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# LINEARITY OF BEST PREDICTORS FOR NON-ADJACENT RECORD VALUES

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*SUMMARY.* Let $\{\mathbf{X}_n, n \geq 1\}$ be a sequence of independent and identically distributed random variables with absolutely continuous distribution function. Suppose $\mathbf{X}_{U(k)},\ k = 1, 2, \ldots$ be the upper record values of $\{\mathbf{X}_n, n \geq 1\}$. A complete solution of the problem of determining the distribution by the linearity of the regression of $\mathbf{X}_{U(m+2)}$ with respect to $\mathbf{X}_{U(m)}$ is given. It is shown that the class of possible distributions consists of exponential, power function and Pareto type. Equivalently, the best unbiased predictors of $\mathbf{X}_{U(m+2)}$ given $\mathbf{X}_{U(m)}$ is linear only for this class.

## 1. **Introduction**

Let $\{\mathbf{X}_n\}, n \geq 1$, be a sequence of independent and identically distributed (i.i.d.) random variables (rvs) with absolutely continuous (with respect to the Lebesgue measure) cumulative distribution function (cdf) F. We will denote by $f$ the corresponding probability density function (pdf). Set $\mathbf{Y}_n = \max(\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n)$ for $n \geq 1$. We say $\mathbf{X}_j$ is an upper record value of $\{\mathbf{X}_n\}$ if $\mathbf{Y}_j > \mathbf{Y}_{j-1}$. By definition $\mathbf{X}_1$ is an upper record value. In this paper we will call the upper record values as record values. The indices at which the record values occur are given by the record value times $U(n)$ where $U(1) = 1$ and $U(n) = \min\{k|k > U(n-1), \mathbf{X}_k > \mathbf{X}_{U(n-1)}\},\ n > 1$. We will define $R(x) = -ln\overline{F}(x),\ \overline{F}(x) = 1 - F(x)$. The conditional pdf of $\mathbf{X}_{U(m+k)}$ given $\mathbf{X}_{U(m)}$ is (see Ahsanullah (1995)) as given below :

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$$f_{m+k|m}(x,y) = \frac{1}{\Gamma(k)}(R(y)-R(x))^{k-1}f(y)(\overline{F}(x))^{-1}, \quad -\infty < x < y < \infty$$
$$= 0, \text{ otherwise },$$
$$\ldots (1.1)$$

We will denote by $EXP(\lambda, \gamma)$, the shifted exponential distribution distribution with the pdf as given in the following form :

$$f(x) = \lambda e^{-\lambda(x-\gamma)}, \quad x > \gamma, \ \lambda > 0$$
$$= 0, \quad \text{otherwise} \qquad \ldots (1.2)$$

Similarly, we denote by $POW(\theta, \mu, v)$ the power function distribution with the pdf as of the following form :

$$f(x) = \frac{\theta(v-x)^{\theta-1}}{(v-\mu)^{\theta}}, \quad -\infty < \mu < x < v < \infty, \quad \theta > 0,$$
$$= 0, \text{ otherwise.} \qquad \ldots (1.3)$$

For the Pareto distribution, $PAR(\theta, \mu, \delta)$, we take the following pdf :

$$f(x) = \frac{\theta(\mu+\delta)^{\theta}}{(x+\delta)^{\theta+1}}, \ \theta > 0, x > \mu, \quad \mu + \delta > 0$$
$$= 0, \text{ otherwise.} \qquad \ldots (1.4)$$

Using the conditional pdf of $\mathbf{X}_{U(m+k)}$ given $\mathbf{X}_{U(m)}$, it can be shown that for the above three distributions

$$E(\mathbf{X}_{U(m+k)}|\mathbf{X}_{U(m)}) = aX_{U(m)} + b \qquad \ldots (1.5)$$

for some constants a and b. Equivalently, for these three distributions, the best unbiased predictor of $\mathbf{X}_{U(m+k)}$ given $\mathbf{X}_{U(m)}$ coincides a.s. with the best linear unbiased predictor (BLUP). It is interesting to know, as pointed out by Nagaraja (1977), that the relation (1.5) characterizes the exponential, power function and the Pareto type distributions for $k = 1$. Forguson (1967) obtained a characterization of the same class of distributions by the BUP=BLUP for adjacent order statistics. Nagaraja (1988) showed that if BUP= BLUP for $\mathbf{X}_{U(m+1)}$ given $\mathbf{X}_{U(m)}$ and $\mathbf{X}_{U(m)}$ given $\mathbf{X}_{U(m+1)}$, then the parent distribution is exponential. The BUP and BLUP for record values were also considered in Ahsanullah (1980) and Dunsmore (1983).

In this paper we show that the best unbiased predictor and the best linear predictor of record values coincide a.s. only for the exponential, power function and Pareto type distribution for $k = 2$. Equivalently these three distributions are the only distributions that are characterized by the relation (1.5) with $k = 2$. It is an open problem whether this characterization also holds for $k > 2$.

## 2. Main Results

For the exponential, power function and Pareto distribution,

$$E(\mathbf{X}_{U(m+2)}|\mathbf{X}_{U(m)}) = a\mathbf{X}_{U(m)} + b,$$

where the constants $a$ and $b$ are of the following form :

(a) $a = 1$ and $b = \frac{2}{\lambda}$ for the exponential distribution, $EXP(\lambda, \gamma)$,

(b) $a = \left(\frac{\theta}{\theta+1}\right)^2$ and $b = \frac{2\theta}{(\theta+1)^2}v$ for the power function distribution, $POW(\theta, \mu, v)$,

(c) $a = \left(\frac{\theta}{\theta-1}\right)^2$ and $b = \frac{2\theta}{(\theta-1)^2}\delta$ for the power function distribution, $PAR(\theta, \mu, \delta)$, $\theta > 1$,

To prove the main result, we need the following Lemma which is also of independent interest as a new characterization of the exponential distribution.

LEMMA 2.1 *Let* $\{\mathbf{X}_n, n \geq 1\}$ *be i.i.d. rvs with absolutely continuous cdf* $F$. *Assume that there is an* $\epsilon > 0$ *such that* $E(exp((c+\epsilon)\mathbf{X}_1)) < \infty$. *Then* $\mathbf{X}_1 \in EXP(\lambda, \gamma)$ *iff*

$$E[\exp(c\mathbf{X}_{U(m+2)})|\mathbf{X}_{U(m)}] = a\ \exp(c\mathbf{X}_{U(m)}) \qquad \ldots(2.1)$$

*with* $\lambda = \frac{c\sqrt{a}}{\sqrt{a}-1}$ *and* $a \in (0,1)$ *if* $c < 0$ *and* $a > 1$ *if* $c > 0$.

PROOF. It is easy to verify that if $\mathbf{X}_1 \in EXP(\lambda, \gamma)$ then (2.1) holds.

Conversely, suppose (2.1) holds true. Since $\mathbf{X}_{U(m+2)} \geq \mathbf{X}_{U(m)}$ a.s. then it follows easily that $a \in (0,1)$ if $c < 0$ and $a > 1$ if $c > 0$. Let $\gamma = \inf\{x|F(x) > 0\}$ and $\eta = \sup\{x|F(x) < 1\}$. Using again (2.1), we get from (1.1)

$$\int_x^\eta e^{cy}\{R(y) - R(x)\}f(y)dy = ae^{cx}.\overline{F}(x) \qquad \ldots(2.2)$$

almost everywhere with respect to the distribution of $\mathbf{X}_1$. Since $F$ is continuous, it follows that (2.2) holds for any $x \in (\gamma, \eta)$ and $F'(x) = f(x)$ for any $x \in (\gamma, \eta)$.

Differentiating both sides of (2.2) with respect to $x$ and simplifying, we have

$$-\int_x^\eta e^{cy}f(y)dy = ace^{cx}\frac{(\overline{F}(x))^2}{f(x)} - ae^{cx}\overline{F}(x) \qquad \ldots(2.3)$$

If we differentiate both sides of (2.3) with respect to $x$ and simplify, then we obtain the following equation

$$f'(x)\overline{F}^2(x) - cf(x)\overline{F}^2(x) + 3\overline{F}(x)f^2(x) + \frac{1-a}{ac}f^3(x) = 0 \qquad \ldots(2.4)$$

Let $y = \overline{F}$(i.e, $f = -y', f' = -y''$), then (2.4) yields

$$-y''y^2 + cy'y^2 + 3(y')^2y - \delta(y')^3 = 0, \qquad \dots(2.5)$$

where $\delta = \frac{1-a}{ac}$. Substituting $u(y) = y'$ in (2.5), we have

$$-u'y^2 + cy^2 + 3uy - \delta u^2 = 0. \qquad \dots(2.6)$$

Let $u(y) = w(y) - \lambda y$, where $\lambda$ is a real constant. Rewriting the equation (2.6) in terms of $w$, we get

$$-w'y^2 + \lambda y^2 + cy^2 + 3wy - 3\lambda y^2 - \frac{1-a}{ac}(w^2 - 2w\lambda w\ y + \lambda^2 y^2) = 0. \qquad \dots(2.7)$$

We choose $\lambda = \frac{c\sqrt{a}}{\sqrt{a}-1}$. Then we have from (2.7)

$$-y^2w' + (3 + 2\lambda\delta)wy - \delta w^2 = 0, \qquad \dots(2.8)$$

The equation (2.8) is of Bernoulli type. If $w = 0$, then $y' = u(y) = -\lambda y$ and hence

$$y = \overline{F}(x) = ke^{-2\lambda x} \qquad \dots(2.9)$$

from any $x \in (\gamma, \eta)$ and $k$ is a positive constant. The boundary conditions $\overline{F}(\gamma) = 1$ and $\overline{F}(\eta) = 0$, imply that $\eta = \infty$ ad

$$y = \overline{F}(x) = e^{-\lambda(x-\gamma)}, x \geq \gamma. \qquad \dots(2.10)$$

The solution of (2.8) for $w \not\equiv 0$, is

$$w = \frac{2(1+\lambda\delta)Y^{2(1+\lambda\delta)+1}}{2D(1+\lambda\delta) + \delta y^{2(1+2\delta)}}, \qquad \dots(2.11)$$

where $D$ is an arbitrary constant. Writing in terms of $y$, we have

$$-y' = y\frac{\lambda D y^B + C}{Dy^B + A}, \qquad \dots(2.12)$$

where $A = \frac{\delta}{2(1+\lambda\delta)}, C = \lambda A - 1,\ B = -\frac{\delta}{A}$. The equation (2.12) implies

$$f(x) = \overline{F}(x)\frac{\lambda D(\overline{F}(x))^B + C}{D(\overline{F}(x))^B + A}. \qquad \dots(2.13)$$

Let us consider all possible cases.

*First case :* $c < 0$. Then $a \in (0,1)$ and $\lambda > 0$. If $w = 0$, then from (2.9), we have $\overline{F}(x) = ke^{-\lambda x}$. Since $F$ is a probability distribution function we must have

$$\overline{F}(x) = e^{-\lambda(x-\gamma)}, x \geq \gamma. \qquad \dots(2.14)$$

If $w \neq 0$, then $A > 0, C < 0$ and $B > 0$. Thus if $x \to \eta$, then the right hand side of (2.13) is negative but the left hand side of (2.13) is always positive. Thus the only solution for $c < 0$ is the one given by (2.14).

*Second case :* $c > 0$. Then $a > 1$ and again $\lambda > 0$. If $w \equiv 0$, then as in the first case the only solution is given by (2.14). Observe that $\lambda > c$, hence the integrability condition is satisfied.

If $w \not\equiv 0$, then $A > 0, B > 0$ and $C > 0$. Thus from (2.12), we get

$$\overline{F}(x) = e^{-\theta(x-y)} \left( \frac{\lambda D \overline{F}^B + C}{\lambda D + C} \right)^{-\frac{\theta}{c}}, \qquad \ldots (2.15)$$

where $\theta = \frac{c\sqrt{a}}{\sqrt{a}+1}$, $x \geq \gamma$, Let $H(D, y)$ be the second factor of the right hand side of (2.15), $y = \overline{F}$. If $D < 0$, then $H(D, y) \geq H(D, 1)$ and if $D > 0$, then $H(D, y) \geq H(D, 0)$. Let $M_0 = \min\{H(D, 1)\text{and } H(D, 0)\}$. Thus $\overline{F}(x) \geq M_0 e^{-\theta x}$. The integrability condition implies that $\theta > c + \epsilon$ which is contradictory since $c > 0$. Hence the only solution of (2.12) is the one given by (2.14). □

We now give the main characterization theorem.

THEOREM 2.1. *Let $\{\mathbf{X}_n, n \geq 1\}$ be i.i.d. rvs with absolutely continuous cdf $F$. Assume that for some $\epsilon > 0, E(\mathbf{X}^{1+\epsilon})$ exists. If*

$$E(\mathbf{X}_{U(m+2)} | \mathbf{X}_{U(m)}) = a\mathbf{X}_{U(m)} + b, \qquad \ldots (2.16)$$

*then only the following three cases are possible :*

1. *If $a < 1$, then $\mathbf{X}_1 \in POW(\theta, \mu, v)$, where $\theta = \frac{\sqrt{a}}{2-\sqrt{a}}, v = \frac{b}{1-a}$ and $\mu$ is an arbitrary real number with $\mu < v$.*
2. *If $a > 1$, then $\mathbf{X}_1 \in PAR(\theta, \mu, \delta)$, where $\theta = \frac{\sqrt{a}}{\sqrt{a}-1}, \delta = \frac{b}{a-1}$ and $\mu > \frac{b}{1-a}$.*
3. *If $a = 1$, then $\mathbf{X}_1 \in EXP(\lambda, \mu)$, where $\lambda = \frac{2}{b}$ and $\mu$ is an arbitrary real number.*

PROOF. Let $\mu = \inf\{x | F(x) > 0\}$ and $v = \sup\{x | F)x) < 1\}$.

Suppose $a < 1$. Then the assumption(2.16) implies that for all $x \in (\mu, v)$ we have $x \leq ax + b$. Thus $v = \frac{b}{1-a} < \infty$. Let us define the random variable $Z_n$ by $Z_n = -ln(v - \mathbf{X}_n), n \geq 1$.

Let $Z_{U(I)}, Z_{U(2)}, \ldots$ be the record values of $Z_n, n \geq 1$. Then the condition (2.16) reduces to

$$E(exp(-Z_{U(m+2)}) | Z_{U(m)}) = exp(-Z_{U(m)}). \qquad (2.17)$$

Hence by Lemma 2.1, $Z_1 \in EXP(\lambda, \gamma)$, where $\lambda = \frac{\sqrt{a}}{1-\sqrt{a}}$ and $\gamma$ is an arbitrary number.

Thus $\mathbf{X}_1 \in POW(\theta, \mu, v)$, where $\theta = \frac{\sqrt{a}}{\sqrt{a}-1}, v$ is as given above and $\mu = v - e^{-\gamma}$.

Consider $a > 1$. Let $\delta = \frac{b}{a-1}$ and consider the record values of the sequence $Z_n = ln(\delta + X_n), n \geq 1$. Then the condition (2.16) again reduces to (2.17). Hence by Lemma 2.1, $Z_1 \in EXP(\lambda, \gamma)$ where $\lambda = \frac{a}{1-\sqrt{a}}$ and $\gamma$ is an arbitrary number. Thus $\mathbf{X}_1 \in PAR(\theta, \mu, \delta)$ where $\theta = \frac{\sqrt{a}}{\sqrt{a}-1}, \delta = \frac{b}{a-1}$ and $\mu = e^{\gamma} - \delta$.

Consider $a = 1$, then using (1.1), proceeding similarly as in the proof of Lemma 2.1, we get

$$\int_x^y y(\{R(y) - R(x)\}f(y)dy = (x + b)\overline{F}(x), \qquad \ldots (2.18)$$

for any $x \in (\mu, v)$. Differentiating both sides of (2.18), we get

$$-\int_x^y yf(y)dy = -(x + b)\overline{F}(x) + \frac{\overline{F}(x))^2}{f(x)}. \qquad \ldots (2.19)$$

Differentiating both sides of (2.19), with respect to $x$, we obtain on simplification

$$\overline{F}^2(x)f'(x) + 3\overline{F}(x)f^2 - bf^3(x) = 0. \qquad \ldots (2.20)$$

Now substituting $y = \overline{F}(x)$(i.e. $f(x) = -y', f' = y''$) the equation (2.20) reduces to

$$-y''y^2 + 3yy'^2 + by'^3 = 0. \qquad \ldots (2.21)$$

Upon substitution $u(y) = y'$ the equation(2.21) reduces to the Bernoulli type finally yielding

$$y' = u = \frac{2y^3}{2D - by^2}, \qquad \ldots (2.22)$$

where $D$ is an arbitrary constant. Rewriting (2.22), we get

$$f(x) = \frac{2(\overline{F}(x))^3}{b(\overline{F}(x))^2 - 2D}. \qquad \ldots (2.23)$$

Since $f(x) \geq 0$ for all $x$ and $\overline{F}(x) \to 0$ as $x \to \eta$ it follows that we must have $D \leq 0$. Finally the solution of (2.23) is given by

$$-\frac{b}{2}ln\overline{F}(x) - \frac{D}{2}(\overline{F}(x))^{-2} = x + d \qquad \ldots (2.24)$$

where $d$ is an arbitrary real number. Since $\lim_{x\to\mu} \overline{F}(x) = 1$, we have $\mu > -\infty$ and $\mu + d = -D/2 \geq 0$. Thus we can write (2.24) as

$$\overline{F}(x) = exp[-\frac{2}{b}(x + d) + \frac{2(\mu + d)}{b(\overline{F}(x))^2}]. \qquad \ldots (2.25)$$

Suppose $v < \infty$, then $\lim_{x\to v} \overline{F}(x) = 0$ but the right hand side of (2.25) remains finite, while the left hand side tends to $\infty$. Hence $v = \infty$. Now by

Markov's inequality $\overline{F}(x) \leq \frac{m}{x^{l+\epsilon}}$ for sufficiently large positive $x$, where $m$ is a positive constant.

Thus for large $x$ we have

$$\overline{F}(x) \geq exp[-\frac{2}{b}(x+d) + \frac{2}{b}(\mu+d)\frac{x^{2+2\epsilon}}{m^2}]. \qquad \ldots (2.26)$$

If $\mu + d > 0$, then the right hand side of (2.26) is unbounded for $x \to \infty$ which contradicts the assumption that $0 \leq \overline{F}(x) \leq 1$. Hence $\mu + d = 0$ and thus from (2.25), we get

$$\overline{F}(x) exp[-\frac{2}{b}(x-\mu)], x > \mu.$$

$\square$

## References


AHSANULLAH, M (1980). Linear prediction of record values for the two parameter exponential distribution. *Ann. Inst. Stat. math*, **32**, 363-368.

AHSANULLAH, M (1995). Record Statistics. Nova Science Publisher, Inc, Commack NY.

DUNSMORE, I.R. (1983) The finite occurrence of records. *Ann. Inst. Stat. Math*, **35**, 267-277.

FEGRSON, T.S. (1967). On charactering distributions by properties of order statistics. *Sankhyā* A **29**, 265-278.

GOLOMB, M. AND SHANKO, M. (1965). *Elements of Ordinary Differential Equations.* Second Edition. McGraw-Hill Book Company, New York, NY.

NAGARAJA, H.N. (1977). On characterizations based on record values. *Austral. J. Statist.* **20**, 176-182.

NAGARAJA H.N. (1988). Some characterizations of continuous distributions based on regressions of adjacent order statists and record values. *Sankhyā* A **50**, 70-73.



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