Y| > \varepsilon/2)) \leq E(|X|^r I(|X| > \varepsilon/2)) + E(|Y|^r I(|Y| > \varepsilon/2)).$$

Now to obtain (9) it suffices to put both the inequalities in (10). \square

Proof of Theorem 5. Observe that the assumption (5) and the Tchebyshev inequality yield

$$P(|S_n| > \lambda) \leq \lambda^{-2} \sum_{i,j=1}^{k_n} E(X_{n,i} X_{n,j}) \rightarrow \lambda^{-2} \sigma^2$$

as $n \rightarrow \infty$. Hence, the sequence of distributions of S_n , $n = 1, 2, \dots$, is tight. By the Prokhorov theorem, each subsequence contains a weakly convergent subsequence. Consequently, without loss of generality, we can assume that S_n converges in distribution. To prove that the limit distribution is normal $N(0, \sigma^2)$ it suffices to show that

$$\lim_{n \rightarrow \infty} E(\exp(itS_n)) = \exp\left(-\frac{t^2 \sigma^2}{2}\right) \tag{11}$$

in some neighbourhood of the origin.

Consider independent r.v.'s $Y_{n,j}$, $j = 1, 2, \dots, k_n^2$, such that

$$Y_{n,(j-1)k_n+i} \stackrel{d}{=} X_{n,j}, \quad j, i = 1, \dots, k_n,$$

$n = 1, 2, \dots$. It is not difficult to observe that

$$\sum_{i=1}^{k_n^2} E(Y_{n,i}^2) = k_n \sum_{i=1}^{k_n} E(X_{n,i}^2) \rightarrow \tau^2$$

and for any $\varepsilon > 0$

$$\sum_{i=1}^{k_n^2} E(Y_{n,i}^2 I(|Y_{n,i}| > \varepsilon)) = k_n \sum_{i=1}^{k_n} E(X_{n,i}^2 I(|X_{n,i}| > \varepsilon)) \rightarrow 0.$$

Consequently, by the classical Lindeberg theorem (for rowwise independent arrays), $\sum_{i=1}^{k_n^2} Y_{n,i} \xrightarrow{d} N(0, \tau^2)$ as $n \rightarrow \infty$. Hence,

$$\lim_{n \rightarrow \infty} \left(\prod_{1 \leq i \leq k_n} \phi_{n,i}(t) \right)^{k_n} = \exp\left(-\frac{t^2 \tau^2}{2}\right) \tag{12}$$

for any real t .

Define a new collection of independent r.v.'s $Z_{n,i,j}$, $1 \leq i < j \leq k_n$, such that

$$Z_{n,i,j} \stackrel{d}{=} X_{n,i} + X_{n,j}$$

for any $n = 1, 2, \dots$. Since

$$\sum_{1 \leq i < j \leq k_n} E(X_{n,i} + X_{n,j})^2 = (k_n - 2) \sum_{i=1}^{k_n} E(X_{n,i}^2) + \sum_{i,j=1}^{k_n} E(X_{n,i} X_{n,j})$$

then by (5) and (6)

$$\lim_{n \rightarrow \infty} \sum_{1 \leq i < j \leq k_n} E(Z_{n,i,j}^2) = \tau^2 + \sigma^2. \tag{13}$$

On the other hand, by Lemma 2, we have

$$\begin{aligned} & \sum_{1 \leq i < j \leq k_n} E((X_{n,i} + X_{n,j})^2 I(|X_{n,i} + X_{n,j}| > \varepsilon)) \\ & \leq 2 \sum_{1 \leq i < j \leq k_n} [E(X_{n,i}^2 I(|X_{n,i}| > \varepsilon/2)) + E(X_{n,j}^2 I(|X_{n,j}| > \varepsilon/2))] \\ & = 2(k_n - 1) \sum_{1 \leq i \leq k_n} E(X_{n,i}^2 I(|X_{n,i}| > \varepsilon/2)). \end{aligned}$$

Consequently, (8) implies

$$\lim_{n \rightarrow \infty} \sum_{1 \leq i < j \leq k_n} E(Z_{n,i,j}^2 I(|Z_{n,i,j}| > \varepsilon/2)) = 0. \quad (14)$$

Now formulas (13) and (14) via the classical Lindeberg theorem imply $\sum_{1 \leq i < j \leq k_n} Z_{n,i,j} \xrightarrow{d} N(0, \tau^2 + \sigma^2)$. Hence,

$$\lim_{n \rightarrow \infty} \prod_{1 \leq i < j \leq k_n} \phi_{n,i,j}(t, t) = \exp\left(-\frac{t^2(\tau^2 + \sigma^2)}{2}\right) \quad (15)$$

for any real t , where $\phi_{n,i,j}$ is the ch.f. of $(X_{n,i}, X_{n,j})$, $1 \leq i < j \leq k_n$.

Formulas (12) and (15) imply that there is a neighbourhood V of the origin such that for sufficiently large N , $n > N$, $\phi_{n,i}(t)$, $1 \leq i \leq k_n$, and $\phi_{n,i,j}(t, t)$, $1 \leq i < j \leq k_n$, are non-zero for $t \in V$. Hence, by (2), for $k = 2$ we have

$$E(\exp(itS_n)) = \frac{\prod_{1 \leq i < j \leq k_n} \phi_{n,i,j}(t, t)}{(\prod_{1 \leq i \leq k_n} \phi_{n,i}(t))^{k_n - 2}}$$

for $t \in V$ and $n > N$. Applying (12) and (15) to the above formula we conclude the proof by obtaining the relation (11). \square

The general factorizable Lyapounov-type clt obtained in Wesolowski (1994b), being an extension of Theorem 4, was possible due to a new version of the classical Lyapounov clt for rowwise independent arrays given in that paper. This allowed not only to replace 2-factorizability by k -factorizability but also to omit the technical condition (6). Unfortunately, such an approach under the Lindeberg-type condition seems to be difficult and the general factorizable Lindeberg-type clt remains an open problem.

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References

- Darmois, G. (1953), Analyse générale des liaisons stochastiques, *Rev. Inst. Internat. Statist.* **21**, 2–8.
- Feldman, G.M. (1988), Characterization of the Gaussian distribution on groups by independence of linear statistics, *Dokl. Akad. Nauk SSSR* **301**, 558–560 (in Russian).
- Ghurye, S.G. and I. Olkin (1962), A characterization of the multivariate normal distribution, *Ann. Math. Statist.* **33**, 533–541.
- Kagan, A.M. (1987), A class of two-dimensional distributions arising in connection with the Cramér and Darmois–Skitovitch theorems, *Teoriya Veroyatnostey Yeyo Primeneniya* **32**, 349–351 (in Russian).
- Kagan, A.M. (1988a), A multivariate analog of the Cramér theorem on components of Gaussian distribution, in: V.M. Zolotarev and V.V. Kalashnikov, eds., *Stability Problems of Stochastic Models*, Lecture Notes in Mathematics, Vol. 1412 (Springer, New York), pp. 68–77.
- Kagan, A.M. (1988b), New classes of dependent random variables and a generalization of the Darmois–Skitovitch theorem to the case of several linear forms, *Teoriya Veroyatnostey Yeyo Primeneniya* **33**, 305–314 (in Russian).
- Kagan, A.M. (1989), Generalized condition of identity of distributions of random vectors in connection with analytic theory of linear forms in independent random variables, *Teoriya Veroyatnostey Yeyo Primeneniya* **34**, 370–375 (in Russian).
- Kagan, A.M., Yu.V. Linnik and C.R. Rao (1973), *Characterization Problems of Mathematical Statistics* (Wiley, New York).
- Krakowiak, W. (1985), The theorem of Darmois–Skitovitch for Banach valued random variables, *Bull. Polish Academy Sci. Math.* **33**, 77–85.

- Linnik, Yu.V. (1956), A remark on Cramér's theorem on decomposition of the normal law, *Teoriya Veroyatnostey Yeyo Primenenya* **1**, 479-480 (in Russian).
- Lisyanoy, S.V. (1995), On a group analogue of the Kagan theorem, *Teoriya Veroyatnostey Yeyo Primenenya* **40**, 180-183.
- Rao, C.R. (1966), Characterization of random variables in linear structural relations, *Sankhyā Ser. A* **28**, 251-260.
- Rao, C.R. (1973), Characterizations of probability laws by linear functions, *Sankhyā Ser. A* **35**, 265-270.
- Reiersol, O. (1950), Identifiability of a linear relation between variables which are subject to error, *Econometrika* **18**, 375-389.
- Skitovitch, W.P. (1953), On a property of the normal distribution, *Dokl. Akad. Nauk SSSR* **89**, 217-219 (in Russian).
- Wesolowski, J. (1991a), On determination of multivariate distribution by its marginals for the Kagan class, *J. Multivariate Anal.* **36**, 314-319.
- Wesolowski, J. (1991b), Gaussian conditional structure of the second order and the Kagan classification of multivariate distributions, *J. Multivariate Anal.* **39**, 79-86.
- Wesolowski, J. (1993), Factorizable distributions - a new concept of dependence, *Bull. Internat. Statist. Inst. Contributed Papers*, 49th Session, Firenze, pp. 541-542.
- Wesolowski, J. (1994a), The Kagan classification of multivariate distributions and the central limit theorem, in: M.L. Puri and J.P. Vilaplana, eds., *New Trends in Probability and Statistics* (VSP International Publishers, Zeist) pp. 67-70.
- Wesolowski, J. (1994b), The Lyapounov central limit theorem for factorizable arrays, *Proc. Amer. Math. Soc.* **122**, 165-174.