Journal of Multivariate Analysis **77**, 163–174 (2001)
doi:10.1006/jmva.2000.1932, available online at http://www.idealibrary.com on IDEAL®

Identifiability of Modified Power Series Mixtures via Posterior Means

Arjun K. Gupta and Truc T. Nguyen

Bowling Green State University

Yinning Wang

Scheming Plough Research Institute

and

Jacek Wesołowski

Politechnika Warszawska, Warsaw, Poland

Received September 28, 1998; published online April 6, 2001

Problems of specification of discrete bivariate statistical models by a modified power series conditional distribution and a regression function are studied. An identifiability result for a wide class of such mixtures with infinite support is obtained. Also the finite support case within a more specific model is considered. Applications for Poisson, (truncated) geometric, and binomial mixtures are given. From the viewpoint of Bayesian analysis unique determination of the prior by a Bayes estimate of the mean for modified power series mixtures is investigated. © 2001 Academic Press

AMS 1991 subject classifications: 62H05; 62F15.
Key words and phrases: modified power series distribution; identifiability of mixtures; posterior mean; regression function; characterization of probability distributions; Poisson mixture; geometric mixture; binomial mixture.

1. INTRODUCTION

The modified power series distribution ($MPSD(a, g(\theta))$) defined by the probability mass function (pmf) of the form

$$p_k = a(k)[\, g(\theta)]^k / f(\theta), \qquad k \in \{0, 1, ...\},$$

0047-259X/01 $35.00

where a, f, and g are nonnegative functions, $\theta \geqslant 0$, was introduced by Gupta (1974). As a generalization of the power series family it includes a wide variety of discrete probability distributions; consequently it is not only of theoretical but also of applied interest. A recent development in this area was given by Gupta *et al.* (1995).

In this paper we study the identifiability of MPSD mixtures via posterior means. Problems of this kind were considered for the first time by Johnson (1957), where a Poisson mixture and a linear regression model were treated. The development of the area includes, on one side, studying other, possibly more general families of mixtures and, on the other side, considering any consistent posterior mean without specifying its exact form. More recently such questions for discrete models were investigated by, among others, Cacoullos and Papageorgiou (1983), Xekalaki (1983), Papageorgiou (1985), Kyriakoussis (1988), Kyriakoussis and Papageorgiou (1991), Johnson and Kotz (1992), Arnold *et al.* (1993), Sapatinas (1995), Wesołowski (1995, 1996), Papageorgiou and Wesołowski (1997), and Gupta and Wesołowski (1999).

These problems fall in the general area of identifiability and identification of statistical models. The basic reference here is the monograph by Prakasa Rao (1992). Applications of the approach we adopted here in the Bayesian context for studying properties of Bayes estimators of the uniform scale parameter, i.e., in a continuous model, were given recently in Lillo and Martín (1999, 2000)—see also references in these papers for earlier related results.

For a random vector (X, Y) the mixture is defined by the conditional distribution $\mu_{Y|X}$. A starting point of our interest was the following result from Wesołowski (1996): Denote by $\mathscr{P}(\lambda)$ the Poisson distribution with the pmf

$$p(k) = e^{-\lambda} \frac{\lambda^k}{k!}, \qquad k \in \mathbf{N} = \{0, 1, \ldots\},$$

$\lambda > 0$. If $\mu_{Y|X} \stackrel{d}{=} \mathscr{P}(bc^X)$, $b > 0$, $0 < c < 1$, and $E(X \mid Y) = ac^Y$, $a > 0$, then (X, Y) has a bivariate Poisson–Poisson conditionals distribution (i.e., both the conditional distributions are of the Poisson type—see Arnold and Strauss (1991)). Observe that this is a special case of MPSD mixture with $a(k) = b^k/k!$, $g(X) = c^X$, and $f(X) = \exp(bc^X)$. The aim of this paper is to extend the above result by considering the question of unique determination (identifiability) for a wide class of MPSD mixtures by a consistent posterior mean. A special case of such a problem with $g(\theta) = 1/\theta$ was studied in Wesołowski (1995). On the other hand it also complements the results for the power series distribution (PSD) mixtures obtained recently in Sapatinas (1995) and Wesołowski (1995). (Recall that the PSD is included in the MPSD by taking $g(\theta)=\theta$).

In Section 2 the infinite support case is studied and it is shown that the MPSD mixture is identifiable if only the parameter function g is decreasing. Possible forms of the regression functions in the case of a power function g are given. In the finite support case, Section 3, only MPSD mixtures with a power function g are considered. In both the cases the theoretical results are illustrated with examples involving special cases of MPSD mixtures (Poisson, geometric, truncated geometric, binomial) and some exact forms of posterior means. Translating it to the Bayesian language: problems of unique determination of the prior for MPSD mixtures by Bayes estimates of the mean are studied.

2. MPSD MIXTURES WITH INFINITE SUPPORT

Assume that (X, Y) is a discrete random vector such that $S_X = supp(X) \subset \mathbf{N}$ and $S_Y = supp(Y) \subset \mathbf{N}$. If $p(n \mid k)$, $n \in S_Y$, $k \in S_X$, is the mixture kernel for $\mu_{Y \mid X}$ and $m(n) = E(X \mid Y = n)$, $n \in S_Y$, then the identifiability problem, by the Bayes theorem, can be reduced to studying uniqueness of the solution of the following equation

$$\sum_{k \in S_X} [m(n) - k]\, p(n \mid k)\, p_X(k) = 0, \qquad n \in S_Y, \tag{1}$$

where p_X is the unknown pmf of X. Such a general approach was discussed in Arnold *et al.* (1993) revealing essential difficulties, particularly in the case of infinite supports. Here we give a solution for a wide family of MPSD mixtures. It should be emphasized that we follow neither the traditional method of using classical identifiability (see Teicher (1961)) as it is done in Cacoullos and Papageorgiou (1983) and Sapatinas (1995), nor the approach via transformation of X to another random variable (rv) with distribution determined by moments (see Wesołowski (1995)). Instead we develop a new method based on the limiting behaviour of the function m.

THEOREM 1. *Let (X, Y) be a random vector with $S_X = S_Y = \mathbf{N}$ and $E(X) < \infty$. Assume that $\mu_{Y \mid X} = MPSD(a, g(X))$, where g is a decreasing function. Then the joint distribution of (X, Y) is uniquely determined by the regression function $E(X \mid Y)$.*

Proof. The general equation (1) in the case of an MPSD mixture takes the form

$$\sum_{k=0}^{\infty} [m(n) - k]\, u^n(k)\, v(k) = 0, \qquad n \in \mathbf{N}, \tag{2}$$

where $u(k)=g(k)/g(0)<1$, $v(k)=f(0)\,p_X(k)/[f(k)\,p_X(0)]$, $k\in\mathbf{N}$. Obviously, $u(0)=v(0)=1$ and u is decreasing. Rewrite (2) as

$$m(n)+m(n)\sum_{k=1}^{\infty}u^n(k)\,v(k)-\sum_{k=1}^{\infty}ku^n(k)\,v(k)=0,$$

$n\in\mathbf{N}$. Observe that, since $f(k)\geqslant a(0)$, $\forall k\geqslant 0$, then for sufficiently large n's the second term is majorized by $m(n)\,u^n(1)$ and the absolute value of the third by $u^n(1)\,E(X)$. Since $\lim_{n\to\infty}u^n(1)=0$ we obtain that $\lim_{n\to\infty}m(n)=0$.

Now dividing (2) by u^n (1) one gets the identity

$$\frac{m(n)}{u^n(1)}+[m(n)-1]\,v(1)+\sum_{k=2}^{\infty}[m(n)-k]\left(\frac{u(k)}{u(1)}\right)^n v(k)=0, \qquad n\in\mathbf{N}.$$

Again taking $n\to\infty$, similarly as in the above argument, we conclude that the limit exists and

$$v(1)=\lim_{n\to\infty}\frac{m(n)}{u^n(1)}.$$

Similarly, after obtaining $v(i)$, $i=1,\ldots,k-1$, one can divide (2) by $u^n(k)$ to obtain the general recurrence formula

$$v(k)=\frac{1}{k}\lim_{n\to\infty}\sum_{i=0}^{k-1}[m(n)-i]\left(\frac{u(i)}{u(k)}\right)^n v(i), \qquad k=2,3,\ldots.$$

Consequently the sequence $(v(k))_{k=0,1,\ldots}$ is uniquely determined by the function m. By the definition of v it follows that p_X is identifiable up to a multiplicative factor $p_X(0)$, which can be determined by the normalizing condition $\sum_{i=0}^{\infty}p_X(i)=1$. Finally the joint distribution of (X, Y) is characterized by $\mu_{Y\mid X}$ and p_X. ∎

Theorem 1 immediately gives identifiability of the joint distribution with Poisson or geometric conditional distributions considered in Arnold and Sarabia (1991).

For the Poisson mixture the following extension of the main result of Wesołowski (1996) (see also Section 1) holds:

COROLLARY 1. *Let (X, Y) be a random vector such that $E(X)<\infty$ and $\mu_{Y\mid X}=\mathscr{P}(bc^X)$, $b>0$, $0<c<1$. Then the joint distribution of (X, Y) is uniquely determined by $E(X\mid Y)$.*

Denote by $ge(p)$ the geometric distribution defined by the pmf

$$p(k)=(1-p)\,p^k, \qquad k\in\mathbf{N}.$$

For the geometric mixture, via Theorem 1, we have

COROLLARY 2. *Let (X, Y) be a random vector such that $E(X) < \infty$ and $\mu_{Y|X} = ge(bc^X)$, $b, c \in (0, 1)$. Then the joint distribution of (X, Y) is uniquely determined by $E(X \mid Y)$.*

Both the above corollaries complement recent results on identifiability of the Poisson and geometric mixtures via the form of $E(c^X \mid Y)$ or $E(c^{-X} \mid Y)$ obtained in Wesołowski (1995).

It follows from the proof of Theorem 1 that we have the following recurrence relations for p_X:

$$p_X(k) = \frac{f(k)}{k} \lim_{n \to \infty} \sum_{i=0}^{k-1} [m(n) - i] \left(\frac{g(i)}{g(k)}\right)^n \frac{p_X(i)}{f(i)}, \qquad k = 1, 2, ..., \tag{3}$$

which for any given function m allows us to get an exact expression for p_X. So our result not only lies in identifiability but also gives a method how to find the joint distribution. Of course in many cases computations involved in (3) might be difficult and laborious. However some special and interesting cases can be quite easily settled; e.g., see below.

Formula (3) says that the distribution is uniquely determined by limiting properties of the function m (as the argument tends to infinity). On the other hand p_X and $\mu_{Y|X}$ uniquely determine the function m. It means that m is completely defined by its limiting properties (contained in (3)). Now we are going to follow this direction more thoroughly.

Define

$$c_{i,j}(n) = [m(n) - j] \left(\frac{g(j)}{g(i)}\right)^n$$

for $j = 0, ..., i-1, i$, $n = 1, 2, ...$. Then (3) implies

$$k \frac{p_X(k)}{f(k)} = \sum_{i=0}^{k-1} b_{k-i} \frac{p_X(i)}{f(i)}, \qquad k = 1, 2, ..., \tag{4}$$

where b_j, $j = 1, 2, ...$, are defined by

$$b_j = \lim_{n \to \infty} \left(c_{j,0}(n) + \sum_{i=1}^{j-1} (c_{j,i}(n) - b_{j-i}) B_i \right),$$

$$j = 2, 3, ..., \quad b_1 = \lim_{n \to \infty} c_{1,0}(n),$$

$$B_j = \frac{1}{j} \sum_{i=0}^{j-1} b_{j-i} B_i, \qquad j = 1, 2, ..., \quad B_0 = 1.$$

The definition of b's is correct since all the limits exist.

Now applying (4) to (2) we arrive at the following representation of the regression function

$$\begin{aligned} m(n) &= \frac{\sum_{k=1}^{\infty} k(g(k))^n \, p_X(k)/f(k)}{\sum_{k=0}^{\infty} (g(k))^n \, p_X(k)/f(k)} \\ &= \frac{\sum_{k=1}^{\infty} (g(k))^n \sum_{i=0}^{k-1} b_{k-i} p_X(i)/f(i)}{\sum_{k=0}^{\infty} (g(k))^n \, p_X(k)/f(k)} \\ &= \frac{\sum_{k=0}^{\infty} [(g(k))^n \, p_X(k)/f(k)] \sum_{i=1}^{\infty} b_i (g(i+k)/g(k))^n}{\sum_{k=0}^{\infty} (g(k))^n \, p_X(k)/f(k)}, \end{aligned} \tag{5}$$

where the last equation follows by changing the order of summation and then changing the variable in the numerator.

From here until the end of the section we consider $g(x) = \theta^x$, $\theta \in (0, 1)$.

PROPOSITION 1. *If* $\mu_{Y \mid X} = MPSD(a, \theta^X)$, $\theta \in (0, 1)$, *then*

$$E(X \mid Y) = \sum_{i=1}^{\infty} b_i \theta^{iY}, \tag{6}$$

where b_i, $i = 1, 2, ...,$ *satisfy the identities* (4).

Proof. It suffices to observe that $g(i+k)/g(k) = \theta^i$ for all k's in (5). ▮

Consequently any special form of $E(X \mid Y)$ given by (6) allows us to calculate exact values for p_X. In some special cases the calculations can be easily performed. For example:

1. If $E(X \mid Y) = b_1 \theta^Y$, i.e., $b_2 = b_3 = \cdots = 0$, then (4) implies

$$p_X(k) = \frac{b_1^k}{k!} \frac{f(k)}{f(0)} \, p_X(0), \qquad k = 1, 2, ... \tag{7}$$

2. If $E(X \mid Y) = q\theta^Y/(1 - q\theta^Y)$, $q \in (0, 1)$, i.e.,

$$E(X \mid Y) = \sum_{i=1}^{\infty} (q\theta^Y)^i,$$

then $b_i = q^i$, $i = 1, 2, ...,$ and (4) implies

$$p_X(k) = q^k \frac{f(k)}{f(0)} \, p_X(0), \qquad k = 1, 2, ... \tag{8}$$

In the next two remarks we apply the above considerations for characterizing some Poisson and geometric mixtures (L and K denote suitable normalizing constants):

Remark 1 (Poisson mixtures). Consider a Poisson mixture $\mu_{Y|X} = \mathscr{P}(bc^X)$, $b>0$, $0<c<1$, which is an MPSD mixture with $f(k) = \exp(bx^k)$, $k=0, 1, ...$ We study two cases:

Case 1. Assume that $E(X \mid Y) = ac^Y$, $a>0$. Then (7) gives

$$p_X(k) = L\frac{a^k \exp(bc^k)}{k!}, \qquad k=0, 1, ...$$

Consequently

$$P(X=k,\ Y=n) = K\frac{a^k b^n c^{kn}}{k!\, n!}, \qquad k, n=0, 1, ...,$$

which is the bivariate Poisson–Poisson conditionals distribution derived as the only distribution for which both the conditional distributions are Poisson in Obrechkoff (1938) (see also Arnold *et al.* (1992)). Thus we have just reproved Wesołowski's (1996) result. Similar characterizations of this distribution involving the same mixture and either of $E(c^{\pm X} \mid Y) = \exp[a(c^{\pm 1}-1)\, c^Y]$ have been given recently in Wesołowski (1995).

Case 2. Assume now that $E(X \mid Y) = ac^Y/(1-ac^Y)$, $0<a<1$. Consequently (8) implies that

$$p_X(k) = La^k e^{bc^k}, \qquad k=0, 1, ...$$

Hence

$$P(X=k,\ Y=n) = K\frac{a^k b^n c^{kn}}{n!}, \qquad k, n=0, 1, ...$$

which is the bivariate geometric–Poisson conditionals distribution. Again it follows from Arnold and Strauss (1991) that it is the only bivariate distribution for which $\mu_{Y|X} = \mathscr{P}(s(X))$ and $\mu_{X|Y} = ge(t(Y))$, where $ge(p)$, $0<p<1$, is the geometric distribution defined earlier. Observe that in our case $t(Y) = ac^Y$ and $E(Y \mid X) = bc^X$.

Remark 2 (Geometric mixtures). Consider a geometric mixture $\mu_{Y|X} = ge(bc^X)$, b, $c \in (0, 1)$, which is an MPSD mixture with $f(k) = 1/(1-bc^k)$, $k=0, 1, ...$ Again two cases will be studied:

Case 1. Assume that $E(X \mid Y) = ac^Y$, $a > 0$. Then (7) gives

$$p_X(k) = L \frac{a^k}{(1 - bc^k)\, k!}, \qquad k = 0, 1, \ldots$$

and consequently (X, Y) has the Poisson–geometric conditionals distribution defined in Case 2 of Remark 1. Obviously $\mu_{X \mid Y} = \mathscr{P}(ac^Y)$ and $E(Y \mid X) = bc^X/(1 - bc^X)$.

Case 2. Assume that $E(X \mid Y) = ac^Y/(1 - ac^Y)$, $0 < a < 1$. Then (8) implies that

$$p_X(k) = L \frac{a^k}{1 - bc^k}, \qquad k = 0, 1, \ldots.$$

Consequently

$$P(X = k, Y = n) = Ka^k b^n c^{kn}, \qquad k, n = 0, 1, \ldots$$

which is the bivariate geometric–geometric conditionals distribution, as introduced in Arnold and Strauss (1991) (see also Arnold *et al.* (1992)). Recall that it is the only distribution with both conditional distributions of the geometric form. Other characterizations of this distribution involving the same mixture and either of $E(c^{\pm} \mid Y) = (1 - ac^Y)/(1 - ac^{Y \pm 1})$ were given recently in Wesołowski (1995).

3. MPSD MIXTURES WITH FINITE SUPPORT

In this section we treat the case $S_X = S_Y = \{0, 1, \ldots, M\}$, where M is a given positive integer. Then (1) defines a system of linear equations

$$\sum_{k=1}^{M} [m(n) - k]\, p(n \mid k)\, w(k) = -m(0)\, p(n, 0), \qquad n = 0, 1, \ldots, M,$$

with unknown $w(k) = p_X(k)/p_X(0)$, $k = 1, 2, \ldots, M$. Consequently the mixture is identifiable if the coefficient matrix $[[m(n) - k]\, p(n \mid k)]_{k, n = 1, \ldots, M}$ is nonsingular (for more general comments see Arnold *et al.* (1993)). Hence the solution of the identifiability problem depends on some additional, not only the consistency, properties of the function m. It also seems to be the case in the general finite MPSD mixture. However, if g is a power function it appears that the finite MPSD mixture is identifiable without any further restrictions on m.

THEOREM 2. *Let (X, Y) be a random vector with $S_X = S_Y = \{0, 1, ..., M\}$. Assume that $\mu_{Y|X} = MPSD(a, g(X))$, where g is a power function. Then the joint distribution of (X, Y) is uniquely determined by any consistent regression function $E(X \mid Y)$.*

Proof. Assume that the mixture is not identifiable, i.e., there are two different pmfs p_1 and p_2 on S_X such that

$$m(n) = \frac{\sum_{k=1}^{M} kt^{kn} p_1(k)/f(k)}{\sum_{k=0}^{M} t^{kn} p_1(k)/f(k)} = \frac{\sum_{j=1}^{M} jt^{jn} p_2(j)/f(j)}{\sum_{j=0}^{M} t^{jn} p_2(j)/f(j)}$$

for any $n = 0, 1, ..., M$, where t is a given positive parameter. Consequently

$$\sum_{k=1}^{M} \sum_{j=0}^{M} kt^{n(k+j)} \frac{p_1(k)\, p_2(j)}{f(k)\, f(j)} = \sum_{k=0}^{M} \sum_{j=1}^{M} jt^{n(k+j)} \frac{p_1(k)\, p_2(j)}{f(k)\, f(j)},$$

which can be rewritten in the form

$$\sum_{i=1}^{M} t^{in} z_i = 0, \qquad n = 0, 1, ..., M, \tag{9}$$

where

$$z_i = \sum_{j=1}^{i} \frac{p_1(j)\, p_2(i-j) - p_1(i-j)\, p_2(j)}{f(i-j)\, f(j)}, \qquad i = 1, 2, ..., M. \tag{10}$$

Observe that (9) is a system of linear equations with unknown z_i, $i = 1, 2, ..., M$, and nonsingular coefficient matrix $[t^{kn}]_{k, n = 1, ..., M}$. Consequently $z_i = 0$, $i = 1, ..., M$.

The next step lies in proving that there exist numbers A, B, and k, $i = 1, ..., M$, such that $p_1(i) = k_i A$ and $p_2(i) = k_i B$, $i = 1, ..., M$. Here we use induction with respect to i. For $i = 1$ it is easily seen that (10) and $z_1 = 0$ imply $p_1(1)/p_1(0) = p_2(1)/p_2(0) = k_1$. Now assume that the result holds for $i = 1, ..., m$ and consider $i = m + 1$. Since, by induction assumption,

$$p_1(j)\, p_2(i-j) = k_j k_{i-j} AB = p_1(i-j)\, p_2(j), \quad j = 1, ..., i, \quad i = 1, ..., m$$

then again $z_{m+1} = 0$ via (10) implies that

$$p_1(m+1)\, p_2(0) = p_1(0)\, p_2(m+1).$$

Consequently p_1 and p_2 differ only by a constant factor A/B, which must be 1, via the normalizing condition. Finally the joint distribution of (X, Y) is unique. ∎

Now some special cases will be considered.

Denote by $b(M, p)$ the binomial distribution defined by

$$p_k = \binom{M}{k} p^k (1-p)^{M-k}, \qquad k = 0, 1, ..., M,$$

where $p \in (0, 1)$ and M is a positive integer. Let

$$p(q, t, \theta) = \frac{qt^\theta}{qt^\theta + 1 - q} \tag{11}$$

for some $0 < q$, $t < 1$ and $\theta > 0$. Then $b(M, p(q, t, \theta))$ is an MPSD distribution with $a(k) = q^k(1-q)^{M-k}$, $g(\theta) = t^\theta$ and $f(\theta) = (qt^\theta + 1 - q)^{-M}$. Consequently Theorem 2 implies

Corollary 3. *Let (X, Y) be a random vector with $S_X = S_Y = \{0, 1, ..., M\}$ and $\mu_{Y|X} = b(M, p(q, t, X))$, where $p(q, t, X)$ is defined in (11). Then the joint distribution of (X, Y) is uniquely determined by $E(X \mid Y)$.*

Identifiability of this type of binomial mixture by the form of $E(t^{\pm X} \mid Y)$ has been proved recently in Wesołowski (1995). The above result can be used to give a new characterization of a kind of bivariate binomial–binomial conditionals distribution originally derived in Arnold and Strauss (1991) as the only distribution for which both conditional distributions are binomial.

Corollary 4. *If $\mu_{Y|X} = b(M, p(q_2, t, X))$ and $E(X \mid Y) = Mp(q_1, t, Y)$, where $0 < q_1, q_2, t < 1$, and M is a positive integer then (X, Y) has the bivariate binomial–binomial conditionals distribution defined by the pmf*

$$P(X = i, Y = j) = K \binom{M}{i} \binom{M}{j} q_1^i q_2^j (1-q_1)^{M-i} (1-q_2)^{M-j} t^{ij},$$

$i, j = 0, 1, ..., M.$

The above result complements a recent characterization of the general binomial–binomial conditionals distribution based on $\mu_{Y|X} = b(n_2, p(q_2, t, X))$ and any one of

$$E(t^{\pm X} \mid Y) = \left(\frac{1 - q_1 + q_1 t^{Y \pm 1}}{1 - q_1 + q_1 t^Y} \right)^{n_1},$$

where $0<q_1, q_2, t<1$, and n_1, n_2 are positive integers, given in Wesołowski (1995).

Similarly identifiability of a truncated geometric mixture can be established:

COROLLARY 5. *Assume that (X, Y) is a random vector such that*

$$P(Y=n \mid X=k)=\frac{(1-p^k)\,p^{kn}}{1-p^{k(M+1)}}, \qquad k, n=0, 1, ..., M,$$

where $0<p<1$ and M is a positive integer. Then $E(X \mid Y)$ uniquely determines the joint distribution of (X, Y).

Proof. It follows immediately from Theorem 2 since $\mu_{Y \mid X}$ is an MPSD mixture with $g(\theta)=p^\theta$, $a(k)=1$ and $f(k)=(1-p^k)/(1-p^{k(M+1)})$, $k=0, 1, ..., M$. ∎

ACKNOWLEDGMENTS

The authors are indebted to the referees whose suggestions helped to improve the exposition of the material.

REFERENCES

B. C. Arnold, E. Castillo, and J. M. Sarabia, Conditionally specified distributions, Lecture Notes in Statistics, Vol. 73, Springer-Verlag, Berlin, 1992.

B. C. Arnold, B. C. Castillo, and J. M. Sarabia, Conditionally specified models: structure and inference, *in* "Multivariate Analysis: Future Directions, 2" (C. M. Coudras and C. R. Rao, Eds.), pp. 441–450, Elsevier Science, Amsterdam, 1993.

B. C. Arnold and D. J. Strauss, Bivariate distributions with conditionals in prescribed exponential families, *J. Roy. Statist. Soc. B* **53** (1991), 365–375.

T. Cacoullos and H. Papageorgiou, Characterizations of discrete mixtures by a conditional distribution and a regression function, *Ann. Inst. Statist. Math.* **35** (1983), 95–103.

R. C. Gupta, Modified power series distribution and some of its applications, *Sankhyā Ser. B* **36** (1974), 288–298.

P. L. Gupta, R. C. Gupta, and R. C. Tripathi, Inflated modified power series distributions with applications, *Comm. Statist. Theory Methods* **24** (1995), 2355–2374.

A. K. Gupta and J. Wesołowski, Discrete uniform mixtures via posterior means, *Test* **8**, (1999), 399–409.

N. L. Johnson, Uniqueness of a result in the theory of accident proneness, *Biometrika* **44** (1957), 530–531.

N. L. Johnson and S. Kotz, Non-existence of certain polynomial regression in random sum theory, *Sankhyā* **54** (1992), 455–459.

A. Kyriakoussis, Characterization of bivariate discrete distributions, *Sankhyā*, *Ser. A* **50** (1988), 286–287.

A. Kyriakoussis and H. Papageorgiou, Characterizations of logarithmic series distribution, *Statist. Neerlandica* **45** (1991), 1–8.

R. E. Lillo and M. Martín, Characterization of Bayes estimators of the uniform scale parameter, *Statist. Decisions* **17** (1999), 31–47.

R. E. Lillo and M. Martín, Identifiability of differentiable Bayes estimators of the uniform scale parameter, *Ann. Inst. Statist. Math.*, in press (2000).

N. Obrechkoff, Sur la loi de Poisson, la série de Charlier et les équations linéaires aux différences finies du premier ordre à coefficients constants, *in* "La Statistique Mathematique" (E. Dodal, Ch. Jordan, and N. Obrechkoff, Eds.), Actualités Scientifiques et Industrielles, Vol. 740, pp. 35–64, Herman and Cie, Paris, 1938.

H. Papageorgiou and J. Wesołowski, Posterior mean identifies the prior distribution in NB and related models, *Statist. Probab. Lett.* **36** (1997), 127–134.

B. L. S. Prakasa Rao, "Identifiability in Stochastic Models; Characterization of Probability Distributions," Academic Press, Boston, 1992.

T. Sapatinas, Identifiability of mixtures of power series distributions and related characterizations, *Ann. Inst. Statist. Math.* **47** (1995), 447–459.

H. Teicher, Identifiability of mixtures, *Ann. of Math. Statist.* **32** (1961), 244–248.

J. Wesołowski, Bivariate discrete measures via a power series conditional distribution and a regression function, *J. Multivariate Anal.* **55** (1995), 219–229.

J. Wesołowski, A new conditional specification of the bivariate Poisson conditionals distribution, *Statist. Neerlandica* **50** (1996), 390–393.

E. Xekalaki, Infinite divisibility, completeness and regression properties of the uni-variate generalized Waring distribution, *Ann. Inst. Statist. Math.* **35** (1983), 279–289.