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Limiting behavior of random permanents

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Abstract

Limit distributions of random permanents of increasing order are obtained by studying the asymptotic behavior of their variances and applying some well-known results from the asymptotic theory of random elementary symmetric polynomials.

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1. Introduction

Denote by $A=[a_{i,j}]$ an $m\times n$ real matrix with $n\geqslant m$. Then a permanent of the matrix A is defined by

$$\mathrm{Per}(A)=\sum_{(i_1,\dots,i_m):\ \{i_1,\dots,i_m\}\subset\{1,\dots,n\}} a_{1,i_1}\dots a_{m,i_m}.$$

Here we study asymptotic properties of random permanents with iid entries as m and n tend to ∞, i.e., we are interested in a matrix $X=[X_{i,j}]$, where $X_{i,j}$, $i=1,\dots,m$, $j=1,\dots,n$ are iid square integrable random variables. Denote $\mu=E(X_{1,1})$, $\sigma^2=\mathrm{Var}(X_{1,1})$ and assume that $\mu\neq 0$. Additionally, denote the coefficient of variation of $X_{1,1}$ by γ, i.e., $\gamma=\sigma/\mu$. We are interested in the limiting distribution of

$$\frac{1}{\binom{n}{m}m!}\,\mathrm{Per}(X)$$

as $m\to\infty$, $n\to\infty$ in such a way that $m/n\to\lambda\geqslant 0$.

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Problems of this kind for permanents of one dimensional projection matrices (i.e., matrices with all rows identical, equal to the same random vector with iid entries) and some related themes have been studied by many authors; see, for instance, Székely (1982); van Es and Helmers (1988), Borovskikh and Korolyuk (1994), Korolyuk and Borovskikh (1992, 1995), Kaneva (1995), and Kaneva and Korolyuk (1996). On the other hand, the scheme we are interested in here has been considered in earlier papers by Girko (1971), Rempala (1996), and Rempala and Gupta (1998) under additional assumption that there exist positive numbers $c < d$ such that $P(c < X_{i,j} < d) = 1$ for any i, j (cf. also Girko, 1990; Chapters 2 and 7). Here, developing a new approach, we are able to get rid of these artificial restrictions.

2. The variance of a random permanent

First, let us derive the formula for the variance of $\mathrm{Per}(X)$ in our setting.

Proposition 1.

$$\mathrm{Var}\left(\frac{\mathrm{Per}(X)}{\binom{n}{m} m!}\right) = \mu^{2m} \sum_{k=1}^{m} \frac{\binom{m}{k}}{\binom{n}{k} k!} \gamma^{2k}. \tag{1}$$

Proof. Let us fix an arbitrary product $X_{1,i_1} \ldots X_{m,i_m}$ among the summands present in the definition of $\mathrm{Per}(X)$. Since a permanent function is invariant under interchanging any two rows or columns, without loss of generality we may assume that this fixed summand consists of the diagonal entries only, i.e., that we have $i_k = k$ for $k = 1, \ldots, m$. For any number $l = 1, \ldots, m-1$, let us find the number of products in $\mathrm{Per}(X)$ having exactly l factors in common with $X_{1,1} \ldots X_{m,m}$. First, we fix l factors in $\binom{m}{l}$ ways. If we assume that $X_{1,1}, \ldots, X_{l,l}$ are fixed, then the remaining factors, in the products we are looking for, have to be of the form $X_{l+1,j_{l+1}}, \ldots, X_{m,j_m}$, where $j_r \neq r$, $r = l+1, \ldots, m$. Finding the number of such products is equivalent to computing the number of summands in a permanent of the matrix of dimensions $(m-l) \times (n-l)$ which do not contain any diagonal entry. To this end, we subtract the number of all summands having at least one factor being the diagonal entry, from the total number of all summands in that permanent. Using the exclusion–inclusion formula we get that this number equals to

$$\binom{n-l}{m-l}(m-l)! - \sum_{j=1}^{m-l}(-1)^{j+1}\binom{m-l}{j}\binom{n-l-j}{m-l-j}(m-l-j)!,$$

where the absolute value of the jth member of the above sum denotes the number of products having exactly j factors being the diagonal entries (equal to the number of choices of j positions on the diagonal) multiplied by the number of products of $m-l-j$ factors from the outside of the diagonal (equal to number of products in the permanent of the matrix of dimensions $(m-l-j) \times (n-l-j)$). Observe that the above expression can be written in a more compact form as

$$\sum_{j=0}^{m-l}(-1)^{j}\binom{m-l}{j}\binom{n-l-j}{m-l-j}(m-l-j)!.$$

Consequently, the number of pairs of products in $\mathrm{Per}(X)$ with exactly l factors in common equals to

$$\binom{n}{m} m! \binom{m}{l} \sum_{j=0}^{m-l}(-1)^{j}\binom{m-l}{j}\binom{n-l-j}{m-l-j}(m-l-j)!.$$

Now we are in a position to compute Var(Per(X)). We have

$$\operatorname{Var}(\operatorname{Per}(X)) = \binom{n}{m} m![\operatorname{Var}(Y_1 \dots Y_m)$$
$$+\sum_{l=1}^{m-1}\binom{m}{l}\operatorname{Cov}(Y_1\dots Y_l R_1(l), Y_1\dots Y_l R_2(l))\sum_{j=0}^{m-l}(-1)^j\binom{m-l}{j}\binom{n-l-j}{m-l-j}(m-l-j)!],$$

where $R_1(l)=Z_1\dots Z_{m-l}$, $R_2(l)=V_1\dots V_{m-l}$, $l=1,\dots,m-1$ and $Y_i,\ Z_i,\ V_i,\ i=1,\dots,m$ are iid r.v.'s independent of X and $Y_1 \stackrel{D}{=} X_{1,1}$. Note that

$$\operatorname{Var}(Y_1\dots Y_m) = \mu^{2m}[(1+\gamma^2)^m - 1],$$

$$\operatorname{Cov}(Y_1\dots Y_l R_1(l), Y_1\dots Y_l R_2(l)) = \mu^{2m}[(1+\gamma^2)^l - 1], \quad l=1,\dots,m-1.$$

Consequently,

$$\operatorname{Var}(\operatorname{Per}(X))$$
$$=\binom{n}{m} m!\mu^{2m}\sum_{l=1}^{m}\binom{m}{l}\sum_{j=0}^{m-l}(-1)^j\binom{m-l}{j}\binom{n-l-j}{m-l-j}(m-l-j)![(1+\gamma^2)^l-1]$$
$$=\binom{n}{m} m!\mu^{2m}\sum_{l=1}^{m}\binom{m}{l}\sum_{j=0}^{m-l}(-1)^j\binom{m-l}{j}\binom{n-l-j}{m-l-j}(m-l-j)!\sum_{k=1}^{l}\binom{l}{k}\gamma^{2k}$$

which, upon changing the order of summation and obvious cancellations, leads to

$$\operatorname{Var}(\operatorname{Per}(X)) = \binom{n}{m} m!\mu^{2m}\sum_{k=1}^{m}\gamma^{2k}\frac{m!}{k!}\sum_{j=0}^{m-k}(-1)^j\frac{1}{j!}\sum_{r=0}^{m-k-j}\frac{1}{r!}\binom{n-k-j-r}{m-k-j-r}.$$

Now, by Lemma 1 (see below), it follows that

$$\sum_{j=0}^{m-k}(-1)^j\frac{1}{j!}\sum_{r=0}^{m-k-j}\frac{1}{r!}\binom{n-k-j-r}{m-k-j-r} = \binom{n-k}{m-k},$$

which applied to the formula for Var(Per(X)) results in

$$\operatorname{Var}(\operatorname{Per}(X)) = \left(\binom{n}{m}m!\right)^2\mu^{2m}\sum_{k=1}^{m}\frac{\binom{m}{k}}{\binom{n}{k}k!}\gamma^{2k}. \qquad \square$$

Lemma 1. *For any positive integers $n \geqslant m$*

$$\sum_{j=0}^{m}(-1)^j\frac{1}{j!}\sum_{r=0}^{m-j}\frac{1}{r!}\binom{n-j-r}{m-j-r} = \binom{n}{m}. \tag{2}$$

Proof. First, consider the case $n=m$, i.e., we want to prove the formula

$$\sum_{j=0}^{m}(-1)^j\frac{1}{j!}\sum_{r=0}^{m-j}\frac{1}{r!} = 1 \tag{3}$$

for any positive integer m. To this end, we apply induction with respect to m. For $m=1$ the identity is obvious. Consider now $m+1$, assuming that (3) holds for m. Then, we have

$$\sum_{j=0}^{m+1}(-1)^j\frac{1}{j!}\sum_{r=0}^{m+1-j}\frac{1}{r!}=\sum_{j=0}^{m}(-1)^j\frac{1}{j!}\sum_{r=0}^{m-j}\frac{1}{r!}+\sum_{j=0}^{m}(-1)^j\frac{1}{j!(m+1-j)!}+(-1)^{m+1}\frac{1}{(m+1)!},$$

and by the induction assumption it follows that:

$$\sum_{j=0}^{m+1}(-1)^j\frac{1}{j!}\sum_{r=0}^{m+1-j}\frac{1}{r!}=1+(m+1)!\sum_{j=0}^{m+1}\binom{m+1}{j}(-1)^j=1.$$

Now, we apply the induction argument (with respect to $n=m,m+1,\dots$) to prove the identity (2). Up to now we have checked (2) for $n=m$. Assume now that (2) holds for some n and any $m\leqslant n$. Consider the identity for $n+1$. Then, by the recurrence formula for the Newton symbols we get

$$\sum_{j=0}^{m}(-1)^j\frac{1}{j!}\sum_{r=0}^{m-j}\frac{1}{r!}\binom{n+1-j-r}{m-j-r}=\sum_{j=0}^{m}(-1)^j\frac{1}{j!}\sum_{r=0}^{m-j}\frac{1}{r!}\binom{n-j-r}{m-j-r}$$
$$+\sum_{j=0}^{m-1}(-1)^j\frac{1}{j!}\sum_{r=0}^{m-1-j}\frac{1}{r!}\binom{n-j-r}{m-1-j-r}.$$

Therefore, the induction assumption leads to

$$\sum_{j=0}^{m}(-1)^j\frac{1}{j!}\sum_{r=0}^{m-j}\frac{1}{r!}\binom{n+1-j-r}{m-j-r}=\binom{n}{m}+\binom{n}{m-1}=\binom{n+1}{m}. \qquad \square$$

Consider now the kth-order elementary symmetric polynomials of l variables ($l\geqslant k>0$):

$$S_l^{(k)}=\frac{1}{\binom{l}{k}\mu^k}\sum_{\{i_1,\dots,i_k\}\subset\{1,\dots,l\}}X_{i_1}\dots X_{i_k},$$

where $X_1,\dots,X_l$ are square integrable iid r.v.'s. As in the above considerations, denote $\mu=E(X_1)\neq 0$, $\sigma^2=\mathrm{Var}(X_1)$ and $\gamma=\sigma/\mu$. Observe that

$$S_l^{(k)}=\frac{1}{\binom{l}{k}k!\mu^k}\,\mathrm{Per}(X'),$$

where X' is a $k\times l$ matrix with all identical rows equal to $[X_1,\dots,X_l]$.

Proposition 2.

$$\mathrm{Var}\left(\frac{\mathrm{Per}(X')}{\binom{l}{k}k!}\right)=\mu^{2k}\sum_{j=1}^{k}\frac{\binom{k}{j}^2}{\binom{l}{j}}\gamma^{2j}. \tag{4}$$

Proof. The formula is not new (cf. e.g., van Es and Helmers, 1988) but to make the paper more self-contained we give here its elementary derivation. Similarly as in the proof of Proposition 1 we compute the number of pairs of products having exactly j, $j=1,\dots,k-1$, factors identical. But in the present setting, due to symmetry present in $\mathrm{Per}(X')$, it is much easier – for a given fixed product and fixed j that number is simply $\binom{l-k}{k-j}$. Consequently,

$$\mathrm{Var}(\mathrm{Per}(X'))=\binom{l}{k}[\mathrm{Var}(X_1\dots X_k)+\sum_{j=1}^{k-1}\binom{k}{j}\binom{l-k}{k-j}\mathrm{Cov}(X_1\dots X_jR_1(j),X_1\dots X_jR_2(j))],$$

where $R_1(j)=Z_1\dots Z_{k-j}$ and $R_2(j)=V_1\dots V_{k-j}$, $j=1,\dots,k-1$, for iid r.v.'s Z_j, V_j, $j=1,\dots,k-1$, which are independent of X_j's and $Z_1\overset{\mathrm{d}}{=}X_1$. Hence, applying the formulas for the variance and the covariance obtained in the proof of Proposition 1, we get

$$\mathrm{Var}(\mathrm{Per}(X'))=\binom{l}{k}\mu^{2k}\sum_{j=1}^{k}\binom{k}{j}\binom{l-k}{k-j}\sum_{i=1}^{j}\binom{j}{i}\gamma^{2i}.$$

Upon changing the order of summation and some elementary algebra (including the hypergeometric summation formula) we have

$$\mathrm{Var}(\mathrm{Per}(X'))=\binom{l}{k}\mu^{2k}\sum_{i=1}^{k}\binom{k}{i}\gamma^{2i}\sum_{j=0}^{k-i}\binom{k-i}{j}\binom{l-k}{k-i-j}$$

$$=\binom{l}{k}\mu^{2k}\sum_{i=1}^{k}\binom{k}{i}\binom{l-k}{k-i}\gamma^{2i}=\left(\binom{l}{k}k!\right)^2\mu^{2k}\sum_{i=1}^{k}\frac{\binom{k}{i}^2}{\binom{l}{i}}\gamma^{2i}.\qquad\square$$

3. Limit theorems

The limit behavior of $S_l^{(k)}$ as $l-k\to\infty$ and $k^2/l\to 0$ has been studied by many authors. For instance, it was proved in van Es and Helmers (1988) that

$$\frac{S_l^{(k)}-E(S_l^{(k)})}{\sqrt{\mathrm{Var}(S_l^{(k)})}}\xrightarrow{\mathrm{D}}\mathscr{N},$$

where $\mathscr{N}$ is a standard normal random variable.

On the other hand, Borovskikh and Korolyuk (1992) showed that, if $k^2/l\to\lambda>0$, then

$$S_l^{(k)}\xrightarrow{\mathrm{D}}\exp(\sqrt{\lambda}\gamma\mathscr{N}-\lambda\gamma^2/2).$$

Let us consider the matrix $X=[X_{i,j}]$, as defined above, and let us denote by $\boldsymbol{X}$ the $m\times mn$ matrix with all identical rows of the form

$$[X_{1,1},\dots,X_{1,n},X_{2,1},\dots,X_{2,n},\dots,X_{m,1},\dots,X_{m,n}]=[Y_1,\dots,Y_{mn}],$$

i.e., the matrix of m replicas of the vectorization of X, where $X_{i,j}=Y_{(i-1)n+j}$, $i=1,\dots,m$, $j=1,\dots,n$. Note that

$$\mathrm{Per}(\boldsymbol{X})=m!\sum_{\{i_1,\dots,i_m\}\subset\{1,\dots,mn\}}Y_{i_1}\dots Y_{i_m}.$$

Lemma 2.

$$\mathrm{Cov}\left(\frac{\mathrm{Per}(\boldsymbol{X})}{\binom{mn}{m}m!},\frac{\mathrm{Per}(X)}{\binom{n}{m}m!}\right)=\frac{\mathrm{Var}(\mathrm{Per}(\boldsymbol{X}))}{\binom{mn}{m}^2 m!^2}. \tag{5}$$

Proof. Observe that for any product $Y_{j_1}\ldots Y_{j_m}$, $\{j_1,\ldots,j_m\}\subset\{1,\ldots,mn\}$,

$$\mathrm{Cov}(\mathrm{Per}(\boldsymbol{X}),Y_{j_1}\ldots Y_{j_m})=\mathrm{Cov}(\mathrm{Per}(\boldsymbol{X}),Y_1\ldots Y_m)=\alpha$$

is constant. Hence,

$$\mathrm{Cov}(\mathrm{Per}(\boldsymbol{X}),\mathrm{Per}(X))=\binom{n}{m}m!\alpha$$

and

$$\mathrm{Var}(\mathrm{Per}(\boldsymbol{X}))=\binom{mn}{m}m!\alpha.$$

Solving the above two equations for α and equating the resulting formulas gives (5). □

Now we are ready to state our main results.

Theorem 1. *If $m,n\to\infty$ in such a way that $m/n\to\lambda>0$ then*

$$\frac{\mathrm{Per}(X)}{\binom{n}{m}m!\mu^m}\xrightarrow{\mathrm{D}}\exp(\sqrt{\lambda}\gamma\mathcal{N}-\lambda\gamma^2/2).$$

Proof. Observe that by the result of Borovskikh and Korolyuk (1992) with $l=mn$ and $k=m$ ($k^2/l=m^2/(mn)=m/n\to\lambda>0$) it follows that

$$\frac{\mathrm{Per}(\boldsymbol{X})}{\binom{mn}{m}m!\mu^m}\xrightarrow{\mathrm{D}}\exp(\sqrt{\lambda}\gamma\mathcal{N}-\lambda\gamma^2/2).$$

Consequently, to prove our result, it is enough to show that

$$\Delta(m,n)=\mathrm{Var}\left(\frac{\mathrm{Per}(X)}{\binom{n}{m}m!\mu^m}-\frac{\mathrm{Per}(\boldsymbol{X})}{\binom{mn}{m}m!\mu^m}\right)\to 0$$

as $m,n\to\infty$. By Lemma 2 it follows that

$$\Delta(m,n)=\mathrm{Var}\left(\frac{\mathrm{Per}(X)}{\binom{n}{m}m!\mu^m}\right)-\mathrm{Var}\left(\frac{\mathrm{Per}(\boldsymbol{X})}{\binom{mn}{m}m!\mu^m}\right)$$

and Propositions 1 and 2 imply

$$\Delta(m,n)=\sum_{j=1}^{m}\binom{m}{j}\left(\frac{1}{\binom{n}{j}j!}-\frac{\binom{m}{j}}{\binom{mn}{j}}\right)\gamma^{2j}.$$

Observe that $\forall j=1,\ldots,m$ the inequality

$$\binom{mn}{j}\geqslant\binom{m}{j}\binom{n}{j}j!$$

follows immediately from the fact that $(m-k)(n-k)\leqslant mn-k$, $k=0,\dots,j-1$ and thus

$$(mn)(mn-1)\dots(mn-j+1)\geqslant(mn)[(m-1)(n-1)]\dots[(m-j+1)(n-j+1)].$$

Consequently, each member of the sum defining $\Delta(m,n)$ is non-negative. Observe also that for any $n\geqslant m\geqslant j\geqslant 1$

$$\binom{m}{j}\leqslant\binom{n}{j}.$$

For any $\varepsilon>0$ choose now m_0 large enough to have

$$\sum_{j=m_0+1}^{\infty}\frac{\gamma^{2j}}{j!}<\varepsilon.$$

Then, for sufficiently large $m>m_0$, we have

$$\begin{aligned}\Delta(m,n)&=\sum_{j=1}^{m_0}\binom{m}{j}\left(\frac{1}{\binom{n}{j}j!}-\frac{\binom{m}{j}}{\binom{mn}{j}}\right)\gamma^{2j}+\sum_{j=m_0+1}^{m}\binom{m}{j}\left(\frac{1}{\binom{n}{j}j!}-\frac{\binom{m}{j}}{\binom{mn}{j}}\right)\gamma^{2j}\\&\leqslant\sum_{j=1}^{m_0}\binom{m}{j}\left(\frac{1}{\binom{n}{j}j!}-\frac{\binom{m}{j}}{\binom{mn}{j}}\right)\gamma^{2j}+\sum_{j=m_0+1}^{m}\frac{\binom{m}{j}}{\binom{n}{j}j!}\gamma^{2j}\\&\leqslant\sum_{j=1}^{m_0}\binom{m}{j}\left(\frac{1}{\binom{n}{j}j!}-\frac{\binom{m}{j}}{\binom{mn}{j}}\right)\gamma^{2j}+\sum_{j=m_0+1}^{m}\frac{\gamma^{2j}}{j!}\\&\leqslant\sum_{j=1}^{m_0}\binom{m}{j}\left(\frac{1}{\binom{n}{j}j!}-\frac{\binom{m}{j}}{\binom{mn}{j}}\right)\gamma^{2j}+\varepsilon.\end{aligned}$$

Since the number of summands in the above expression is finite, it suffices to show that each of them converges to zero as $m,n\to\infty$. To this end observe that for any $j=1,\dots,m_0$ we have

$$\begin{aligned}\binom{m}{j}\left(\frac{1}{\binom{n}{j}j!}-\frac{\binom{m}{j}}{\binom{mn}{j}}\right)&=\frac{m(m-1)\dots(m-j+1)}{n(n-1)\dots(n-j+1)j!}\\&\quad\times\left[1-\frac{[mn][(m-1)(n-1)]\dots[(m-j+1)(n-j+1)]}{(mn)(mn-1)\dots(mn-j+1)}\right]\\&\leqslant\left[1-\frac{(1-1/m)(1-1/n)}{1-1/(mn)}\cdots\frac{(1-(j-1)/m)(1-(j-1)/n)}{1-(j-1)/(mn)}\right]\leqslant\varepsilon\end{aligned}$$

for sufficiently large m and n. □

The next result is a permanent version of the classical central limit theorem for iid random variables (the classical result for the sums of iid r.v.'s is obtained by taking $m=1$ below).

Theorem 2. *If $n-m\to\infty$ in such a way that $m/n\to 0$, then*

$$\frac{1}{\gamma}\sqrt{\frac{n}{m}}\left(\frac{\mathrm{Per}(X)}{\binom{n}{m}\,m!\mu^m}-1\right)\xrightarrow{\mathrm{D}}\mathcal{N}.$$

Proof. Let us first show that

$$\frac{\sqrt{\mathrm{Var}\left(\frac{\mathrm{Per}(X)}{\binom{n}{m}m!\mu^m}\right)}}{\gamma\sqrt{m/n}}\to 1,$$

as $n-m\to\infty$ and $m/n\to 0$. To this end observe (see Proposition 1) that the square of the left-hand side of the above expression can be rewritten as

$$1+\frac{1}{\gamma^2}\frac{n}{m}\sum_{k=2}^{m}\frac{m(m-1)\dots(m-k+1)}{n(n-1)\dots(n-k+1)}\gamma^{2k}=1+\sum_{k=2}^{m}\frac{(m-1)\dots(m-k+1)}{(n-1)\dots(n-k+1)}(\gamma^2)^{k-1}.$$

Take now m and n large enough to have $\gamma^2 m/n<1$. Then, the second part of the above expression is majorized by

$$\sum_{k=2}^{m}\left(\frac{m}{n}\right)^{k-1}(\gamma^2)^{k-1}=\gamma^2\frac{m}{n}\frac{1-((m/n)\gamma^2)^m}{1-(m/n)\gamma^2}\leqslant\gamma^2\frac{m}{n}\frac{1}{1-(m/n)\gamma^2}\to 0$$

as $n-m\to\infty$ and $m/n\to 0$.

In view of the above, it suffices to prove that

$$\frac{1}{\sqrt{\mathrm{Var}\left(\frac{\mathrm{Per}(X)}{\binom{n}{m}m!\mu^m}\right)}}\left(\frac{\mathrm{Per}(X)}{\binom{n}{m}\,m!\mu^m}-1\right)\xrightarrow{\mathrm{D}}\mathcal{N}. \tag{6}$$

By the result of van Es and Helmers (1988) (with $l=mn$ and $k=m$, i.e. $k^2/l=m/n\to 0$) it follows that:

$$\frac{1}{\sqrt{\mathrm{Var}\left(\frac{\mathrm{Per}(\boldsymbol{X})}{\binom{mn}{m}m!\mu^m}\right)}}\left(\frac{\mathrm{Per}(\boldsymbol{X})}{\binom{mn}{m}\,m!\mu^m}-1\right)\xrightarrow{\mathrm{D}}\mathcal{N}.$$

Hence, to prove the convergence (6) it is enough, similarly as in the proof of Theorem 1, to show that

$$\varDelta(m,n)=\mathrm{Var}\left(\frac{\frac{\mathrm{Per}(X)}{\binom{n}{m}m!\mu^m}}{\sqrt{\mathrm{Var}\left(\frac{\mathrm{Per}(X)}{\binom{n}{m}m!\mu^m}\right)}}-\frac{\frac{\mathrm{Per}(\boldsymbol{X})}{\binom{mn}{m}m!\mu^m}}{\sqrt{\mathrm{Var}\left(\frac{\mathrm{Per}(\boldsymbol{X})}{\binom{mn}{m}m!\mu^m}\right)}}\right)\to 0$$

as $n-m\to\infty$ and $m/n\to 0$. Observe that $\varDelta(m,n)$ can be rewritten (using Lemma 2) in the form

$$\varDelta(m,n)=2\left(1-\sqrt{\mathrm{Var}\left(\frac{\mathrm{Per}(\boldsymbol{X})}{\binom{mn}{m}\,m!\mu^m}\right)\Big/\,\mathrm{Var}\left(\frac{\mathrm{Per}(X)}{\binom{n}{m}\,m!\mu c^m}\right)}\right)$$

and therefore to prove that $\Delta(m,n) \to 0$ it suffices to show that

$$\operatorname{Var}\left(\frac{\operatorname{Per}(\boldsymbol{X})}{\binom{mn}{m}\, m!\mu^m}\right) \Big/ \operatorname{Var}\left(\frac{\operatorname{Per}(X)}{\binom{n}{m}\, m!\mu^m}\right) \to 1.$$

To this end, let us observe that, by the formulas from Propositions 1 and 2, we have

$$1 - \operatorname{Var}\left(\frac{\operatorname{Per}(\boldsymbol{X})}{\binom{mn}{m}\, m!\mu^m}\right) \Big/ \operatorname{Var}\left(\frac{\operatorname{Per}(X)}{\binom{n}{m}\, m!\mu^m}\right) = \frac{\sum_{k=1}^m \left(\binom{m}{k} / \binom{n}{k} k! - \binom{m}{k}^2 / \binom{mn}{k}\right) \gamma^{2k}}{\sum\limits_{k=1}^m \frac{\binom{m}{k}}{\binom{n}{k} k!} \gamma^{2k}}$$

$$\leqslant \frac{n}{m} \gamma^{-2} \sum_{k=2}^m \frac{\binom{m}{k}}{\binom{n}{k} k!} \gamma^{2k},$$

where the last inequality follows by taking only the first member of the sum in the denominator and the observation (see the proof of Theorem 1) that all the summands in the numerator are nonnegative and the one for $k = 1$ vanishes. Observe that

$$\frac{n}{m} \gamma^{-2} \sum_{k=2}^m \frac{\binom{m}{k}}{\binom{n}{k} k!} \gamma^{2k} = \gamma^{-2} \sum_{k=2}^m \frac{(m-1)(m-2)\ldots(m-k+1)}{(n-1)(n-2)\ldots(n-k+1)} \gamma^{2k} \leqslant \sum_{k=2}^m \left(\frac{m}{n}\gamma^2\right)^{k-1} \to 0$$

as $n - m \to \infty$ and $m/n \to 0$ – see the first part of this proof. Consequently, $\Delta(m,n) \to 0$ and the proof is completed. □

As an example of the application of our results obtained above, we consider the problem of counting perfect matchings in a bipartite random graph.

Example (*Counting matchings in a bipartite random graph*). Let $G_{m,n,p} = (V_1, V_2; E)$ be a bipartite random graph with $V_1 = \{r_1, r_2, \ldots, r_m\}$, $V_2 = \{c_1, c_2, \ldots, c_n\}$, $(m \leqslant n)$ and $E \subset V_1 \times V_2$. Assume that the edges occur independently with a fixed probability $0 < p < 1$. In this setting, the reduced adjacency matrix of $G_{m,n,p}$ is a random $m \times n$ matrix $X = [X_{i,j}]$ of independent Bernoulli $B(p)$ random variables. If $m = n$ it is well known (cf. e.g., Brualdi and Ryser, 1991; chapter 7) that the number of *perfect matchings* in $G_{n,n,p}$, say $h(G,n,p)$, satisfies $h(G,n,p) = \operatorname{Per}(X)$. Extending the concept of a perfect matching to the case when $m < n$, we shall say that a matching is *fully saturating* if it saturates the set V_1. Denoting the number of fully saturating matchings by $H(G,m,n,p)$ we have again $H(G,m,n,p) = \operatorname{Per}(X)$. Observe that in the notation of this section, for Bernoulli random variables we have $\mu = p$, $\gamma = \sqrt{(1-p)/p}$, and $\operatorname{Per}(\boldsymbol{X}) \stackrel{D}{=} m! \binom{S_{mn}}{m}$, where S_{mn} is a binomial $b(mn, p)$ random variable. From the proof of Theorem 1 it follows that if $n, m \to \infty$ and $m/n \to \lambda$ then

$$\frac{H(G,m,n,p)}{\binom{n}{m} m!\, p^m} - \frac{\binom{S_{mn}}{m}}{\binom{mn}{m} p^m} \stackrel{P}{\to} 0,$$

and, in particular,

$$\frac{H(G,m,n,p)}{\binom{n}{m} m!\, p^m} \stackrel{D}{\to} \exp(\sqrt{\lambda}\gamma\mathcal{N} - \lambda\gamma^2/2). \tag{7}$$

On the other hand, if $n-m\to\infty$ and $m/n\to 0$ then, by the argument used in the proof of Theorem 2,

$$\sqrt{\frac{mp}{n(1-p)}}\left(\frac{H(G,m,n,p)}{\binom{n}{m}m!\,p^m}-\frac{\binom{S_{mn}}{m}}{\binom{mn}{m}\,p^m}\right)\xrightarrow{\text{P}}0$$

and, in particular,

$$\sqrt{\frac{mp}{n(1-p)}}\left(\frac{H(G,m,n,p)}{\binom{n}{m}m!\,p^m}-1\right)\xrightarrow{\text{D}}\mathcal{N}.$$

In the case $m=n$ the relation (7) has been noted, in a slightly different form, by Janson (1994).

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