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## POISSON PROCESS VIA MARTINGALE AND RELATED CHARACTERISTICS

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### Abstract

The classical martingale characterizations of the Poisson process were obtained for point process or purely discontinuous martingale i.e. under additional assumptions on properties of trajectories. Here our aim is to search for related characterizations without relying on properties of trajectories. Except for a new martingale characterization, results based on conditional moments jointly involving the past and the nearest future are presented.

*Keywords:* Poisson process; martingales; conditional variance; characterizations; conditioning given the future; conditioning given the past

AMS 1991 Subject Classification: Primary 60E99

### 1. Introduction

Let  $X = (X_t)_{t \geq 0}$  be a real stochastic process defined on a probability space  $(\Omega, \mathcal{F}, P)$ , adapted to some filtration  $(\mathcal{F}_t)_{t \geq 0}$  and  $Y = (Y_t)_{t \geq 0}$ , where  $Y_t = X_t - t \forall t \geq 0$ . The celebrated Watanabe martingale characterization theorem states that if  $X$  is a point process and  $(Y_t, \mathcal{F}_t)_{t \geq 0}$  is a martingale, then  $X$  is a Poisson process — see Watanabe (1964). Related martingale characterizations were obtained for the doubly stochastic Poisson process in Brémaud (1981), for spatial Poisson processes in Ivanoff (1985) and Merzbach and Nualart (1986) and for set-indexed Poisson processes in Ivanoff and Merzbach (1993). If a structure of square integrable martingales is taken into account, then it is well known that if  $(Y_t, \mathcal{F}_t)_{t \geq 0}$  is a purely discontinuous martingale with jumps equal to +1, and  $(Y_t^2 - t, \mathcal{F}_t)_{t \geq 0}$  is also a martingale, then  $X$  is a Poisson process — see for instance Elliott (1982), Chapter 12.

Here our aim is to search for characterizations of the Poisson process based only on properties of conditional and ordinary moments. Such an approach to the identification of stochastic processes goes back to the early 1980s and the development of the research in this area has been recently surveyed by Prakasa Rao (1998). In Bryc (1995), Chapter 8.4, applications to random vibrations and epidemics are outlined. Bryc (1998) presents an application of such characterizations to the identification of stationary random fields.

An approach to martingale characterizations of the Poisson process on the real line, without taking into account properties of trajectories, was presented in Wesolowski (1990a). Instead, martingale properties for polynomial processes up to the third order were assumed. Another characterization involving conditional moments up to the second order with respect not only to the past but, additionally, jointly with respect to the past and the future was given in Bryc (1987). Here also no attention was paid to trajectory properties.

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Conditional moments were also used to characterize the Poisson process in the renewal theory setting by Huang *et al.* (1994), extending earlier results by Çinlar and Jagers (1973), Holmes (1974) and Gupta and Gupta (1986). Another characterization based on the joint moments structure has been given by Fang (1991).

The aim of the present note is to push ahead the subject by giving (i) a straightforward extension of the martingale characterization from Wesolowski (1990a) — considering the second and third ordinary moments instead of two of the martingale conditions (Section 2), (ii) a Poisson version of the martingale and reverse martingale characterization of the Wiener process obtained in Wesolowski (1990b) (Section 3), (3) a version of the Bryc (1987) result involving only the conditional variance with respect to the past and the nearest future (Section 4).

Throughout the paper stochastic processes are identified with the measures they generate in the space of all functions  $\mathbb{R}^{\mathbb{R}_+}$ .

### 2. Past conditioning

A non-decreasing process  $X = (X_t)_{t \geq 0}$  is a Poisson process if only

$$(Y_t, \mathcal{F}_t)_{t \geq 0}, \quad (Y_t^2 - t, \mathcal{F}_t)_{t \geq 0} \quad \text{and} \quad (Y_t^3 - 3tY_t - t, \mathcal{F}_t)_{t \geq 0}$$

are martingales, where  $Y_t = X_t - t \forall t \geq 0$  — see Wesolowski (1990a). An immediate consequence is that in the class of thrice-integrable non-decreasing processes with independent increments the Poisson process is uniquely determined only by its moments up to the third order:  $E(X_t) = t, E(X_t^2) = t^2 + t, E(X_t^3) = t^3 + 3t^2 + t, t \geq 0$ .

Here we give a straightforward extension of the above-mentioned martingale characterization by considering the second and third moments instead of the second and third order martingale properties.

**Theorem 2.1.** *Assume that  $X$  is a non-decreasing process. Let  $(Y_t, \mathcal{F}_t)_{t \geq 0}$  be a martingale. If  $E(Y_t^2) = E(Y_t^3) = t \forall t \geq 0$  then  $X$  is a Poisson process.*

The proof will be based on the following version of the Poisson central limit theorem for row-wise dependent triangular arrays obtained by Beška *et al.* (1982).

**Theorem 2.2.** *Let  $\{Z_{n,k}, k = 1, \dots, n; n \geq 1\}$  be a double sequence of non-negative random variables adapted to a row-wise increasing double sequence of  $\sigma$ -fields  $\{\mathcal{G}_{n,k}, k = 1, \dots, n; n \geq 1\}$  and let  $\mathcal{G}_0 \subset \mathcal{G}_{n,0} \subset \mathcal{G}_{n,1}, \forall n \geq 1$ , also be  $\sigma$ -fields. If for  $n \rightarrow \infty$*

$$\max_{1 \leq k \leq n} E(Z_{n,k} | \mathcal{G}_{n,k-1}) \xrightarrow{P} 0, \tag{1}$$

$$\sum_{k=1}^n E(Z_{n,k} | \mathcal{G}_{n,k-1}) \xrightarrow{P} a > 0, \tag{2}$$

$$\sum_{k=1}^n E(Z_{n,k} I(|Z_{n,k} - 1| > b | \mathcal{G}_{n,k-1})) \xrightarrow{P} 0, \quad \forall b > 0, \tag{3}$$

then the conditional distributions  $S_n = \sum_{k=1}^n Z_{n,k}$  given  $\mathcal{G}_0$  are weakly convergent to the Poisson law with the parameter  $a$ .

*Proof of Theorem 2.1.* For any  $s < t$  consider a sequence of divisions of the interval  $[s, t]$ :

$$s = t_{n,0} < t_{n,1} < \dots < t_{n,n} = t, \quad n = 1, 2, \dots,$$

such that

$$\lim_{n \rightarrow \infty} \max_{1 \leq j \leq n} (t_{n,j} - t_{n,j-1}) = 0.$$

Define double sequences of random variables  $(Z_{n,k}, k = 1, \dots, n; n \geq 1)$  and  $\sigma$ -fields  $(\mathcal{G}_{n,k}, k = 1, \dots, n; n \geq 1)$  by

$$Z_{n,k} = X_{t_{n,k}} - X_{t_{n,k-1}}, \quad \mathcal{G}_{n,k} = \mathcal{F}_{t_{n,k}}, \quad k = 0, 1, \dots, n; \quad n \geq 1.$$

Additionally, write  $\mathcal{G}_0 = \mathcal{F}_s$ . Consider now any  $k = 0, 1, \dots, n, n \geq 1$ . Then the martingale property implies

$$E(Z_{n,k} | \mathcal{G}_{n,k-1}) = t_{n,k} - t_{n,k-1}.$$

Consequently the conditions (1) and (2) (with  $a = t - s$ ) of Theorem 2.2 are satisfied.

We want to show now that (3) is also fulfilled. Observe that for any  $\epsilon > 0$  and for any  $b > 0$  we have, via the Markov inequality

$$\begin{aligned} P\left(\left|\sum_{k=1}^n E(Z_{n,k} I(|Z_{n,k} - 1| > b | \mathcal{G}_{n,k-1}))\right| > \epsilon\right) \\ \leq \epsilon^{-1} E\left(\sum_{k=1}^n E(Z_{n,k} I(|Z_{n,k} - 1| > b | \mathcal{G}_{n,k-1}))\right) = \epsilon^{-1} \sum_{k=1}^n E(Z_{n,k} I(|Z_{n,k} - 1| > b)). \end{aligned}$$

Since  $xI(|x-1| > b) \leq x(x-1)^2/b^2$  for any non-negative  $x$  ( $Z_{n,k} \geq 0$  a.s. by the assumption) then

$$P\left(\left|\sum_{k=1}^n E(Z_{n,k} I(|Z_{n,k} - 1| > b | \mathcal{G}_{n,k-1}))\right| > \epsilon\right) \leq \epsilon^{-1} b^{-2} \sum_{k=1}^n E(Z_{n,k} (Z_{n,k} - 1)^2).$$

Now the form of the first three moments of  $Y_t$  implies that

$$E(X_t - X_s)(X_t - X_s - 1)^2 = (t - s)^2(t - s + 1)$$

for any  $0 \leq s < t$ . Consequently

$$\sum_{k=1}^n E(Z_{n,k} (Z_{n,k} - 1)^2) \leq \max_{1 \leq k \leq n} (t_{n,k} - t_{n,k-1})(t - s)^2 \xrightarrow{n \rightarrow \infty} 0.$$

Finally we conclude that all the assumptions of Theorem 2.2 are satisfied. Observe that  $S_n = X_t - X_s \forall n = 1, 2, \dots$ . Hence the conditional distribution of  $X_t - X_s$  given the past is Poisson with parameter  $t - s$  for any  $0 \leq s < t$ , which means that  $X$  is a Poisson process.

### 3. Separate past and (the nearest) future conditioning

Assume, as above, that a process  $X = (X_t)_{t \geq 0}$  is adapted to a non-decreasing sequence of  $\sigma$ -fields  $\mathcal{F} = (\mathcal{F}_t)_{t \geq 0}$ . Assume additionally that  $X$  is also adapted to another sequence of  $\sigma$ -fields  $\mathcal{G} = (\mathcal{G}_t)_{t \geq 0}$  which is non-increasing. Recall that  $(X_t, \mathcal{G}_t)_{t > 0}$  is called a reverse martingale if it is integrable and  $E(X_s | \mathcal{G}_t) = X_t$  for any  $0 < s < t$ .

An analogue of the celebrated Lévy characterization of the Wiener process, proved in Wesolowski (1990b) (see also Wise (1992)) states that: if  $(X_t, \mathcal{F}_t)_{t \geq 0}$ ,  $(X_t^2 - t, \mathcal{F}_t)_{t \geq 0}$  are martingales and  $(X_t/t, \mathcal{G}_t)_{t > 0}$ ,  $((X_t^2 - t)/t^2, \mathcal{G}_t)_{t > 0}$  are reverse martingales then  $X$  is a Wiener process. Here we present a version of this result for the Poisson process.

For the process  $X$  define, as in the previous section, the process  $Y = (Y_t)_{t \geq 0}$  by  $Y_t = X_t - t \forall t \geq 0$ .

**Theorem 3.1.** *Let  $X$  be a square integrable non-decreasing process. If*

$$(Y_t, \mathcal{F}_t)_{t \geq 0}, \quad (Y_t^2 - t, \mathcal{F}_t)_{t \geq 0}$$

*are martingales and*

$$(Y_t/t, \mathcal{G}_t)_{t > 0}, \quad ((Y_t^2 - Y_t - t)/t^2, \mathcal{G}_t)_{t > 0}$$

*are reverse martingales then  $X$  is a Poisson process.*

*Proof.* Observe that the assumptions imply for any  $0 < s < t$  that  $E(Y_t | Y_s) = Y_s$ ,  $E(Y_s | Y_t) = (s/t)Y_t$ ,  $\text{var}(Y_t | Y_s) = t - s$  and  $\text{var}(Y_s | Y_t) = (s(t - s)/t^2)Y_t$ . Consequently it follows from Bryc (1985) that  $E(Y_t^3)$  exists  $\forall t \geq 0$ .

Also it follows easily that  $E(Y_t) = 0$  and  $E(Y_t^2) = t \forall t \geq 0$ . Using basic properties of conditioning we can now compute

$$E(Y_s Y_t^2) = E[E(Y_s | \mathcal{G}_t) Y_t^2] = \frac{s}{t} E(Y_t^3).$$

On the other hand we also have

$$E(Y_s Y_t^2) = E[Y_s E(Y_t^2 | \mathcal{F}_s)] = E[Y_s (Y_s^2 - (t - s))] = E(Y_s^3).$$

Now apply the same approach to  $E(Y_s^2 Y_t)$ :

$$\begin{aligned} E(Y_s^3) &= E[Y_s^2 E(Y_t | \mathcal{F}_s)] = E(Y_s^2 Y_t) = E[E(Y_s^2 | \mathcal{G}_t) Y_t] \\ &= s^2 E\left[ E\left( \frac{Y_s^2 - Y_s - s}{s^2} \mid \mathcal{G}_t \right) Y_t \right] + E[E(Y_s + s | \mathcal{G}_t) Y_t] \\ &= \frac{s^2}{t^2} E[(Y_t^2 - Y_t - t) Y_t] + \frac{s}{t} E(Y_t^2) + s E(Y_t) = \frac{s^2}{t^2} E(Y_t^3) - \frac{s^2}{t} + s. \end{aligned}$$

Taking into account these relations we get the identity

$$\frac{s}{t} E(Y_t^3) = \frac{s^2}{t^2} E(Y_t^3) - \frac{s^2}{t} + s \quad \forall 0 \leq s < t.$$

This yields  $E(Y_t^3) = t \forall t \geq 0$ . The result now follows from Theorem 2.1.

Observe that instead of reverse martingality conditions it suffices to consider only the nearest future, i.e. for any  $0 \leq s < t$

$$E(Y_s | Y_t) = \frac{s}{t} Y_t,$$

and

$$\text{var}(Y_s | Y_t) = \frac{s(t-s)}{t^2} Y_t.$$

Also it follows from the above proof and Theorem 2.1 that the assumption about  $(Y_t^2 - t, \mathcal{F}_t)_{t \geq 0}$  can be reduced to

$$\text{var}(Y_t | Y_s) = t - s$$

for any  $0 \leq s < t$ .

#### 4. Joint past and nearest future conditioning

Although martingale characterizations are not a very recent device, investigations which take into account conditional moments given the past and the future states jointly appeared first in 1980s. In a series of papers (Plucińska (1983), Wesolowski (1984), Bryc (1985)) it was proved that the Gaussian process is uniquely determined by its second order conditional structure (linearity of regressions and non-randomness of conditional variances) — see also Chapter 8 in the monograph Bryc (1995). The ideas developed originally for Gaussian processes were then transformed for the Poisson process by Bryc (1987), where it was proved that a square integrable process  $X = (X_t)_{t \geq 0}$  with  $E(X_t) = t$  and  $\text{cov}(X_s, X_t) = \min\{s, t\}$ ,  $\forall s, t \geq 0$ , is Poisson if only the conditions

$$E(X_s | X_{r_1}, \dots, X_{r_n}, X_r) = \alpha_1 X_r + \beta_1, \tag{4}$$

$$E(X_s | X_{r_1}, \dots, X_{r_n}, X_r, X_t) = \alpha_2 X_r + \beta_2 X_t + \eta, \tag{5}$$

$$\text{var}(X_s | X_{r_1}, \dots, X_{r_n}, X_r) = \delta, \tag{6}$$

$$\text{var}(X_s | X_{r_1}, \dots, X_{r_n}, X_r, X_t) = \gamma(X_t - X_r) \tag{7}$$

hold for any  $0 \leq r_1 < \dots < r_n \leq r < s < t$ ,  $n = 1, 2, \dots$ , where  $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma, \delta$  are some real constants (depending possibly on  $r_1, \dots, r_n, r, s, t$ ).

This result was slightly refined in Wesolowski (1988) by reducing the assumptions to (4), (5) and (7). Also similar results were given for other processes in Wesolowski (1989, 1993). Here we will further reduce the assumptions of Bryc's theorem, first to (5) and (7), and then show that the essence is hidden in (7) only.

**Theorem 4.1.** *Assume that  $X = (X_t)_{t \geq 0}$  is a square integrable process such that  $E(X_t) = t$  and  $\text{cov}(X_s, X_t) = s \forall 0 \leq s \leq t$ . If the conditions (5) and (7) hold then  $X$  is a Poisson process.*

*Proof.* Obvious computations give

$$\alpha_2 = \frac{t-s}{t-r}, \quad \beta_2 = \frac{s-r}{t-r}, \quad \eta = 0.$$

Our aim is to show that (4) holds with  $\alpha_1 = 1$  and  $\beta_1 = s - r$ . To this end write

$$Y = (X_{r_1}, \dots, X_{r_n})$$

and consider

$$E[E(X_s | Y, X_r) - (X_r + s - r)]^2 = E[E(E(X_s | Y, X_r, X_t) | Y, X_r) - (X_r + s - r)]^2,$$

where  $t > s$  is arbitrary. This from (5) is equal to

$$\begin{aligned} E\left[E\left(\frac{t-s}{t-r}X_r + \frac{s-r}{t-r}X_t \mid Y, X_r\right) - (X_r + s - r)\right]^2 \\ = E\left[E\left(\frac{t-s}{t-r}X_r + \frac{s-r}{t-r}X_t - X_r - (s - r) \mid Y, X_r\right)\right]^2. \end{aligned}$$

Consequently the Jensen inequality implies

$$\begin{aligned} E[E(X_s | Y, X_r) - (X_r + s - r)]^2 \\ \leq E\left\{E\left[\left(\frac{t-s}{t-r}X_r + \frac{s-r}{t-r}X_t - X_r - (s - r)\right)^2 \mid Y, X_r\right]\right\} \\ = E\left(\frac{t-s}{t-r}X_r + \frac{s-r}{t-r}X_t - X_r - (s - r)\right)^2 \\ = \left(\frac{s-r}{t-r}\right)^2 E(X_t - X_r - (t - r))^2 = \frac{(s-r)^2}{t-r}. \end{aligned}$$

Now as  $t \rightarrow \infty$  we get from the above inequality

$$E[E(X_s | Y, X_r) - (X_r + s - r)]^2 = 0,$$

which means that (4) really holds. Now the result follows from Wesolowski (1988), where it was shown that (4), (5) and (7) jointly imply (6).

Further simplifications of Theorem 4.1 are possible by deleting (5) and assigning instead the exact value for  $\gamma$ , i.e.

$$\gamma = \frac{(t-s)(s-r)}{(t-r)^2}.$$

Then the Poisson process is characterized by a single condition (7), since then the condition (5) may be recovered by using the minimizing property of the regression function. It is well known that  $\phi(Y, X_r, X_t) = E(X_s | Y, X_r, X_t)$  is the  $P$ -unique function minimizing the functional  $E(X_s - \phi(Y, X_r, X_t))^2$  and the minimum is equal to  $E(\text{var}(X_s | Y, X_r, X_t))$ . Hence it suffices to prove that if (5) is satisfied then for any  $Y$  and  $r < s < t$

$$E(\text{var}(X_s | Y, X_r, X_t)) = \frac{(t-s)(s-r)}{(t-r)^2} E(X_t - X_r) = \frac{(t-s)(s-r)}{t-r}.$$

But under (5)

$$\begin{aligned}
 & E(\text{var}(X_s | Y, X_r, X_t)) \\
 &= E\left(X_s - \frac{t-s}{t-r}X_r - \frac{s-r}{t-r}X_t\right)^2 \\
 &= E\left[(X_s - s) - \frac{t-s}{t-r}(X_r - r) - \frac{s-r}{t-r}(X_t - t)\right]^2 \\
 &= \text{var}(X_s) + \left(\frac{t-s}{t-r}\right)^2 \text{var}(X_r) + \left(\frac{s-r}{t-r}\right)^2 \text{var}(X_t) - 2\frac{t-s}{t-r} \text{cov}(X_r, X_s) \\
 &\quad - 2\frac{s-r}{t-r} \text{cov}(X_s, X_t) + 2\frac{(t-s)(s-r)}{(t-r)^2} \text{cov}(X_r, X_t).
 \end{aligned}$$

Consequently applying the formulas for the first moments and covariances one easily gets

$$\begin{aligned}
 & E(\text{var}(X_s | Y, X_r, X_t)) \\
 &= s + \left(\frac{t-s}{t-r}\right)^2 r + \left(\frac{s-r}{t-r}\right)^2 t - 2\frac{t-s}{t-r}r - 2\frac{s-r}{t-r}s + 2\frac{(t-s)(s-r)}{(t-r)^2}r \\
 &= \frac{(t-s)(s-r)}{t-r}.
 \end{aligned}$$

Now the result follows from the uniqueness of the function minimizing  $E(X_s - \phi(Y, X_r, X_t))^2$ .

Observe that only condition (5), together with the form of ordinary moments of the first and second order was taken into account in the proof of Theorem 4.1 and the above considerations. Consequently the same argument leads to analogous extensions of characterizations of the Wiener, gamma and other processes — see Wesolowski (1989, 1990b, 1993).

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