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Linearity of regression for non-adjacent record values

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Abstract

Let $X_1, X_2, \ldots$ be a sequence of iid random variables having a continuous distribution; by $R_1, R_2, \ldots$ denote the corresponding record values. All the distributions allowing linearity of regressions either $E(R_{m+k}|R_m)$ or $E(R_m|R_{m+k})$ are identified.

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1. Introduction

Let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed (iid) random variables (rvs) with a continuous distribution function (df) F. Let us define $U(1) = 1$ and for $n > 1$

$$U(n) = \min\{k > U(n-1)\colon X_k > X_{U(n-1)}\}\,.$$

The record value sequence (R_n) is then defined by

$$R_n = X_{U(n)}, \quad n = 1, 2, \ldots\,.$$

In this paper we are interested in linearity of regression of a record statistic on another one, not necessarily adjacent. Usually such investigations are preceded by discovering their order statistics counterparts. According to such a scheme, Nagaraja (1977), following Ferguson (1967) for order statistics, proved that if $E(R_{m+1}|R_m) = aR_m + b$ for some real a and b and for some $m \geqslant 1$, then, except for a change of location and scale, only the following three cases are possible:

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(1) $a=1$ and $F(x)=1-\mathrm{e}^{-x}$, $x>0$ (exponential type distribution),
(2) $a>1$ and $F(x)=1-x^{\theta}$, $x>1$ (Pareto type distribution),
(3) $a<1$ and $F(x)=1-(-x)^{\theta}$, $-1<x<0$ (power type distribution), where $\theta=a/(1-a)$.

Dually, Nagaraja (1988), making use of some analogy between conditional distribution of record values and order statistics distribution and exploiting Ferguson (1967) result, observed that if $E(R_m|R_{m+1})=cR_{m+1}+d$, then, except for a change of location and scale:

(4) $c=1$ and $F(x)=1-\mathrm{e}^{-\mathrm{e}^{x}}$, $x\in\mathbb{R}$ (negative Gumbel-type distribution),
(5) $c>1$ and $F(x)=1-\mathrm{e}^{-(-x)^{\theta}}$, $x<0$ (negative Fréchet-type distribution),
(6) $c<1$ and $F(x)=1-\mathrm{e}^{-x^{\theta}}$, $x>0$ (Weibull distribution),

where $\theta=c/[n(1-c)]$. Recall that (4)–(6) are asymptotic distributions of suitably normalized minima of iid rvs. A similar duality for order statistics is rather obvious – it suffices to take negatives of the original observations. For record values it needs some more effort and a useful work could be a thorough explanation of a bright idea introduced in Nagaraja (1988), which helps to settle the problem – this is going to be done in the Section 4, while investigating the dual problem in the non-adjacent case.

Up to late 1990s there was no visible progress in studying linearity of regression for non-adjacent order statistics. In Wesołowski and Ahsanullah (1997) the first result of this kind appeared for the spacing equal to 2. It was followed by the analogous result for the record values, namely, under the assumption of absolute continuity it was proved in Ahsanullah and Wesołowski (1998) that the linearity of regression $E(R_{m+2}|R_m)=aR_m+b$ holds only for the family of distributions (1)–(3) identified in Nagaraja (1977). Both the results were based on solutions of some differential equations for the densities – which approach will not be followed here. Lōpez-Blāzquez and Moreno-Rebollo (1997) considered the problem for non-adjacent order statistics and record values under some stringent smoothness assumptions on the df F. Unfortunately their solution of differential equation, being a crucial point of the proof, raises some reservations. In Dembińska and Wesołowski (1998) the problem of linearity of regression for any, possibly non-adjacent, order statistics has been completely resolved under the mild and natural assumption of continuity of F. The present paper develops some of the ideas of that paper in the case of linearity of regression for, not necessarily adjacent, record values. Both the cases: (1) $E(R_{m+k}|R_m)=aR_m+b$ and (2) $E(R_m|R_{m+k})=cR_{m+k}+d$ are considered. Direct results, given in Section 2, are rather easy, and need only simple computations, which are skipped. The families of distributions include all the distributions contained in (1)–(3) and (4)–(6), respectively. Section 3 covers the converse case (1), by applying an integrated Cauchy functional equation methodology, while the dual converse case (2), is solved in Section 4 by a careful explanation of the idea taken from Nagaraja (1988) combined with a result for the order statistics dual to that from Dembińska and Wesołowski (1998).

Let us point out that a related question based on independence of $R_{m+k}-R_m$ and R_m was considered in Dallas (1981) and Nayak (1981) (see, also Rao and Shanbhag (1986) for a correct version of the proof). The recent monograph by Arnold et al. (1998), especially Chapters 4.2.2, 4.3.2 and 4.5, can be consulted for a review of the state of art in the area of characterization of distributions by properties of records up to mid-1990s.

2. Linearity of regression

Let us start with a detailed definitions of six basic distributions making appearance in the context of linearity of regressions for record values. In the definitions the shift and location parameters are included to observe their relation with the slope and intercept coefficients of the regression lines.

Denote by $\mathscr{POW}(\theta;\mu,\nu)$ a power distribution defined by the probability density function (pdf)

$$f(x)=\frac{\theta(\nu-x)^{\theta-1}}{(\nu-\mu)^{\theta}}I_{(\mu,\nu)}(x),$$

where $\theta>0$, $-\infty<\mu<\nu<\infty$ are some constants. By $\mathscr{PAR}(\theta;\,\mu,\,\delta)$ denote the Pareto distribution with the pdf

$$f(x)=\frac{\theta(\mu+\delta)^{\theta}}{(x+\delta)^{\theta+1}}I_{(\mu,\infty)}(x),$$

where $\theta>0$, and μ, δ are some real constants such that $\mu+\delta>0$. By $\mathscr{EXP}(\lambda,\gamma)$ denote the exponential distribution with the pdf

$$f(x)=\lambda\exp(-\lambda(x-\gamma))I_{(\gamma,\infty)}(x),$$

where $\lambda>0$ and γ are some real constants. By $\mathscr{NG}(\beta,\gamma)$ denote the negative Gumbel distribution with the pdf

$$f(x)=\beta\exp(\beta(x-\gamma))\exp(-\mathrm{e}^{\beta(x-\gamma)}),$$

where $\beta>0$ and γ are some real constants. By $\mathscr{NF}(\theta;\gamma,\alpha)$ denote the negative Fréchet distribution with the pdf

$$f(x)=\frac{\theta(\alpha-\gamma)^{\theta}}{(\alpha-x)^{\theta+1}}\exp\left(-\left(\frac{\alpha-\gamma}{\alpha-x}\right)^{\theta}\right)I_{(-\infty,\alpha)}(x),$$

where $\theta>0$, and α, γ are some real constants such that $\alpha-\gamma>0$. Finally, by $\mathscr{W}(\theta;\mu,\gamma)$ denote the Weibull distribution with the pdf

$$f(x)=\frac{\theta(x-\mu)^{\theta-1}}{(\gamma-\mu)^{\theta}}\exp\left(-\left(\frac{x-\mu}{\gamma-\mu}\right)^{\theta}\right)I_{(\mu,\infty)}(x),$$

where $\theta>0$, and μ, γ are some real constants such that $\gamma>\mu$.

The conditional pdf of R_{m+k} given R_m is (see, for instance, Ahsanullah, 1995) as given below:

$$f_{R_{m+k}|R_m=x}(y)=\begin{cases}\frac{1}{\Gamma(k)}[R(y)-R(x)]^{k-1}\frac{f(y)}{1-F(x)} & \text{for } -\infty<x<y<+\infty,\\ 0 & \text{otherwise,}\end{cases}\tag{1}$$

where $R(x) = -\log(1 - F(x))$.

Consequently,

$$E(R_{m+k}|R_m = x) = \int_x^{\infty} y \frac{1}{\Gamma(k)} [R(y) - R(x)]^{k-1} \frac{f(y)}{1 - F(x)} \, \mathrm{d}y \tag{2}$$

for [L] a.a. $x \in (l_F, r_F)$ (where $l_F = \inf\{x\colon F(x) > 0\}$ and $r_F = \sup\{x\colon F(x) < 1\}$ and [L] denotes the Lebesgue measure).

It can be easily verified, by making use of (2), that for the exponential, power and Pareto distributions

$$E(R_{m+k}|R_m) = aR_m + b, \tag{3}$$

where the constants a and b have the following forms:

1. For $\mathscr{EXP}(\lambda, \gamma)$

$$a = 1, \quad b = \frac{k}{\lambda}.$$

2. For $\mathscr{PAR}(\theta; \mu, \delta)$, $\theta > 1$ and

$$a = \left(\frac{\theta}{\theta - 1}\right)^k > 1, \quad b = \delta \frac{\theta^k - (\theta - 1)^k}{(\theta - 1)^k}.$$

3. For $\mathscr{POW}(\theta; \mu, v)$

$$a = \left(\frac{\theta}{\theta + 1}\right)^k < 1, \quad b = v \frac{(\theta + 1)^k - \theta^k}{(\theta + 1)^k}.$$

Using the formulas for the joint density function of R_m and R_{m+k}, see for instance again Ahsanullah (1995), it follows that

$$f_{R_m|R_{m+k}=y}(x) = \begin{cases} \frac{(m+k)!}{m!(k-1)!} \frac{1}{R^{m+k}(y)} R^m(x)[R(y) - R(x)]^{k-1} \frac{f(x)}{1-F(x)} & -\infty < x < y < +\infty, \\ 0 & \text{otherwise.} \end{cases}$$

Consequently

$$E(R_m|R_{m+k} = y) = \frac{(m+k)!}{m!(k-1)!} \frac{1}{R^{m+k}(y)} \int_{-\infty}^{y} x R^m(x)[R(y) - R(x)]^{k-1} \times \frac{f(x)}{1 - F(x)} \, \mathrm{d}x \tag{4}$$

for [L] a.a. $x \in (l_F, r_F)$.

It can be easily verified, using (4) above, that for the negative Gumbel, negative Fréchet and Weibull distributions

$$E(R_m|R_{m+k}) = cR_{m+k} + d,$$

where the constants c and d have the following forms:

1. For $\mathscr{NG}(\beta, \gamma)$
$$c = 1, \quad d = \frac{(m+k)!}{\beta m!} \sum_{i=0}^{k-1} \frac{(-1)^{i+1}}{i!(k-1-i)!(m+i+1)^2} < 0.$$
2. For $\mathscr{NF}(\theta; \gamma, \alpha)$ if $\theta > 1/(m+1)$
$$c = \frac{\Gamma(m+k+1)\Gamma(m-\frac{1}{\theta}+1)}{\Gamma(m+1)\Gamma(m-\frac{1}{\theta}+k+1)} > 1, \quad d = \alpha(1-c).$$
3. For $\mathscr{W}(\theta; \mu, \gamma)$
$$c = \frac{\Gamma(m+k+1)\Gamma(m+\frac{1}{\theta}+1)}{\Gamma(m+1)\Gamma(m+\frac{1}{\theta}+k+1)} < 1, \quad d = \mu(1-c).$$

3. Characterizations by linearity of regression of R_{m+k} on R_m

The following result shows that the examples of distributions (exponential, Pareto and power) given in the preceding section are the only for which linearity of regression (3) holds.

Theorem 1. *Let $X_1, X_2, \ldots$ be iid rvs with a continuous df such that $E(|R_{m+k}|) < \infty$, where m and k are fixed positive integers. If $E(R_{m+k}|R_m) = aR_m + b$, for some real numbers a and b, then only the following three cases are possible*:

1. *$a = 1$ and $X_1 \sim \mathscr{EXP}(\lambda, \gamma)$ where $\lambda = k/b > 0$ and γ is an arbitrary real number*,
2. *$a > 1$ and $X_1 \sim \mathscr{PAR}(\theta; \mu, \delta)$ where $\theta = \sqrt[k]{a}/(\sqrt[k]{a}-1) > 1$, $\delta = b/(a-1)$ and μ is a real number such that $\mu + \delta > 0$*,
3. *$0 < a < 1$ and $X_1 \sim \mathscr{POW}(\theta; \mu, \nu)$ where $\theta = \sqrt[k]{a}/(1-\sqrt[k]{a})$, $\nu = b/(1-a)$ and μ is a real number such that $\mu < \nu$*,

Remark. The condition $E(|R_{m+k}|) < \infty$ holds if $E(X_1)$ and $E(X_1^+(\ln X_1^+)^{m+k-1})$ are finite (see Nagaraja, 1978).

Before, we give the proof of the above result let us recall, following Rao and Shanbhag (1986), an important result concerning possible solutions of an extended version of the integrated Cauchy functional equation (for a wide review on applications of integrated Cauchy functional equations for characterization problems consult the monograph of Rao and Shanbhag, 1994).

Theorem 2. *Consider the integral equation*
$$\int_{\mathbb{R}_+} H(x+y)\mu(\mathrm{d}y) = H(x) + c \quad \textit{for } [L] \textit{ a.a. } x \in \mathbb{R}_+,$$
where c is a real constant, μ is a non-arithmetic σ-finite measure on $\mathbb{R}_+$ such that $\mu(\{0\}) < 1$ and $H: \mathbb{R}_+ \mapsto \mathbb{R}_+$ is a Borel measurable, either non-decreasing or

non-increasing function that is not identically equal to a constant $[L]$ *a.e. Then* $\exists \eta \in \mathbb{R}$ *such that*

$$\int_{\mathbb{R}_+} \exp(\eta x)\mu(\mathrm{d}x) = 1$$

and H *has the form*

$$H(x) = \begin{cases} \gamma + \alpha(1 - \exp(\eta x)) & \textit{for } [L] \textit{ a.a. } x \quad \textit{if } \eta \neq 0 \\ \gamma + \beta x & \textit{for } [L] \textit{ a.a. } x \quad \textit{if } \eta = 0 \end{cases}$$

where α, β, γ *are some constants. If* $c = 0$ *then* $\gamma = -\alpha$ *and* $\beta = 0$.

Now we are ready to prove our main result.

Proof of Theorem 1. Using a version of (1) for the conditional df one can write

$$E(R_{m+k}|R_m = x) = \int_x^\infty y \frac{1}{\Gamma(k)\bar{F}(x)}[R(y) - R(x)]^{k-1} d[-\bar{F}(y)]$$

F a.e., where $\bar{F} = 1 - F$. Consequently, the linearity of regression assumption, (3) implies that

$$\frac{1}{\Gamma(k)} \int_x^\infty y \left[\log\left(\frac{\bar{F}(x)}{\bar{F}(y)}\right)\right]^{k-1} d\left[-\frac{\bar{F}(y)}{\bar{F}(x)}\right] = ax + b \tag{5}$$

for F-almost all x's. Following the argument applied in Ferguson (1967) it follows, that (l_F, r_F) is the support of the distribution defined by F and F is strictly increasing in this interval. Notice also that since both the sides of (5) are continuous with respect to x we can assume that the equation holds for any $x \in (l_F, r_F)$.

Substituting $t = \bar{F}(y)/\bar{F}(x)$, i.e. $y = \bar{F}^{-1}(t\bar{F}(x))$ (observe that $\bar{F}^{-1}$ exists because $\bar{F}$ is strictly decreasing in (l_F, r_F)) in (5) one gets

$$\frac{1}{\Gamma(k)} \int_0^1 \bar{F}^{-1}(t\bar{F}(x))(-\log(t))^{k-1}\,\mathrm{d}t = ax + b.$$

Observe that the left-hand side of the above equation is an increasing function of x. Consequently $a > 0$. Now substitute $\bar{F}(x) = w$, hence $x = \bar{F}^{-1}(w)$, and thus

$$\frac{1}{\Gamma(k)} \int_0^1 \bar{F}^{-1}(tw)(-\log(t))^{k-1}\,\mathrm{d}t = a\bar{F}^{-1}(w) + b, \quad w \in (0, 1).$$

Divide both sides by a and substitute once again $t = \mathrm{e}^{-u}$ and $w = \mathrm{e}^{-v}$. Then

$$\frac{1}{a\Gamma(k)} \int_0^\infty \bar{F}^{-1}(\mathrm{e}^{-(u+v)})u^{k-1}\mathrm{e}^{-u}\,\mathrm{d}u = \bar{F}^{-1}(\mathrm{e}^{-v}) + \frac{b}{a}, \quad v \in (0, +\infty).$$

Now let $G(v) = \bar{F}^{-1}(\mathrm{e}^{-v})$. Consequently,

$$\int_{\mathbb{R}_+} G(u + v)\mu(\mathrm{d}u) = G(v) + \frac{b}{a}, \quad v \in (0, +\infty),$$

where μ is a finite measure on $\mathbb{R}_+$, which is absolutely continuous with respect to the $[L]$ measure and is defined by

$$\mu(\mathrm{d}u) = \frac{1}{a\Gamma(k)} u^{k-1} \mathrm{e}^{-u}\, \mathrm{d}u.$$

Observe that G is strictly increasing on $[0, \infty)$ since it is a superposition of two strictly decreasing functions. Consequently, the assumptions of Theorem 2 are fulfilled. Hence, since G is continuous it follows that

$$G(v) = \begin{cases} \gamma + \alpha(1 - \exp(\eta v)) & \text{if } \eta \neq 0, \\ \gamma + \beta v & \text{if } \eta = 0, \end{cases} \tag{6}$$

$v > 0$, where α, β, γ, η are some constants and

$$\int_{\mathbb{R}_+} \exp(\eta x)\mu(\mathrm{d}x) = 1. \tag{7}$$

From (7) we get

$$1 = \frac{1}{a\Gamma(k)} \int_0^\infty x^{k-1} \mathrm{e}^{(\eta-1)x}\, \mathrm{d}x,$$

which means that $\eta < 1$, since the integral at the right-hand side has to converge. Consequently

$$\frac{1}{a} = (1 - \eta)^k. \tag{8}$$

Obviously, there is a unique $\eta < 1$ that fulfils (8): $\eta = 1 - \sqrt[k]{1/a}$.

Observe that

1. $a = 1$ iff $\eta = 0$,
2. $a > 1$ iff $0 < \eta < 1$,
3. $a < 1$ iff $\eta < 0$.

Coming back to (6), for a non-zero η, we can write:

$$\bar{F}^{-1}(\mathrm{e}^{-x}) = \gamma + \alpha(1 - \mathrm{e}^{\eta x})$$

which implies $\bar{F}(z) = ((\alpha + \gamma - z)/\alpha)^{-1/\eta}$, for z such that $\alpha(\alpha + \gamma - z) > 0$.

Consider now three possible cases:

1. If $a = 1$ and $\eta = 0$ then from (6) we get:

   $$\bar{F}^{-1}(\mathrm{e}^{-x}) = \gamma + \beta x.$$

   Hence $\beta > 0$ and then $\bar{F}(z) = \mathrm{e}^{-(z-\gamma)/\beta} = \mathrm{e}^{-\lambda(z-\gamma)}$ for $z > \gamma$, where $\lambda = 1/\beta > 0$. Hence $X_1 \sim \mathscr{EXP}(\lambda, \gamma)$, where

- $\lambda = k/b$ by the observation from Section 2,
- γ is a real number.

2. If $a > 1$ and $\eta > 0$ then

$$\bar{F}(z) = \left(\frac{-\alpha}{z-(\alpha+\gamma)}\right)^{1/\eta} = \left(\frac{\gamma-(\alpha+\gamma)}{z-(\alpha+\gamma)}\right)^{1/\eta} = \left(\frac{\mu+\delta}{z+\delta}\right)^{\theta}$$

for $z > \mu$, where $\delta = -(\alpha+\gamma), \mu = \gamma, \theta = \frac{1}{\eta} > 0$.
Thus $X_1 \sim \mathscr{PAR}(\theta; \mu, \delta)$, where

- $\theta = \frac{1}{1-\sqrt[k]{1/a}} = \frac{\sqrt[k]{a}}{\sqrt[k]{a}-1}$,
- $\delta = b/(a-1)$,
- μ is a real number such that $\mu + \delta > 0$.

3. If $0 < a < 1$ and $\eta < 0$ then

$$\bar{F}(z) = \left(\frac{\alpha+\gamma-z}{\alpha+\gamma-\gamma}\right)^{-1/\eta} = \left(\frac{v-z}{v-\mu}\right)^{\theta}$$

for $z \in (\mu, v)$, where $v = \alpha+\gamma,\ \mu = \gamma,\ \theta = -1/\eta > 0$.
Thus $X_1 \sim \mathscr{POW}(\theta; \mu, v)$, where

- $\theta = -\frac{1}{1-\sqrt[k]{1/a}} = \frac{\sqrt[k]{a}}{1-\sqrt[k]{a}}$,
- $v = b/(1-a)$,
- $\mu < v$ is a real number. $\square$

4. Characterizations by linearity of regression of R_m on R_{m+k}

In this section we will determine the distributions for which

$$E(R_m|R_{m+k}) = cR_{m+k} + d, \tag{9}$$

where m and k are some positive integers. From (9) and (4) one gets immediately

$$\frac{(m+k)!}{m!(k-1)!}\frac{1}{R^{m+k}(y)}\int_{-\infty}^{y} xR^m(x)[R(y)-R(x)]^{k-1}\,\mathrm{d}(R(x)) = cy + d, \tag{10}$$

F a.e. Now we are going to apply an idea, relating the conditional distribution of records to distribution of order statistics (see, for instance, Lemma 4.3.3 in Arnold et al. (1998)) having its origins in Nagaraja (1988). This is a clever device but the original argument seems to benefit from more clarifications: First it is observed there that the conditional joint distribution of $R_1, \ldots, R_n$ given R_{n+1} is identical with the joint distribution of the order statistics from a random sample of size n from the df

$$F_1(x|y) = \begin{cases} R(x)/R(y), & x < y, \\ 1, & x \geqslant y. \end{cases}$$

while one possibly should be rather interested in relations between the conditional distribution of records, say R_n given R_{n+1} and a conditional distribution of order statistics, say $T_{k:n}$ given $T_{k+1:n}$, from some observations on T with a df related possibly somehow to F. Then it is claimed that the reference to an analoguous result for order statistics (Ferguson, 1967, in that case) brings the solution. Beneath, we give the detailed argument, which confirms in general Nagaraja's intuitions.

Note that Eq. (10) is equivalent to the set of equations

$$\frac{(m+k)!}{m!(k-1)!}\frac{1}{Q_z^{m+k}(y)}\int_{l_F}^{y} xQ_z^m(x)[Q_z(y)-Q_z(x)]^{k-1}\,\mathrm{d}[Q_z(x)]=cy+d$$

for $y\in(l_F,r_F)$ and $\forall z\in(l_F,r_F)$, where

$$Q_z(y)=\begin{cases} R(y)/R(z) & \text{for } y\leqslant z,\\ 1 & \text{for } y>z. \end{cases}$$

is a continuous df. Thus, condition (9) is equivalent to the set of conditions:

$$E(Y^{(z)}_{m+1:n}|Y^{(z)}_{m+1+k:n})=cY^{(z)}_{m+1:n}+d \tag{11}$$

$\forall z\in(l_F,r_F)$, where $Y_1^{(z)},Y_2^{(z)},\ldots$ are iid rvs with the df Q_z and n is a natural number such that $n\geqslant k+m+1$.

To determine the distributions for which (11) holds we will use the following auxiliary result, which is a version of the main theorem from Dembińska and Wesołowski (1998).

Lemma 1. *Assume that $X_1,X_2,\ldots$ are iid rvs with a common continuous df F. Let $E(|X_{k:n}|)<\infty$. If for some $k\leqslant n-r$ and some real c and d*

$$E(X_{k:n}|X_{k+r:n})=cX_{k+r:n}+d, \tag{12}$$

then only the following three cases are possible:

1. $c=1$ *and* $F(x)=\mathrm{e}^{\lambda(x+\gamma)}$ *for* $x\leqslant-\gamma$, $F(x)=1$ *for* $x>-\gamma$, *where* γ *and* $\lambda>0$ *are some real numbers* (*the negative exponential distribution*).
2. $c>1$ *and* $F(x)=((\mu+v)/(-x+v))^{\theta}$ *for* $x<-\mu$, $F(x)=1$ *for* $x\geqslant-\mu$, *where* $\theta>0$, v, μ *are some real numbers*, $\mu+v>0$ (*the negative Pareto distribution*).
3. $0<c<1$ *and* $F(x)=0$ *for* $x\leqslant-\sigma$, $F(x)=((\sigma+x)/(\sigma-\mu))^{\theta}$ *for* $x\in(-\sigma,-\mu)$ *and* $F(x)=1$ *for* $x\geqslant-\mu$, *where* $\theta>1/k$, σ, μ *are some real numbers such that* $\mu<\sigma$ (*the negative power distribution*).

Proof. Putting $Y_k=-X_k$ condition (12) turns into

$$E(Y_{n-k+1:n}|Y_{n-k-r+1:n})=cY_{n-k-r+1:n}-d \tag{13}$$

(because $X_{k:n}=-Y_{n-k+1:n}$). Now the result follows immediately from Theorem 1 of Dembińska and Wesołowski (1998). □

Thus, if (11) holds then only the following three cases are possible:

1. $c=1$ and $Q_z(y)=\mathrm{e}^{\lambda(y-z)}$ for $y\leqslant z$, $\forall z\in(l_F,r_F)$. Then $\forall z\in(l_F,r_F)$ and $\forall y\leqslant z$

$$\mathrm{e}^{-\lambda z}\log(1-F(z))=\mathrm{e}^{-\lambda y}\log(1-F(y)).$$

Hence $\forall z \in (l_F, r_F)$

$$e^{-\lambda z}\log(1 - F(z)) = \text{const}.$$

Consequently $F(y) = 1 - e^{-e^{\lambda(y-\gamma)}} \forall y \in \mathbb{R}$, where $\lambda > 0$ and $\gamma \in \mathbb{R}$. Thus, F is a negative Gumbel-type df.

2. $c > 1$ and $Q_z(y) = ((-z+v)/(-y+v))^\theta$ for $y < z$, $\forall z \in (l_F, r_F)$. Then, as in point (1) it follows that $F(y) = 1 - e^{-((\alpha-\gamma)/(\alpha-y))^\theta}$, for $y \in (-\infty, \alpha)$, where $\theta > 1/(m+1)$ and $\alpha > \gamma$; consequently F is a negative Fréchet-type df.
3. $0 < c < 1$ and $Q_z(y) = ((\sigma + y)/(\sigma + z))^\theta$ for $y \in (-\sigma, z)$, $\forall z \in (l_F, r_F)$. Then, similarly as above it follows that $F(y) = 1 - e^{-((y-\mu)/(\sigma-\mu))^\theta}$, for $y \in (\mu, \infty)$ where $\theta > 0$ and $\sigma > \mu$, i.e. F is a Weibull type df.

The above discusion proves the result dual to Theorem 1.

Theorem 3. *Let $X_1, X_2, \ldots$ be iid rvs with a continuous df such that $E(|R_m|) < \infty$, where m is a positive integer. If for some positive integer k*

$$E(R_m | R_{m+k}) = cR_{m+k} + d,$$

where c and d are real numbers, then only the following three cases are possible:

1. $c = 1$ *and* $X_1 \sim \mathcal{NG}(\beta, \gamma)$,
2. $0 < c < 1$ *and* $X_1 \sim \mathcal{NF}(\theta; \gamma, \alpha)$,
3. $c > 1$ *and* $X_1 \sim \mathcal{W}(\theta; \mu, \gamma)$.

The relations between c, d in the above theorem and the parameters of the respective distributions are as described in Section 2.

Finally, let us point out that essentially the proof of Theorem 3 makes also use of Theorem 2 (due to Rao and Shanbhag, 1986), since it was exploited for proving the main result in Dembińska and Wesołowski (1998).

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