

## MUTUAL CHARACTERIZATIONS OF THE GAMMA AND THE GENERALIZED INVERSE GAUSSIAN LAWS BY CONSTANCY OF REGRESSION

By VANAMAMALAI SESHADRI  
*McGill University, Montreal, Canada*  
and

JACEK WESOŁOWSKI  
*Politechnika Warszawska, Warszawa, Poland*

*SUMMARY.* Let  $X$  and  $Y$  be non-negative independent random variables. Characterizations of the generalized inverse Gaussian and gamma distributions through the constancy of regression of  $X^{-1} - (X + Y)^{-1}$  on  $X + Y$  are considered.

### 1. Introduction

Tweedie (1957), in a fundamental paper devoted to the inverse Gaussian distribution, proved that if  $X_1, \dots, X_n$  are independent identically distributed (iid) inverse Gaussian random variables (rv's) then  $\sum_{i=1}^n X_i$  and  $\sum_{i=1}^n X_i^{-1} - n^2(\sum_{i=1}^n X_i)^{-1}$  are independent. For the converses of this one can consult Khatri (1962) and Letac and Seshadri (1985). In Seshadri (1983) the inverse Gaussian law is characterized by the constancy of regression of  $\sum_{i=1}^n X_i^{-1} - n^2(\sum_{i=1}^n X_i)^{-1}$  on  $\sum_{i=1}^n X_i$  under appropriate moment conditions. For a review of characterization problems for the inverse Gaussian distribution, see Seshadri (1993).

For the gamma law an important characterization through an independence property is due to Lukacs (1955), based on the independence of  $Y_1/Y_2$  and  $Y_1 + Y_2$ , where  $Y_1, Y_2$  are independent random variables. The regression version of this statement was obtained in Wesolowski (1990) (see also Hall and Simons (1969), Wang (1981) and Li, Huang and Huang (1994) for related results).

These are separate results for the inverse Gaussian and the gamma distributions. A characterization of the generalized inverse Gaussian (GIG) that involves the gamma distribution was given by Letac and Seshadri (1983). It goes as follows: if  $X$  has the same law as  $(X + Y)^{-1}$ , where  $X$  and  $Y$  are independent and  $Y$  is gamma, then  $X$  follows the GIG law.

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Let us denote by  $\mu_{-p,a,b}$  the GIG law defined by the density:

$$f(x) = C_1 x^{-p-1} \exp(-(ax + b/x)/2) I_{(0,\infty)}(x);$$

and by  $\gamma_{p,a/2}$  the gamma law with the density

$$g(y) = C_2 y^{p-1} \exp(-ay/2) I_{(0,\infty)}(y),$$

where  $C_1$  and  $C_2$  are appropriate norming constants and  $p, a, b$  are positive numbers.

Recently Matsumoto and Yor (1998) observed that if  $X \sim \mu_{-p,a,b}$  and  $Y \sim \gamma_{p,a/2}$  are independent then  $U = 1/(X+Y)$  and  $V = 1/X - 1/(X+Y)$  are also independent with distributions  $\mu_{-p,b,a}$  and  $\gamma_{p,b/2}$ , respectively. (Actually their result was proved for the case  $a = b$ .) The converse of this statement, relying only on the independence property and being a simultaneous characterization of the GIG and gamma laws, has been proved in Letac and Wesolowski (2000). A closely related functional equation has been studied recently in Wesolowski (2000).

Observe that if we consider the scaled rv's  $\tilde{X} = (a/b)^{1/2} X$ ,  $\tilde{Y} = (a/b)^{1/2} Y$ , then  $\tilde{X} \sim \mu_{-p,\sqrt{ab},(ab)^{1/2}}$ ,  $\tilde{Y} \sim \gamma_{p,(ab)^{1/2}/2}$  and for  $\tilde{U} = 1/(\tilde{X} + \tilde{Y})$  and  $\tilde{V} = 1/\tilde{X} - \tilde{U}$  it follows immediately that  $\tilde{U} \sim \mu_{-p,(ab)^{1/2},(ab)^{1/2}}$  and  $\tilde{V} \sim \gamma_{p,(ab)^{1/2}/2}$ .

In the present paper we replace the independence of  $U$  and  $V$  by the constancy of regression of  $V$  on  $U$  under suitable moment conditions. Also a dual problem for the constancy of regression of  $1/V$  on  $U$  will be considered. The price we pay for relaxing independence is that the characterization is not simultaneous because we have to assume one law to obtain the other.

## 2. Constancy of Regression of $V$ on $U$

Let  $X$  and  $Y$  be two independent positive rv's, such that  $E(1/X)$ ,  $E(X)$  and  $E(Y)$  are finite. For  $U = 1/(X+Y)$  and  $V = 1/X - 1/(X+Y)$  we consider the following constancy of regression condition, namely

$$E(V|U) = c,$$

where  $c = E(V)$ . Since  $V = Y/(X(X+Y))$  equivalently we can write this condition as

$$E(Y/X|X+Y) = c(X+Y). \quad (1)$$

Observe that (1) is equivalent to

$$E\left(\frac{Y}{X}e^{s(X+Y)}\right) = cE\left((X+Y)e^{s(X+Y)}\right), \quad (2)$$

for any  $s$ , for which both sides are finite. To see this, use conditioning with respect to  $(X+Y)$  on the left side of (2). Then (2) follows from (1) directly and (1) follows from (2) by the uniqueness of the Laplace transform since (2) implies

$$E(E(Y/X|X+Y)e^{s(X+Y)}) = E(c(X+Y)e^{s(X+Y)})$$

for any  $s \leq 0$  and at both sides we have the Laplace transforms of finite measures on  $(0, \infty)$ , which are equal.

Define now

$$h_X(s) = E(X^{-1}e^{sX}), \quad L_Y(s) = E(e^{sY}), \quad s \in \Theta,$$

where  $\Theta = \{s : h_X(s) < \infty, L_Y(s) < \infty\}$  and  $(-\infty, 0] \subset \Theta$ . Then (2) can be rewritten as

$$h_X(s)L'_Y(s) = c h''_X(s)L_Y(s) + c h'_X(s)L'_Y(s), \quad s \in \Theta. \quad (3)$$

We start with a characterization of the GIG law assuming that  $Y$  has a gamma distribution.

**THEOREM 1.** *Assume  $E(1/X)$ ,  $E(X)$  are finite and that  $Y \sim \gamma_{p,a/2}$  for some positive numbers  $p$  and  $a$ . If the regression of  $V$  on  $U$  is constant, i.e. (1) holds, then  $X \sim \mu_{-p,a,b}$ , where  $b = 2p/c$  and  $c = E(V)$ .*

**PROOF.** Since  $Y \sim \gamma_{p,a/2}$

$$L_Y(s) = \left(\frac{a}{a-2s}\right)^p, \quad L'_Y(s) = \frac{2pa^p}{(a-2s)^{p+1}}, \quad s \in (-\infty, a/2).$$

Hence (3) takes the form

$$(a-2s)h''_X(s) + 2ph'_X(s) - bh_X(s) = 0 \quad (4)$$

with  $b = 2p/c$ , and the above equation holds in some set  $\Theta_1$  containing  $(-\infty, 0]$ . Define now a function  $w$  by

$$h_X(s) = (a-2s)^{(p+1)/2}w(s), \quad s \in \Theta_1.$$

Consequently

$$\begin{aligned} h'_X(s) &= -(p+1)(a-2s)^{(p-1)/2}w(s) + (a-2s)^{(p+1)/2}w'(s), \\ h''_X(s) &= (p^2-1)(a-2s)^{(p-3)/2}w(s) - 2(p+1)(a-2s)^{(p-1)/2}w'(s) \\ &\quad + (a-2s)^{(p+1)/2}w''(s), \quad s \in \Theta_1. \end{aligned}$$

Plugging these into equation (4) we obtain

$$(a-2s)w''(s) - 2w'(s) - [(p+1)^2/(a-2s) + b]w(s) = 0, \quad s \in \Theta_1. \quad (5)$$

Now for  $z = ((a-2s)b)^{1/2}$  and for the function  $v$  defined by  $v(z) = w(s)$ , we get

$$\begin{aligned} \frac{dw}{ds} &= -\frac{dv}{dz} \frac{b}{z}, \\ \frac{d^2w}{ds^2} &= \frac{d^2v}{dz^2} \left(\frac{b}{z}\right)^2 - \frac{dv}{dz} \frac{b^2}{z^3}, \end{aligned}$$

$z > 0$ . Inserting the above in (5) we arrive at the modified Bessel equation

$$z^2 v''(z) + z v'(z) - (z^2 + (p+1)^2) v(z) = 0, \quad z > 0.$$

The space of the solutions of the equation is spanned by two modified Bessel functions  $K_{(p+1)}$  and  $I_{(p+1)}$  (see, for instance, Abramovitz and Stegun (1964)).

Going back to the formula connecting  $h_X$  and  $w$  we see that, since  $h_X$  is a Laplace transform of a bounded measure on  $(0, \infty)$ , it goes to zero as  $s$  goes to  $-\infty$ ; consequently the same is true for  $w$ . Now by the definition of  $v$  it follows that  $\lim_{z \rightarrow \infty} v(z) = 0$ . But  $I_{(p+1)}(z)$  explodes as  $z \rightarrow \infty$ , and hence the only possible solution is the modified Bessel function of the third kind  $K_{(p+1)} = K_{-(p+1)}$ .

We conclude that  $h_X(s) = c_3(a - 2s)^{(p+1)/2} K_{-(p+1)}((b(a - 2s))^{1/2})$ ,  $s \in \Theta_1$ , where  $c_3$  is a suitable constant. Hence  $\Theta_1 = (-\infty, a/2)$  and  $h_X$  is the Laplace transform of the function  $x^{-1}f(x)$ , where  $f$  is the density of  $\mu_{-p,a,b}$ .  $\square$

Now we consider a parallel situation assuming that the distribution of  $X$  is GIG and seek the distribution of  $Y$ .

**THEOREM 2.** *Assume that  $X \sim \mu_{-p,a,b}$  for some positive numbers  $p$ ,  $a$  and  $b$  and that  $E(Y)$  is finite. If the regression of  $V$  on  $U$  is constant, and equal to  $2p/b$ , i.e. (1) holds with  $c = 2p/b$ , then  $Y \sim \gamma_{p,a/2}$ .*

**PROOF.** Since  $X \sim \mu_{-p,a,b}$

$$h_X(s) = c_4(b(a - 2s))^{(p+1)/2} K_{-(p+1)}((b(a - 2s))^{1/2}), \quad s < a/2,$$

where  $c_4$  is an appropriate constant.

Let  $z = (b(a - 2s))^{1/2}$ . Then  $z' = -b/z$  and  $z'' = -b^2/z^3$ . Now, to avoid cumbersome notation we will suppress the indices of the  $K$  function. Then

$$h_X(s) = c_4 z^{p+1} K(z),$$

$$h'_X(s) = -c_4 b [(p+1)z^{p-1} K(z) + z^p K'(z)],$$

$$h''_X(s) = c_4 b^2 z^{p-3} [z^2 K''(z) + (2p+1)z K'(z) + (p^2 - 1)K(z)],$$

(the primes on the left sides denote derivatives taken with respect to  $s$ , while those on the right sides are derivatives taken with respect to  $z$ ). But from the modified Bessel equation we have  $z^2 K''(z) + z K' - (z^2 + (p+1)^2) K(z) = 0$  and this then gives

$$h''_X(s) = c_4 b^2 z^{p-3} [(z^2 + 2p(p+1))K(z) + 2pz K'(z)], \quad s \in \Theta_1,$$

where  $\Theta_1$  is some set containing  $(-\infty, 0]$ . We plug the above formulas into (3) to obtain

$$\begin{aligned} L'_Y(s) & (z^{p+1} K(z) + 2p[(p+1)z^{p-1} K(z) + z^p K'(z)]) \\ & = 2pb L_Y(s) z^{p-3} [(z^2 + 2p(p+1))K(z) + 2pz K'(z)]. \end{aligned}$$

After obvious simplifications we arrive at

$$z^2 L'_Y(s) = 2pb L_Y(s), \quad s \in \Theta_1.$$

Now since  $z^2 = b(a - 2s)$  we immediately obtain  $L_Y(s) = (a/(a - 2s))^p$ ,  $s \in \Theta_1$ . Hence  $\Theta_1 = (-\infty, a/2)$  and consequently  $L_Y$  is the Laplace transform of the  $\gamma_{p,a/2}$  distribution. □

### 3. Constancy of Regression of $1/V$ on $U$

Let  $X$  and  $Y$  be two independent positive rv's such that  $E(X^2)$ ,  $E(1/Y)$  and  $E(Y)$  are finite. With  $U$  and  $V$  defined in Section 2 we consider the constancy of regression of  $1/V$  on  $U$ , i.e. the condition

$$E(V^{-1}|U) = d,$$

where  $d$  is a constant equal to  $E(1/V)$ . An equivalent form reads

$$E(X/Y|X + Y) = d/(X + Y). \tag{6}$$

Just as in Section 2, we get

$$L''_X(s)h_Y(s) + L'_X(s)h'_Y(s) = dL_X(s)h'_Y(s), \tag{7}$$

where

$$L_X(s) = E(e^{sX}), \quad h_Y(s) = E(Y^{-1}e^{sY}), \quad s \in \Theta,$$

and  $\Theta = \{s : L_X(s) < \infty, h_Y(s) < \infty\}$ .

Now assuming that the distribution of  $Y$  is gamma we derive a characterization of the GIG law for  $X$ .

**THEOREM 3.** *Assume that  $E(X^2)$  is finite and  $Y \sim \gamma_{p,a/2}$  for some numbers  $p > 1$  and  $a > 0$ . If the regression of  $1/V$  on  $U$  is constant, i.e. (6) holds, then  $X \sim \mu_{-p,a,b}$ , where  $b = 2(p - 1)d$  and  $d = E(1/V)$ .*

Analogous to Theorem 2 we have

**THEOREM 4.** *Assume that  $X \sim \mu_{-p,a,b}$  for some positive numbers  $p, a$  and  $b$ , and that  $E(1/Y)$ ,  $E(Y)$  are finite. If the regression of  $1/V$  on  $U$  is constant and equal to  $b/(2(p - 1))$ , i.e. (6) holds with  $d = b/(2(p - 1))$ , then  $Y \sim \gamma_{p,a/2}$ .*

Since both the theorems have proofs following the lines of the proofs from the previous section, they are skipped.

In each of the cases of constancy of regression considered in this paper we were not able to obtain simultaneous characterization of both the distributions: GIG and gamma. Then an interesting open question arises, if such a characterization holds true if both the conditions  $E(V|U) = c$  and  $E(V^{-1}|U) = d$  are satisfied. We expect that the answer is in affirmative.

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VANAMAMALAI SESHADRI  
DEPARTMENT OF MATHEMATICS AND  
STATISTICS  
MCGILL UNIVERSITY  
MONTREAL, CANADA  
E-mail: vansesh@hotmail.com

JACEK WESOŁOWSKI  
WYDZIAŁ MATEMATYKI I NAUK  
INFORMACYJNYCH  
POLITECHNIKA WARSZAWSKA  
WARSZAWA, POLAND  
E-mail:wesolo@alpha.mini.pw.edu.pl