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SKETCHES ON DILATION PROBABILITY DISTRIBUTIONS

Dedicated to Professor Kazimierz Urbanik

Abstract. The dilation property allows to define an intriguing family of statistical distributions parameterized by the coefficients of respective dilation equation and the dilation scale. The family includes, except some commonly used probability laws, also a wide range of naturally arising singular distributions, which usually are difficult for statistical analysis. But here due to dilation scheme some progress in developing statistical tools can be expected. The paper describes basic properties of dilation distributions, including an extension of the Kershner-Wintner theorem on infinite Bernoulli convolutions, and indicates possible directions for future studies, including preliminary observations on statistical inference.

1. Introduction

Statistical modelling is almost exclusively built on families of distributions possessing closed formulas for densities (in the absolutely continuous case) or for probability mass functions (in the discrete case) or for distribution functions or at least for characteristic functions or some other transforms. Therefore never or almost never continuous singular (with respect to the Lebesgue measure) distributions are encountered in such settings. They are believed mainly to work as strange examples falling away from what could be useful in applications. The same is true even for distributions which are absolutely continuous but which do not have a closed formula property. The aim of this paper is to break out that scheme and to justify that such break is natural and useful. It is recalled here that families of distributions without closed formulas naturally arise, for instance, in some ruin problems

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or in models involving random jumps. Such families, connected with dilation equations, are mathematically treatable. Also statistical inference can be developed for them. This paper developes, in probabilistic language, theory of dilation distributions involving: probability models leading to dilation distributions, basic distributional properties, simulations and visualizations, basics of estimation and hypothesis testing. The ambition of the paper is to give a base for an additional chapter in standard monographs on families of distributions useful in scientific work.

A revival of interest in dilation equations in recent years is due to an expansion of the wavelet theory. The wavelets are constructed on a base function which is a solution of the dilation equation (dile) of the form

$$f(x) = \sum_k a_k f(2x - k),$$

where (a_k) are some real (or complex) coefficients. Usually it is assumed that $\int f\,dx = 1$, implying $\sum_k a_k = 2$ and $\#\{k : a_k \neq 0\} < \infty$ implying a compact support—see Strang (1989) for an excellent concise introduction. Then a unique solution exists (as a distribution), and its Fourier transform can be easily represented in a form of an infinite product—see Heil and Colella (1994) and Colella and Heil (1994). For a thorough study of more general two-scale difference equations see Daubechies and Lagarias (1991, 1992).

Consider some examples.

The simplest dile has the form:

$$f(x) = 2f(2x),$$

i.e. the only non-zero a_k is $a_0 = 2$ and the solution is the delta function. In probabilistic terms we can rewrite the above equation as:

$$\mu_X = \mu_{X/2},$$

where μ_X denotes a probability distribution of a random variable (rv) X. Consequently the solution is: $P(X = 0) = 1$.

Take another example of a dile:

$$f(x) = f(2x - 1) + f(2x + 1),$$

with $a_{-1} = a_1 = 1$ and the solution $f(x) = 0.5I_{[-1,1]}(x)$, which is a probability density function of a uniform $U([-1,1])$ rv X. The respective equation for its distribution takes the form

$$\mu_X = 0.5\mu_{(X+1)/2} + 0.5\mu_{(X-1)/2}. \tag{1}$$

Hence the following mixture characterization of the uniform $U([-1,1])$ distribution follows immediately:

THEOREM 1. *If for a distribution of a rv X the mixture representation* (1) *holds then X has the uniform distribution on* $[-1, 1]$.

Proof. From (1) it follows that a characteristic function (chf) ϕ of X satisfies

$$\phi(t) = \cos(t/2)\phi(t/2),$$

for any real t. Iterating the above equation we have for any n

$$\phi(t) = \phi(t/2^n) \prod_{k=1}^{n} \cos(t/2^k)$$

which is well known to converge to $\sin(t)/t$, a chf of the $U([-1, 1])$ distribution. ■

Let us consider now a three terms dile of the form

$$f(x) = 0.5f(2x - 1) + f(2x) + 0.5f(2x + 1),$$

i.e. $a_{-1} = a_1 = 0.5$ and $a_0 = 1$. Then $f(x) = (1 - |x|)I_{[-1,1]}(x)$, which is again a pdf of a rv X with a triangular distribution, and a respective equation for its distribution takes the shape

$$\mu_X = 0.25\mu_{(X+1)/2} + 0.5\mu_{X/2} + 0.25\mu_{(X-1)/2}.$$

Observe that $X \stackrel{d}{=} (X_1 + X_2)/2$, where X_1 and X_2 are two independent copies of rv's for which the equation (1) is fulfilled.

In general if a dile

$$f(x) = \sum_k a_k f(2x - k)$$

with $a_k \geq 0$ for any k is given, then a respective probabilistic dilation equation (prodile) has the form

$$\mu_X = \frac{1}{2} \sum_k a_k \mu_{(X+k)/2}.$$

DEFINITION 1. Let μ_X be a probability distribution of a rv X. Then the equation

$$\mu_X = \sum_k c_k \mu_{(X+k)/r},$$

where c_k are positive constants such that $\sum_k c_k = 1$ and $r > 1$, is called a probabilistic dilation equation (prodile). A probability distribution μ_X fulfilling such a prodile is called a dilation distribution with a scale r.

2. Prodiles in probability

Infinite symmetric Bernoulli convolutions. Let $X_1, X_2, \dots$ be independent identically distributed (iid) rv's with $P(X_1 = \pm 1) = 0.5$ and let

$$Y = \sum_{i=1}^{\infty} r^{-i} X_i,$$

where $r > 1$. It was shown by Kershner and Wintner (1936) that for $r > 2$ the distribution of Y is singular. For $r = 2$ the distribution is uniform $U([-1,1])$. But, except some special cases, see for instance Erdös (1939, 1940), nothing is known in general for $r \in (1,2)$. More recent contributions for Bernoulli convolutions can be found in Garsia (1962), Brown and Moran (1973), Lau (1993) and in the Lukacs (1970) monograph. Observe that

$$\begin{aligned} E\left(e^{itY}\right) &= \prod_{i=1}^{\infty} \cos(t/r^i) = \cos(t/r) \prod_{i=1}^{\infty} \cos\left(\frac{t/r}{r^i}\right) \\ &= 0.5\left(e^{-it/r} + e^{it/r}\right) E\left(e^{itY/r}\right) \\ &= 0.5E\left(e^{it(Y-1)/r}\right) + 0.5E\left(e^{it(Y+1)/r}\right). \end{aligned}$$

Consequently μ_Y is a dilation distribution generated by the prodile

$$\mu_Y = 0.5\mu_{(Y-1)/r} + 0.5\mu_{(Y+1)/r}.$$

The brave gambler problem (see, for instance, Billinglsey (1979)). Let X_i denotes an outcome of the game in i-th step, $i = 1, 2, \dots$, i.e. $X_1, X_2, \dots$ are iid rv's with $P(X_1 = 1) = p = 1 - q = 1 - P(X_1 = -1)$. The gambler in each step bets as much as is reasonable to win the whole amount equal 1 as quickly as possible. Hence his asset at any time n is described by

$$S_n = S_{n-1} + W_n X_n,$$

where $S_0 = x \in [0,1]$ and

$$W_n = \begin{cases} S_{n-1} & 0 \le S_{n-1} \le 0.5 \\ 1 - S_{n-1} & 0.5 \le S_{n-1} \le 1. \end{cases}$$

Consequently the probability of winning for starting amount $x \in (0,1)$, denoted by $Q(x)$, fulfills

$$Q(x) = \begin{cases} pQ(2x) & 0 \le x \le 0.5 \\ p + qQ(2x-1) & 0.5 \le x \le 1. \end{cases}$$

Since $Q(x) = 0$ for $x \le 0$ and $Q(x) = 1$ for $x \ge 1$, complement in a natural way the definition, then it follows easily that Q can be considered as a distribution function (df) of a prodile rv X with

$$\mu_X = p\mu_{X/2} + q\mu_{(X+1)/2}.$$

An extension of this scheme involves a game with $N+1$ possible outcomes: $0, 1, \ldots, N$ of respective probabilities $c_0, c_1, \ldots, c_N$, where N is the total capital for the whole game. If the outcome is i then a gambler betting $x \in (0, N)$ wins $N - x$ if $2x - i \geq N$, or $x - i$ (which can be a loss also) if $0 < 2x - i < N$, or $-x$ if $2x - i \leq 0$. Then it follows easily that for $x \in [k/2, (k+1)/2]$

$$Q(x) =$$
$$\begin{cases} c_0 Q(2x) + \ldots + c_k Q(2x - k) & k = 0, \ldots, N-1, \\ c_0 + \ldots + c_{k-N} + c_{k-N+1} Q(2x - k + N - 1) + \ldots + c_N Q(2x - N) & \\ & k = N, \ldots, 2N-1. \end{cases}$$

(observe that with $N = 1$ the earlier scheme is covered). Consequently Q is a df of a rv X with a distribution fulfilling

$$\mu_X = \sum_{k=0}^{N} c_k \mu_{(X+k)/2}.$$

Random jumps. Starting at $X_0 = 1/2$ a particle in each step (1^o) stays where it is with probability $q = 1 - p$, or jumps on the distance 1 to the right with probability p and then (2^o) goes back to the left on half of its distance from 0 (thus it can not leave the interval $(0,1)$). Consequently its position after $n \geq 1$ steps is described by

$$X_n = \frac{X_{n-1} + Y_n}{2},$$

where $Y_1, Y_2, \ldots$ are iid rv's with $P(Y_1 = 0) = q = 1 - P(Y_1 = 1)$. Hence $X_n \xrightarrow{d} X$, and for the chf ϕ of X one gets

$$E(e^{itX}) = \phi(t) = \phi(t/2)(q + pe^{it/2}) = E(qe^{itX/2} + pe^{it(X+1)/2}).$$

Consequently

$$\mu_X = q\mu_{X/2} + p\mu_{(X+1)/2}.$$

Assume now that, starting at $X_0 = 0$, in each step the following possibilities of the movement of the particle are given: 0 with probability $1/3$, ± 1 with probabilities $(1 - 2\alpha)/6$, and ± 2 with probabilities $(1 + 2\alpha)/6$ and then the particle jumps in the direction of the origin reducing its distance to the origin to $1/3$ of the distance after the first phase. Then its position after n steps is described by

$$X_n = \frac{X_{n-1} + Y_n}{3},$$

where $Y_1, Y_2, \ldots$ are iid rv's with $P(Y_1 = 0) = 1/3$, $P(Y_1 = \pm 1) = (1 - 2\alpha)/6$ and $P(Y_1 = \pm 2) = (1 + 2\alpha)/6$, where $\alpha \in [-1/2, 1/2]$ is a given number.

Consequently $X_n \xrightarrow{d} X$ with a prodile distribution concentrated on $(-1,1)$ fulfilling

$$(2)\qquad \mu_X = \frac{1+2\alpha}{6}\mu_{(X-2)/3} + \frac{1-2\alpha}{6}\mu_{(X-1)/3} + \frac{1}{3}\mu_{X/3} + \frac{1-2\alpha}{6}\mu_{(X+1)/3} + \frac{1+2\alpha}{6}\mu_{(X+2)/3}.$$

It is known that then μ_X is absolutely continuous and its density is a generalized de Rham function.

An extension of the above scheme involves a sequence $Y_1, Y_2, \ldots$ of iid rvs with $P(Y_1 = k) = c_k$, $k = -N_1, -N_1+1, \ldots, N_2$, where N_1 and N_2 are positive integers, describing the movement of the particle in the first phase of subsequent steps, while the second phase is its jump in the direction of the origin on $1/r < 1$ of the distance after the first phase. Then its evolution is described by the sequence $X_0 = 0$,

$$X_n = \frac{X_{n-1}+Y_n}{r}, \qquad n = 1, 2, \ldots,$$

converging in distribution to a rv X with a prodile distribution fulfilling

$$\mu_X = \sum_{k=-N_1}^{N_2} c_k \mu_{(X+k)/r}.$$

3. Basic properties of prodile distributions

Consider a rv X with a prodile distribution fulfilling

$$\mu_X = \sum_{k=-N_1}^{N_2} c_k \mu_{(X+k)/r}.$$

Then for the rv $Y = X + N_1/(r-1)$ it follows that for any Borel set A

$$\mu_Y\left(A + \frac{N_1}{r-1}\right) = \sum_{l=0}^{N_1+N_2} c_{l-N_1}\mu_{(Y+l)/r}\left(A + \frac{N_1}{r-1}\right).$$

Consequently any prodile can be reduced to the basic one of the form

$$\mu_X = \sum_{k=0}^{N} c_k \mu_{(X+k)/r},$$

with $c_0 > 0$ and $c_N > 0$. Denote by $DIL(r; c_0, \ldots, c_N)$ any such a prodile distribution.

Smoothness properties

THEOREM 2. *If X is a $DIL(r; c_0, \ldots, c_N)$ rv with a df F_X then*

$$\sup\{x: \; F_X(x) = 0\} = 0, \quad \inf\{x: \; F_X(x) = 1\} = N/(r-1),$$

and F_X is continuous.

Proof. Take any $\epsilon_1 > 0$. Then the prodile implies

$$P\left(X > \frac{N}{r-1} + \epsilon_1\right) = \sum_{k=0}^{N} c_{N-k} P\left(X > \frac{N}{r-1} + k + r\epsilon_1\right). \tag{3}$$

Observe that the assumption

$$P\left(X > \frac{N}{r-1} + N + r\epsilon_1\right) < P\left(X > \frac{N}{r-1} + \epsilon_1\right).$$

is contradictory to (3). Consequently

$$P\left(X > \frac{N}{r-1} + \epsilon_2\right) = P\left(X > \frac{N}{r-1} + \epsilon_1\right),$$

where $\epsilon_2 = N + r\epsilon_1$. Repeating the above argument n times we get

$$P\left(X > \frac{N}{r-1} + \epsilon_{n+1}\right) = P\left(X > \frac{N}{r-1} + \epsilon_1\right),$$

where

$$\epsilon_{n+1} = N + r\epsilon_n = N(1 + r + \ldots + r^{n-1}) + Nr^n\epsilon_1 \to \infty.$$

Hence $P(X > N/(r-1)) = 0$.

Similar approach can be applied to the left end of the support with

$$P(X < -\epsilon) = \sum_{k=0}^{N} c_k P(X < -r\epsilon - k).$$

Since $P(X < -\epsilon) \geq P(X < -r\epsilon - k)$ for any $k = 0, 1, \ldots$, then $P(X < -\epsilon) > P(X < -r\epsilon - N)$ is a contradiction. Hence $P(X < -\epsilon) = P(X < -r\epsilon - N)$ for any $\epsilon > 0$, which implies that $P(X < 0) = 0$.

Consequently $\mathrm{supp}(X) \subset [0, N/(r-1)]$.

Consider now $K = \inf\{x: \; P(X > x) = 0\}$. Assume that $K < N/(r-1)$, i.e. $K = N/(r-1) - \epsilon$ for some $\epsilon > 0$. Then $P(X > K) = 0$ and the prodile implies that $P(X > rK - k) = 0$ for any $k = 0, 1, \ldots, N$. Take $k = N$, which yields $P(X > N/(r-1) - r\epsilon) = 0$, which contradicts the assumption since $N/(r-1) - r\epsilon < K$.

Take now $L = \sup\{x: \; P(X < x) = 0\}$ and assume that $L > 0$. Then for $\epsilon \in (0, (r-1)L)$ it follows that $P(X < L + \epsilon) > 0$ and consequently for

$\delta = (L+\epsilon)/r$ the prodile implies

$$P(X<\delta) \geq c_0 P(X<L+\epsilon) > 0,$$

which is a contradiction since $0<\delta<L$.

To prove that F_X is of the continuous type assume in contrary that $\exists\ x_0 \in [0, N/(r-1)]$ such that $P(X=x_0) = \alpha_0 > 0$. Without loosing generality we can assume that $\alpha_0 = sup\{\alpha :\ \alpha = P(X=x),\ x \in \mathbf{R}\}$. Since

$$\alpha_0 = \sum_{k=0}^{N} c_k P(X = rx_0 - k)$$

then it follows that $\alpha_0 = P(X = rx_0 - k)\ \forall\ k = 0, \dots, N$ such that $c_k > 0$, i.e. at least for $k=0$ and $k=N$. Hence $rx_0 \leq N/(r-1)$ and $rx_0 - N \geq 0$, implying

$$\frac{N}{r} \leq x_0 \leq \frac{N}{r(r-1)}.$$

Consequently $r \leq 2$. Consider now the point rx_0. Applying again the prodile we get that $P(X = r^2 x_0) = \alpha_0$, similarly starting with the point $rx_0 - N$ we get $P(X = r(rx_0 - N) - N) = \alpha_0$. Taking both the conditions into account one gets

$$\frac{N(r+1)}{r^2} \leq x_0 \leq \frac{N}{r^2(r-1)},$$

which implies $r \leq 2^{1/2}$. Proceeding in the same way after subsequent m steps with all points in each step inside $[0, N/(r-1)]$ one gets

$$\frac{N(r^{m-1} + \dots + r + 1)}{r^m} \leq x_0 \leq \frac{N}{r^m(r-1)}$$

which implies that $1 < r \leq 2^{1/m}$, being contradictory for sufficiently large m. ■

It appears that depending on the quantity $N/(r-1)$ the df's of the dilation distributions, being continuous functions can be of two basic kinds: strictly increasing (on the support) or constant at some intervals (and then singular). These two observations are described more thoroughly in the next two results.

THEOREM 3. *Let X be a $DIL(r; c_0, \dots, c_N)$ rv with a df F_X and $c_k > 0$ $\forall k \in \{0, \dots, N\}$. If $N \geq r-1$ then F_X is strictly increasing in* $[0, N/(r-1)]$ $=$ supp(X).

Proof. Let us take any $a, b \in [0, N/(r-1)]$ such that $a<b$. Assume that $P(a \leq X \leq b) = 0$. Then for any n, let us define a sequence $k_1, k_2, \dots, k_n$

of numbers from $\{0, 1, \ldots, N\}$ such that

$$\sum_{i=1}^{n} \frac{k_i}{r^i} \leq a$$

and for any $j = 1, \ldots, n$ we have $k_j = N$ or $k_j < N$ and then

$$\sum_{i=1}^{j} \frac{k_i}{r^i} \leq a < \sum_{i=1}^{j} \frac{k_i}{r^i} + \frac{1}{r^j}.$$

Hence

$$a = \sum_{i=1}^{n} \frac{k_i}{r^i} + \delta_n.$$

We claim now that $0 \leq \delta < N/[(r-1)r^n]$. If $k_n < N$ then it follows immediately that $\delta_n < r^{-n} < N/(r^n(r-1))$. If $k_n = N$ then define $j_0 = \min\{j : k_l = N \,\forall\, l \in \{j, j+1, \ldots, n\}\}$. Hence

$$\sum_{i=1}^{j_0-1} \frac{k_i}{r^i} + \frac{N}{r^{j_0}} + \frac{N}{r^{j_0+1}} + \ldots + \frac{N}{r^n} + \delta_n < \sum_{i=1}^{j_0-1} \frac{k_i}{r^i} + \frac{1}{r^{j_0-1}}$$

and thus

$$\begin{aligned} \delta_n &< \frac{1}{r^n}\left[r^{n-j_0+1} - (r^{n-j_0} + \ldots + 1)N\right] \\ &= \frac{N}{r^n(r-1)}\left[r^{n-j_0+1}(r-1) - r^{n-j_0+1} + 1\right] < \frac{N}{r^n(r-1)}. \end{aligned}$$

Now for $\delta_1 = b - a$, and sufficiently large n we have $\delta_1 > 2\left(\frac{N}{(r-1)r^n} - \delta_n\right)$. For such an n denote now $\epsilon = N/(r-1) - r^n\delta_n$, which is a positive number.

Observe that $P(a \leq X \leq b) = 0$ implies, via subsequent application of the prodile, that

$$P\Big(r^n a - \sum_{j=0}^{n-1} r^j l_j \leq X \leq r^n b - \sum_{j=0}^{n-1} r^j l_j\Big) = 0.$$

Putting in the above formula $l_j = k_{n-j}$ one gets

$$r^n a - \sum_{j=0}^{n-1} r^j l_j = \frac{N}{r-1} - \epsilon, \quad r^n b - \sum_{j=0}^{n-1} r^j l_j > \frac{N}{r-1} + \epsilon.$$

Hence

$$P\left(\frac{N}{r-1} - \epsilon \leq X \leq \frac{N}{r-1}\right) = 0$$

which is contradictory to $\inf\{x: P(X>x)=0\}=N/(r-1)$. Consequently $P(a\le X\le b)>0$ for any $0\le a<b\le N/(r-1)$ and the df F_X is strictly increasing in $[0,N/(r-1)]$. ■

The next result is a straight-forward extension of the Kershner and Wintner (1936) theorem on singular infinite Bernoulli convolutions recalled in Section 2.

THEOREM 4. *Let X be a $DIL(r;c_0,\dots,c_N)$ rv such that $c_k>0$ $\forall k=0,1,\dots,N$, with a df F_X. If $N<r-1$ then the set of points of increase of F_X is nowhere dense in $[0,N/(r-1)]$. More, F_X is singular and is constant on each of the intervals*

$$\left[\sum_{i=1}^{n}\frac{k_i}{r^i}+\frac{N}{r^n(r-1)},\sum_{i=1}^{n}\frac{k_i}{r^i}+\frac{1}{r^n}\right]\subset\left[0,\frac{N}{r-1}\right],$$

where $k_1,\dots,k_n$ are any numbers from $\{0,1,\dots,N\}$ with a restriction that $k_n<N$, and n is any positive integer. The Lebesgue measure of the union of intervals on which F_X is non-increasing equals to $N/(r-1)$.

Proof. If $N<r-1$ then

$$P\left(\sum_{i=1}^{n}\frac{k_i}{r^i}+\frac{N}{r^n(r-1)}<X<\sum_{i=1}^{n}\frac{k_i}{r^i}+\frac{1}{r^n}\right)=$$

$$\sum_{l=0}^{N}c_lP\left(\sum_{i=2}^{n}\frac{k_i}{r^{i-1}}+\frac{N}{r^{n-1}(r-1)}+k_1-l<X<\sum_{i=2}^{n}\frac{k_i}{r^{i-1}}+\frac{1}{r^{n-1}}+k_1-l\right).$$

Observe first that for $l<k_1$ the elements of the above sum are 0 since then the left end of the interval is above 1. On the other hand observe that since $k_n<N$ then

$$\sum_{i=2}^{n}\frac{k_i}{r^{i-1}}+\frac{1}{r^{n-1}}\le\sum_{i=2}^{n}\frac{N}{r^{i-1}}<\frac{N}{r-1}<1.$$

Consequently for $l>k_1$ the right hand side of the interval is below zero. Hence the only nonzero member of the above sum is for $l=k_1$, which yields

$$P\left(\sum_{i=1}^{n}\frac{k_i}{r^i}+\frac{N}{r^n(r-1)}<X<\sum_{i=1}^{n}\frac{k_i}{r^i}+\frac{1}{r^n}\right)$$

$$=c_{k_1}P\left(\sum_{i=1}^{n-1}\frac{k_{i+1}}{r^i}+\frac{N}{r^{n-1}(r-1)}<X<\sum_{i=1}^{n-1}\frac{k_{i+1}}{r^i}+\frac{1}{r^{n-1}}\right).$$

And by the induction argument the probabilities are zero, since for $n=1$

one gets at the right hand side

$$P\left(\frac{N}{r-1} < X < 1\right) = 0.$$

Hence the df F_X is nonincreasing on each of these intervals.

Now observe that for a given n we have $(N+1)^{n-1}N$ intervals and each of them has the length $(r-1-N)/((r-1)r^n)$, $n = 1, 2, \ldots$. Consequently the total measure of the union of these intervals is given by

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{(N+1)^{n-1}N(r-1-N)}{(r-1)r^n} &= \frac{N(r-1-N)}{(N+1)(r-1)} \sum_{n=1}^{\infty} \left(\frac{N+1}{r}\right)^n \\ &= \frac{N(r-1-n)}{(N+1)(r-1)} \frac{N+1}{r} \frac{1}{1-(N+1)/r} \\ &= \frac{N}{r-1}.\end{aligned}$$

To prove that F_X is singular with respect to the Lebesgue measure let us take an arbitrary point of increase x_0 of F_X. We claim that since x_0 can not lay inside the intervals considered above then it has a representation

$$x_0 = \sum_{i=1}^{\infty} \frac{k_i}{r^i},$$

for some sequence of numbers $k_1, k_2, \ldots$ belonging to $\{0, 1, \ldots, N\}$. To prove this claim observe first that x_0 has the representation

$$x_0 = \sum_{i=1}^{\infty} \frac{k_i}{r^i},$$

where k_j's are defined by:

$$\sum_{i=1}^{j} \frac{k_i}{r^i} \leq x_0 < \sum_{i=1}^{j} \frac{k_i}{r^i} + \frac{1}{r^j},$$

$j = 1, 2, \ldots$, with no additional restrictions on k_i's. We will show now that if $\exists\, j$ such that $k_j > N$ then x_0 lies in one of the intervals defined above. Define $j_0 = \min\{j : \ k_j > N\}$. Assume that $j_0 > 1$ and $k_{j_0-1} < N$. Then, by the definition of the sequence (k_i) it follows that $x_0 < \sum_{i=1}^{j_0-1} k_i/r^i + 1/r^{j_0-1}$ and also $x_0 > \sum_{i=1}^{j_0-1} k_i/r^i + N/(r^{j_0-1}(r-1))$, since by $r-1 > N$ it follows that

$$\sum_{i=j_0}^{\infty} k_i/r^i \geq (N+1)/r^{j_0} > N/(r^{j_0-1}(r-1)).$$

Consequently $x_0 \in (\sum_{i=1}^{j_0-1} k_i/r^i + N/(r^{j_0-1}(r-1)), \sum_{i=1}^{j_0-1} k_i/r^i + 1/r^{j_0-1})$, which contradicts the assumption that x_0 is a point of increase.

Consider now the situation in which $k_{j_0-1} = N$. Then define $j' = \min\{j : k_j = k_{j+1} = \ldots = k_{j_0-1} = N\}$ and assume that $j' > 1$. Then, again by the definition it follows that $x_0 < \sum_{i=1}^{j'-1} k_i/r^i + 1/r^{j'-1}$. Also $x_0 > \sum_{i=1}^{j'-1} k_i/r^i + N/(r^{j'-1}(r-1))$, since

$$\sum_{i=j'}^{\infty} k_i/r^i - \sum_{i=j'}^{\infty} N/r^i = \sum_{i=j_0}^{\infty} k_i/r^i - \sum_{i=j_0}^{\infty} N/r^i$$
$$\geq (N+1)/r^{j_0} - N/(r^{j_0-1}(r-1)) > 0.$$

Consequently $x_0 \in (\sum_{i=1}^{j'-1} k_i/r^i + N/(r^{j'-1}(r-1)), \sum_{i=1}^{j'-1} k_i/r^i + 1/r^{j'-1})$. If $j' = 1$ or $j_0 = 1$ then, repeating the above argument we get that $x_0 > N/(r-1)$ which is impossible. This proves our claim about sequence of k_i's defining any point of increase x_0 of the df F. Observe that all the points of the interval $[0, N/(r-1)]$ except of the inner points of the intervals defined in the first part are points of increase. It follows from the fact that the total length of these intervals equals to $N/(r-1)$.

Take now two sequences of points converging to x_0 defined by:

$$x_n = \sum_{i=1}^{n} \frac{k_i}{r^i}, \quad x_n' = x_n + \frac{N}{r^n(r-1)} - \frac{1}{r^n} < x_n,$$

$n = 1, 2, \ldots$. Observe that $F_X(x_n) = F_X(x_n')$ since F_X is constant on $[x_n', x_n]$, $n = 1, 2, \ldots$. Assume that there exists a derivative $F_X'(x_0) = c > 0$. Then

$$\lim_{n\to\infty} \frac{F_X(x_0) - F_X(x_n)}{x_0 - x_n} = \lim_{n\to\infty} \frac{F_X(x_0) - F_X(x_n)}{\alpha_n} = c > 0,$$

where $\alpha_n = \sum_{i=n+1}^{\infty} k_i r^{-i}$. But for the second sequence it follows that

$$\lim_{n\to\infty} \frac{F_X(x_0) - F_X(x_n')}{x_0 - x_n'} = \lim_{n\to\infty} \frac{F_X(x_0) - F_X(x_n)}{\alpha_n + \frac{1}{r^n}[1 - N/(r-1)]} = c\frac{1}{1 + \frac{1-N/(r-1)}{\lim_{n\to\infty}\beta_n}},$$

where

$$\lim_{n\to\infty} \beta_n = \lim_{n\to\infty} r^n \sum_{i=n+1}^{\infty} \frac{k_i}{r^i} \leq N \sum_{i=1}^{\infty} \frac{1}{r^i} = \frac{N}{r-1}.$$

Consequently $\lim_{n\to\infty} \beta_n = \beta < \infty$ and thus the derivative of F_X at x_0 does not exist since

$$\lim_{n\to\infty} \frac{F_X(x_0) - F_X(x_n')}{x_0 - x_n'} \neq \lim_{n\to\infty} \frac{F_X(x_0) - F_X(x_n)}{x_0 - x_n}. \blacksquare$$

The above two theorems were proved under the assumptions that all c_k's are positive. If some of c_k's are zero (recall that $c_0 > 0$ and $c_N > 0$ by definition) then denote by k', k'' such numbers belonging to $\{0, \ldots, N\}$, $k' < k''$, for which

$$k'' - k' = max\{k_2 - k_1 : \; c_{k_1} > 0, \, c_{k_2} > 0, \, c_j = 0 \,\forall j \in \{k_1 + 1, \ldots, k_2 - 1\}\}.$$

Then the analogue of Th. 3 holds with the assumption $N \geq r - 1$ changed to $N \geq (k'' - k')(r - 1)$. Similarly it is conjuctured that an analogue of Th. 4 holds if $N < (k'' - k')(r - 1)$. Partially it has been confirmed recently in Gosk (2000).

Moments and generators. Take now a rv X having a $DIL(r; c_0, \ldots, c_N)$ distribution. Then

$$X \stackrel{d}{=} \frac{X + Y}{r},$$

where Y is a rv independent of X with the distribution $P(Y = k) = c_k$, $k = 0, 1, \ldots, N$. Consequently

$$E(X) = \frac{E(Y)}{r - 1}, \quad Var(X) = \frac{Var(Y)}{r^2 - 1},$$

and any higher moment can be computed recursively from

$$E(X^m) = \frac{\sum_{j=0}^{m-1} \binom{m}{j} E(X^j) E(Y^{m-j})}{r^m - 1}, \qquad m = 2, 3, \ldots.$$

Denote by ϕ_X and ϕ_Y chf's of X and Y, respectively. Then

$$\phi_X(t) = \phi_X(t/r)\phi_Y(t/r) = \phi_X(t/r^n) \prod_{i=1}^{n} \phi_Y(t/r^i)$$

for any $t \in \mathbf{R}$ and any n. Taking limits of both sides for $n \to \infty$ implies

$$\phi_X(t) = \prod_{i=1}^{\infty} \phi_Y(t/r^i).$$

Consequently, knowing a scale r, the distribution of X is uniquely determined by the distribution of Y, which is called the generator of the $DIL(r; c_0, \ldots, c_N)$ distribution.

Convolutions. Consider a rv X_i with a $DIL(r; c_0^{(i)}, \ldots, c_{N_i}^{(i)})$ distribution having a generator Y_i, $i = 1, \ldots, m$, and such that X_i, Y_i, $i = 1, \ldots, m$ are jointly independent. Then

$$E\Big(\exp\Big(it\sum_{i=1}^{m}X_i\Big)\Big)=\prod_{i=1}^{m}\phi_{X_i}(t)=\prod_{i=1}^{m}\phi_{X_i}(t/r)\phi_{Y_i}(t/r)$$
$$=E\Big(\exp(i(t/r)\sum_{i=1}^{m}X_i)\Big)E\Big(\exp(i(t/r)\sum_{i=1}^{m}Y_i)\Big).$$

Hence $X_1+\ldots+X_m$ has a dilation distribution with the generator $Y_1+\ldots+Y_m$, i.e. its distribution is $DIL(r;\hat{c}_0,\ldots\hat{c}_{\hat{N}})$, where $\hat{N}=N_1+\ldots+N_m$ and

$$\hat{c}_k=\sum_{0\le i_1\le N_1,\ldots,0\le i_m\le N_m:\,i_1+\ldots+i_m=k}\ \sum_{j=1}^{m}c_{ij}^{(j)},\quad k=0,1,\ldots,\hat{N}.$$

The above considerations imply the following result:

THEOREM 5. *The family of DIL distributions with a fixed scale r is closed with respect to taking finite convolutions.*

EXAMPLE. If

$$\mu_X=\sum_{k=1}^{N}\binom{N}{k}2^{-N}\mu_{(X+k)/2},$$

then $X\overset{d}{=}X_1+\ldots+X_N$, where $X_1,X_2,\ldots,X_N$ are iid $U([0,1])$ rv's. It follows easily from the above theorem since X is driven by the generator Y with the binomial distribution with the parameters N and $1/2$. Observe that Y is a convolution of N iid Bernoulli rv's with probability of 1 equal to 1/2, and they are dilation generators of the $U([0,1])$ distribution. This scheme can be immediately extended to the generator Y with the binomial $b(N,p)$ distribution for any $p\in(0,1)$.

4. Solving prodiles

Let a prodile

$$\mu_X=\sum_{k=0}^{N}c_k\mu_{(X+k)/r},$$

be given. There are no explicit solutions of such equations available in general in a closed form. Of course always an infinite product representation of its chf can be written. A useful method of obtaining approximate solutions is a cascade algorithm. Here we present its probabilistic version. Start with some df F_0. Then compute recursively

$$F_n(x)=\sum_{k=0}^{N}c_kF_{n-1}(rx-k),\quad n=1,2,\ldots,$$

for any $x\in[0,N/(r-1)]$. If X_n denotes a rv with a df F_n then we have

$$X_n \stackrel{d}{=} \frac{X_{n-1}+Y_1}{r} \stackrel{d}{=} \frac{X_0}{r^n} + \sum_{i=1}^{n} \frac{Y_i}{r^i},$$

where Y_i are independent copies of a generator of the prodile, independent also of the sequence of X_i's. Hence for the chf it follows that

$$\phi_{X_n}(t) = \phi_{X_0}(t/r^n) \prod_{i=1}^{n} \phi_{Y_i}(t/r^i) \to \phi_X(t).$$

Consequently the sequence (X_n) converges in distribution to X. Since F_X is continuous it follows that the cascade algorithm converges to the solution of the prodile.

Since a solution of the prodile has the representation

$$X \stackrel{d}{=} \sum_{j=1}^{\infty} \frac{Y_j}{r^j},$$

where (Y_j) is a sequence of iid generators, then a Monte Carlo technique can be applied also. To this end we define

$$X_m = \sum_{j=1}^{m} \frac{Y_j}{r^j}, \qquad m = 1, 2, \ldots.$$

Obviously $X_m \xrightarrow{d} X$ as $m \to \infty$. Hence the df F_m approximates F_X, and F_m can be estimated by the empirical df $F_{n,m}$ based on a sample of n independent realizations of X_m: $X_{1,m}, \ldots, X_{n,m}$. On the other hand each of $X_{i,m}$ can be produced from independent generators $Y_{j,i,m}$, $j = 1, \ldots, m$, $i = 1, \ldots, n$.

Observe that the generalized de Rham densities, i.e. densities of the DIL distributions fulfilling the prodile (2) can be also obtained using the set of three affine transformations:

$$\begin{aligned} w_1(x,y) &= (x/3, (1/2+\alpha)y + 1/2 - \alpha), \\ w_2(x,y) &= ((2-x)/3, 2\alpha y + 1/2 - \alpha), \\ w_3(x,y) &= ((x+2)/3, (1/2+\alpha)y). \end{aligned}$$

Then $w = (w_1, w_2, w_3)$ can be applied recursively to some points of $\{(x, 1-x) : x \in [0,1]\}$. Finally the resulting curve should be reflected symmetrically for negative x's.

5. Estimation and hypothesis testing for dilation distributions

Here we include some preliminary remarks which can be treated only as a starting point for investigations concerning statistical inference for dilation distributions.

The $DIL(r; c_0, \dots, c_N)$ distributions form a parameter family. Then the question of estimation of the parameters arises. Since even for the absolutely continuous DIL laws there is no general explicit formula for the densities the maximum likelihood estimation is a problem. Instead some estimators based on the empirical df can be developed.

If F is a df of a $DIL(r; c_0, \dots, c_N)$ distribution then it follows from the prodile that

$$F(k/r) = c_0 F(k) + c_1 F(k-1) + \dots + c_{k-1} F(1), \quad k = 1, 2, \dots, N.$$

Hence it suffices to solve the above triangular system of linear equations for c_k's (with the determinant equal to $[F(1)]^N$). Then, taking empirical df for the df, consistent estimators of the parameters $c_0, c_1, \dots, c_{N-1}$ are obtained. For instance

$$\hat{c}_0 = \frac{\hat{F}_n(1/r)}{\hat{F}_n(1)}, \quad \hat{c}_1 = \frac{\hat{F}_n(2/r)\hat{F}_n(1) - \hat{F}_n(1/r)\hat{F}_n(2)}{[\hat{F}_n(1)]^2},$$

and in general

$$\hat{c}_k = \frac{(-1)^k \hat{D}_{n,k}}{[\hat{F}_n(1)]^k}, \quad k = 0, 1, \dots, N-1,$$

where $\hat{D}_{n,k}$ is the determinant of the matrix

$$\begin{bmatrix} \hat{F}_n(1/r) & \hat{F}_n(1) & 0 & 0 & \dots & 0 \\ \hat{F}_n(2/r) & \hat{F}_n(2) & \hat{F}_n(1) & 0 & \dots & 0 \\ \hat{F}_n(3/r) & \hat{F}_n(3) & \hat{F}_n(2) & \hat{F}_n(1) & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & 0 \\ \hat{F}_n((k-1)/r) & \hat{F}_n(k-1) & \hat{F}_n(k-2) & \hat{F}_n(k-3) & \dots & \hat{F}_n(1) \\ \hat{F}_n(k/r) & \hat{F}_n(k) & \hat{F}_n(k-1) & \hat{F}_n(k-2) & \dots & \hat{F}_n(2) \end{bmatrix}.$$

Then $\hat{c}_N = 1 - \hat{c}_0 - \dots - \hat{c}_{N-1}$. Also the moment method can be used since the formula

$$r^k E(X^k) = E(X+Y)^k = \sum_{l=0}^{N} E[(X+l)^k] c_l, \quad k = 1, 2, \dots$$

implies another system of linear equations for c_k's.

If r is not known then first one can estimate it using $\max_{1 \le i \le n} X_i$ as an estimate of the upper limit of the support of X. Consequently the estimate of r can be taken as

$$\hat{r} = \frac{N}{\max_{1 \le i \le n} X_i} + 1.$$

EXAMPLES. For the random jumps $DIL(2; q, p)$ distribution with the first method one gets $\hat{q}_1 = \hat{F}_n(1/2) = 1 - \hat{p}_1$ and with the second $\hat{p}_2 = \bar{X}_n = (X_1 + \dots + X_n)/n = 1 - \hat{q}_2$.

For the generalized de Rham DIL with the one unknown parameter α the formula

$$\alpha = \frac{3F(x) - F(3x) - 0.5[F(3x-2) + F(3x+2) + F(3x-1) + F(3x+1)]}{F(3x-2) + F(3x+2) - F(3x-1) - F(3x+1)},$$

valid for any x ensuring non-zero denominator can be used to construct an estimate. Obviously the quality depends on a suitable choice of x. On the other hand, since $E(X) = E(Y) = 0$, then for applying the moment method the second moment has to be considered. Since $9E(X^2) = E(X^2) + E(Y^2)$, then some elementary algebra leads to the estimate

$$\widehat{\alpha} = \frac{4}{n}\sum_{i=1}^{n} X_i^2 - \frac{5}{6}.$$

Consider again a class of $DIL(2; q, p)$ distributions. Given iid observations $X_1, \ldots, X_{2n}$ the problem is to test the hypothesis that they come from the $DIL(2; q, p)$ distribution with a given q. Since no closed formulas for the df's are available no standard test can be used here. Then a possible approach is to simulate n independent generators $Y_1, \ldots, Y_n$ and define $Z_i = (X_{n+i} + Y_i)/2$, $i = 1, 2, \ldots, n$. Then under the null hypothesis the distributions of Z's and X's are the same, hence on the basis of the double sample $X_1, \ldots, X_n$ and $Z_1, \ldots, Z_n$, which are independent, some rank tests can be used to test the equidistribution. If the parameters q and p are unknown one has to estimate them first. Obviously similar approach can be applied for testing that observations come from any dilation distribution.

6. Dilation distributions driven by arbitrary generators

A possible natural extension of the dilation distribution family is by considering arbitrary generators. Then a rv X with the generalized DIL distribution has to be represented in a form

$$X \stackrel{d}{=} \frac{X+Y}{r},$$

for some real number r and a rv Y if only the above formula makes sense, for instance if the chf ϕ_Y of Y has form $\phi_Y(t) = \phi(rt)/\phi(t)$, for some chf ϕ. Then the respective probability dilation equation has the general form

$$\mu_X = \int_{\mathbf{R}} \mu_{(X+y)/r}\, \mu_Y(dy),$$

or

$$F_X(x) = \int_{\mathbf{R}} F_X(rx - y)\, dF_Y(y), \quad x \in \mathbf{R}.$$

Consequently such a generalized dilation distributions family is parameterized by a scale r and a dilation generator distribution μ_Y.

This family is, obviously, much wider than the previous one and includes many important classes of distributions, as normal, stable or a more general family of self-decomposable laws. Observe that the dilation generator of the normal $\mathcal{N}(m, \sigma^2)$ distribution is again normal $\mathcal{N}((r-1)m, (r^2-1)\sigma^2)$. Similarly for the symmetric α stable distribution $S\alpha S(\beta)$ with the chf $\phi(t) = \exp(-\beta|t|^\alpha)$, the dilation generator is $S\alpha S((r^\alpha - 1)\beta)$.

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