

Linearity of convex mean residual life

Jacek Wesołowski^{a,*}, Arjun K. Gupta^b

^a*Wydział Matematyki i Nauk Informacyjnych, Politechnika Warszawska, 00-661 Warszawa, Poland*

^b*Department of Mathematics and Statistics, Bowling Green State University, Bowling Green, Ohio, USA*

Received 30 May 2000; received in revised form 8 September 2000; accepted 2 February 2001

Abstract

It is shown that linearity of convex mean residual life identifies the distribution up to a scale in the class of distributions with zero mean. On this basis new characterizations of the uniform and, unexpectedly, the Student distribution with two degrees of freedom are obtained. It is observed that, strictly speaking, the conjecture of Nagaraja and Nevzerov about uniqueness property of the convex mean residual life time is incorrect. The condition under study is in obvious relations with linearity of regression of observations with respect to order statistics. © 2001 Elsevier Science B.V. All rights reserved.

MSC: 60E05; 62E10; 62E15

Keywords: Mean residual life time; Characterization; Linearity of regression; Order statistics; Uniform distribution; Student distribution

1. Introduction

Let X_1, X_2, \dots, X_n be independent identically distributed random variables (rv's) with a distribution function (df) F and the support $[a, b]$, $-\infty \leq a < b \leq \infty$. Denote by $X_{i:n}$, $i = 1, \dots, n$, the respective order statistics. Since the Ferguson (1967) paper a lot of work has been devoted to determining the distribution of X by linearity of regression $E(X_{i+k:n}|X_{i:n})$ —for the adjacent case (i.e. $k = \pm 1$) in addition to Ferguson (1967) one can consult Nagaraja (1988a, b), Pakes et al. (1996) or the monograph by Arnold et al. (1992), while the non-adjacent case was considered more recently for instance in Wesołowski and Ahsanullah (1997), López-Blázquez and Moreno-Rebollo (1997) or Dembińska and Wesołowski (1998).

Here, we are interested in the linearity of regression of an observation on the i th order statistic, i.e. $E(X_1|X_{i:n})$, which at the first glance seems to be of similar nature as the problems mentioned above. However it appears that a completely new approach,

* Corresponding author.

E-mail addresses: wesolo@alpha.mini.pw.edu.pl (J. Wesołowski), gupta@math.bgsu.edu (A.K. Gupta).

connected with the study of properties of convex mean residual life function (cmrlf) has to be developed and consequently only a partial answer will be given.

As observed in Nagaraja and Nevzerov (1997) (NN in the sequel):

$$E(X_1|X_{i:n} = x) = \frac{x}{n} + \frac{n-1}{n} [\alpha E(X|X \leq x) + (1-\alpha)E(X|X > x)], \quad x \in (a, b),$$

where $\alpha = (i-1)/(n-1)$, and X is rv with the df F , if only the expectation $E(X)$ is finite. The observation lead these authors to investigate the cmrlf: $M_{X,\alpha}$, $\alpha \in [0, 1]$, defined by

$$M_{X,\alpha}(x) = \alpha E(X|X \leq x) + (1-\alpha)E(X|X > x), \quad x \in (a, b)$$

for any rv X with finite expectation. The basic problem is concerned with identification of the distribution of X by the function $M_{X,\alpha}$. It has been completely solved only in two special cases of $\alpha=0$ or 1 —see for instance Kotlarski (1972), Galambos and Kotz (1978) (Chapter 2.3) or more recent contributions as Zoroa et al. (1990), Galambos and Hagwood (1992) or Lillo and Martin (1999) (an application to uniform mixtures).

In NN the following uniqueness result is proved.

Theorem 1. (Nagaraja and Nevzerov (1997).) *Let X and Y be rv's with a common support $[a, b]$, $-\infty \leq a < b \leq \infty$, and continuous df's F and G , respectively. Assume that $E(X)$ exists and $\exists x_0 \in (a, b)$ such that:*

$$E(X|X \leq x_0) = E(Y|Y \leq x_0) \tag{1}$$

and

$$F(x_0) = G(x_0). \tag{2}$$

Then $F = G$ iff $M_{X,\alpha} = M_{Y,\alpha}$, where $\alpha = F(x_0) \in (0, 1)$.

Additionally it is observed in NN (Remark 2) that (2) holds if $E(X)=E(Y)$. In their Remark 3 these authors say: "... while the assumptions (3.4) ((2) in our setting) and (3.5) ((1) in our setting) play a crucial role in the proof above, the key question is whether the cmrlf $M(x)=\alpha E(X|X \leq x) + (1-\alpha)E(X|X > x)$ uniquely identifies c.d.f. F . When $M(x)$ has a simple form, F can be determined explicitly and in such cases, (3.4) and (3.5) will be clearly unnecessary. We conjecture that the characterization does hold even without these assumptions."

This remark was the starting point of our interest in the subject. First, we discover that if M "has a simple form", even with further restrictions on the family of distributions under considerations, the condition (1) can not be omitted in order to pin down the df F . Consequently, strictly speaking, the NN conjecture is incorrect. On the other hand some scale families of distributions are uniquely determined by M of "a simple form", and in such a sense the NN intuition is correct.

In general identifiability of the distribution of X by the function $M_{X,\alpha}$ does not seem to be a trivial task. Here we restrict our attention to linear $M_{X,\alpha}$, first observing that changes of location and scale of the distribution of X preserve the linearity of the cmrlf.

Consequently linear cmrlf's of the form $M_{X,\alpha}(x) = Ax$ for some constant A are studied within the class of zero mean rv's, which appear to be invariant under scaling. In the case of $\alpha = \frac{1}{2}$ (which for the order statistics means the regression of an observation with respect to the sample median) three special scale families of distributions are characterized by choosing concrete values of A : uniform for $A = \frac{1}{2}$, Student t_2 for $A = 1$ and a related one for $A = \frac{2}{3}$. Also a general uniqueness theorem is derived for any $A \geq \frac{1}{2}$, which appears to be the smallest admissible value. If F_A is a df of the rv X with $M_{X,1/2}(x) = Ax$ then it is shown that X has a symmetric distribution, F_A is absolutely continuous, the limit $\lim_{A \rightarrow \infty} F_A = F$ exists and is a df. The formula for F is obtained. Finally, graphs of selected densities of F_A 's are presented.

2. Linear convex mean residual life function

Observe that

$$M_{\gamma X + \delta, \alpha}(x) = \gamma M_{X, \alpha}((x - \delta)/\gamma) + \delta$$

if $\gamma > 0$ or

$$M_{\gamma X + \delta, \alpha}(x) = -\gamma M_{-X, \alpha}((\delta - x)/\gamma) + \delta$$

if $\gamma < 0$. Since for a continuous type rv X it follows that

$$M_{X, \alpha}(x) = -M_{-X, 1-\alpha}(-x),$$

then if $\gamma < 0$ one has

$$M_{\gamma X + \delta, \alpha}(x) = \gamma M_{X, 1-\alpha}((x - \delta)/\gamma) + \delta.$$

Consequently if $M_{X, \alpha}$ is linear then $M_{\gamma X + \delta, \alpha}$ is also linear for any $\gamma \neq 0$ and any real δ . Moreover, if $M_{X, \alpha}(x) = Ax$, $x \in [a, b]$, then it follows by the above formulas that $M_{\gamma X, \alpha}(x) = Ax$, $x \in [\gamma a, \gamma b]$.

Observe that, following the definition $M = M_{X, \alpha}$ can be written as

$$M(x) = \frac{1 - \alpha}{1 - F(x)}c + \frac{\alpha - F(x)}{F(x)(1 - F(x))} \int_a^x t dF(t), \quad x \in (a, b),$$

where $c = E(X)$. Consequently:

$$\frac{F(x)(1 - F(x))}{\alpha - F(x)}M(x) - c(1 - \alpha)\frac{F(x)}{\alpha - F(x)} = \int_a^x t dF(t), \quad x \in (a, b)$$

and taking differentials one gets:

$$[(1 - 2F(x))(\alpha - F(x))M(x) + F(x)(1 - F(x))M(x) - c\alpha(1 - \alpha) - x(\alpha - F(x))^2] dF(x) = -F(x)(1 - F(x))(\alpha - F(x)) dM(x). \tag{3}$$

Let's restrict further investigation to X 's such that $E(X) = 0$ and $M_{X, \alpha}$ of the form:

$$M_{X, \alpha}(x) = Ax, \quad x \in \text{supp}(X).$$

Then (3) takes the form

$$\frac{A^{-1}(\alpha - F)^2 - (1 - 2F)(\alpha - F) - F(1 - F)}{F(1 - F)(\alpha - F)} dF = \frac{1}{x} dx, \quad x \in (a, b).$$

Integrating both sides, while remembering that F is a df, one gets

$$F^{\alpha/A-1}(x)(1 - F(x))^{(1-\alpha)/A-1}(\alpha - F(x)) = -Kx, \quad x \in (a, b) \tag{4}$$

for any $x \in \text{supp}(X)$ and for any positive constant K . Hence $F(0) = \alpha$. Observe that taking the limits as $x \rightarrow a$ or $x \rightarrow b$ in (4) we get that $a = -\infty = -b$ iff $A > \max\{\alpha, 1 - \alpha\}$ and $-\infty < a < b < \infty$ iff $\alpha = \frac{1}{2}$ and $A = \frac{1}{2}$. Note that the above equation implies that $A < \max\{\alpha, 1 - \alpha\}$ is impossible, unless X is degenerated at zero. Observe also that by changing X into KX one obtains (4) with $K = 1$. Finally, the argument given above can be repeated backwards starting with (4) and ending with $M_{X,\alpha}(x) = Ax, x \in [a, b]$. Consequently (4) is equivalent to the linearity of cmrlf in the support of X .

Starting from this point on we concentrate on the case of $\alpha = \frac{1}{2}$, since this is the only value of α for which we are able to complete a thorough analysis of the family of distributions characterized by linear cmrlf, including explicit form of cdfs for some special values of A .

Now denoting $g(x) = -(1 - 2F(x)), x \in [a, b]$, we get

$$g(x) = x(1 - g^2(x))^B, \quad x \in [-a, a], \quad 0 < a \leq \infty, \tag{5}$$

where $B = 1 - 1/(2A) \in [0, \infty)$, which is equivalent to (4) with the constant K fixed at $K = 2^{1-1/A}$ —in general we should have $2^{1/A-1}Kx$ at the rhs of (5). Fixing the value of K will allow us to have the unique solution of (4), which results in the uniqueness of the respective scale families for the original problem. First we list some special cases in which explicit solutions can be written.

Theorem 2. Assume that X is a rv with zero mean and $\text{supp}(X) = [a, b], -\infty \leq a < b \leq \infty$. If

$$M_{X,1/2}(x) = Ax, \quad x \in (a, b),$$

then $A \geq \frac{1}{2}$ and under a change of scale

(i) if $A = \frac{1}{2}$ then X has the df

$$F(x) = \frac{1}{2}(1 + \min\{\max\{-1, x\}, 1\}),$$

i.e. X has a uniform type distribution in $[-1, 1]$;

(ii) if $A = 1$ then X has the df

$$F(x) = \frac{1}{2} \left(1 + \frac{x}{\sqrt{1+x^2}} \right),$$

i.e. X has a Student t_2 type distribution;

(iii) if $A = \frac{2}{3}$ then X has the df

$$F(x) = \frac{1}{2} \left(1 + \frac{\sqrt{2}x}{\sqrt{\sqrt{4+x^4}+x^2}} \right).$$

Proof. The argument following (4) implies $A \geq \frac{1}{2}$. Since (5) is equivalent to linearity of cmrlf in the class of zero-mean rv's, the proof is based on solving (5) in special cases.

For $A = \frac{1}{2}$ we get $B = 0$ and (5) implies that

$$g(x) = x, \quad x \in [-a, a].$$

Hence $F(x) = 0.5(1+x)$ for $x \in [-a, a]$. Now by properties of the df it follows that $a = 1$, which says that X is uniform in $[-1, 1]$. Observe that this is the only case with finite a . In all the remaining cases, by the remark preceding the formulation of the theorem it follows that $a = \infty$.

Consider now the case of $A = 1$, i.e. $B = \frac{1}{2}$. Then (5) implies that for any $x \in \mathbf{R}$

$$g^2(x) = x^2(1 - g^2(x)).$$

This equation can be easily solved for any $x \in \mathbf{R}$, and in general the solution has the form

$$g(x) = \pm \frac{x}{\sqrt{1+x^2}}, \quad x \in \mathbf{R}.$$

Since g has to be a non-decreasing function it follows that the positive sign in the above solution has to be chosen, whence assertion (ii).

Now for $A = \frac{2}{3}$ we get $B = \frac{1}{4}$ and by (5) it follows that:

$$g^4(x) = x^4(1 - g^2(x)), \quad x \in \mathbf{R}.$$

This equation of the fourth order in g is easily solvable, by solving it for $g^2(x)$. After taking care of the properties of g following from the representation $F = (1+g)/2$ the only solution has the form:

$$g(x) = \frac{\sqrt{2}x}{\sqrt{\sqrt{4+x^4}+x^2}}, \quad x \in \mathbf{R},$$

which is the case (iii).

To end the proof it suffices to observe that in each of the cases $E(X)$ exists and is equal to zero. \square

While we are unable to provide closed formulas for other values of A , the uniqueness up to a change of scale is ensured by

Theorem 3. *In the family of distributions with zero mean the cmrlf of the form*

$$M_{X,1/2}(x) = Ax, \quad x \in \mathbf{R}, \tag{6}$$

uniquely determines the distribution of X up to a scaling factor.

Proof. Since (6) is equivalent to (4), and, after fixing the constant K in (4) at $K = 2^{1-1/A}$, to (5), we can restrict our attention to studying the uniqueness of probabilistic solutions of (5), i.e. solutions g which define a df by the formula $F = (1 + g)/2$.

Observe first that if g is a solution then h defined by $h(x) = -g(-x)$ also solves (5), since

$$h(x) = -(-x)(1 - g^2(-x))^B = x(1 - (-g(-x))^2)^B = x(1 - h^2(x))^B.$$

Consequently it suffices to consider (5) only for non-negative x 's and for $g(x) \in [0, 1]$. Of course, $g(0) = 0$. Further, if we have two solutions for some $x > 0$, say g_1 and g_2 then by (5) we get

$$g_1(1 - g_1^2)^B = g_2(1 - g_2^2)^B.$$

But the function $H : [0, 1) \rightarrow [0, \infty)$ defined by

$$H(x) = x/(1 - x^2)^B, \quad x \in [0, 1),$$

is strictly increasing (check for instance the derivative), which implies that $g_1 = g_2$. Since g is an odd function then $E(X) = 0$ and the unique g satisfying (5) determines the distribution of X . \square

Remark. If additionally the value of $E(X|X \leq x_0)$ is fixed then the unique representative of the scale family is determined (i.e. the unique value of K in (4))—see (1) in the NN theorem. This observation in our setting, i.e. $x_0 = 0$, $F(x_0) = \alpha = \frac{1}{2}$, and $E(X) = 0$, follows from:

$$E(X|X \leq x_0) = \lim_{x \rightarrow x_0} \frac{M(x)F(x)}{\alpha - F(x)} = \lim_{x \rightarrow 0} \frac{Ax}{1 - 2F(x)}$$

and by (4)

$$E(X|X \leq 0) = \lim_{x \rightarrow 0} \frac{Ax}{-Kx[F(x)(1 - F(x))]^{1/(2A)-1}} = -\frac{A}{K}2^{1/A-2}.$$

For instance if in the uniform case, i.e. for $A = \frac{1}{2}$, one has $E(X|X \leq 0) = -L/2$ for some positive constant L , then $K = 1/L$, which means that X has the uniform distribution in $(-L, L)$.

Since $M_{X,1/2}(x) = Ax$ uniquely determines the distribution of X (up to a change of scale) we will relate g 's, F 's and probability measures μ 's to the slope A of the cmrlf by introducing A as a subscript. To have a unique representative of each of the scale families involved we denote by g_A , F_A , X_A and μ_A the versions of respective notions defined by (5), i.e. with $K = 2^{1-1/A}$.

Observe that if we take $B = 1$ (i.e. the impossible case of $A = \infty$) in (5) the resulting quadratic equation for g_∞ brings the solution:

$$g_\infty(x) = \frac{2x}{1 + \sqrt{1 + 4x^2}}, \quad x \in \mathbf{R}.$$

Consequently the df has the form:

$$F_\infty(x) = \frac{1}{2} \left(1 + \frac{2x}{1 + \sqrt{1 + 4x^2}} \right), \quad x \in \mathbf{R}.$$

Consider now the family $\mathcal{M} = \{\mu_A : A \in [1/2, \infty]\}$ of probability measures in $(\mathbf{R}, \mathcal{B}(\mathbf{R}))$.

Theorem 4. *All the probability measures in \mathcal{M} are absolutely continuous and symmetric. The map $A \rightarrow \mu_A$ from $[\frac{1}{2}, \infty]$ onto \mathcal{M} is continuous with respect to weak convergence of probability measures. The probability distributions $(\mu_A)_{A \in [1/2, \infty]}$ truncated to $[0, \infty)$ are stochastically decreasing with A . For $A > \frac{1}{2}$ the moment of the order $2A/(2A - 1)$ does not exist, and any moment of the order $< 2A/(2A - 1)$ is finite.*

Proof. Observe that (5) defines the function $x = x(g)$ mapping $(-1, 1)$ onto \mathbf{R} (or $(-1, 1)$ for $B = 0$) by

$$x(g) = g/(1 - g^2)^B, \quad g \in (-1, 1).$$

Since the function $x = x(g)$ is strictly increasing and differentiable then its inverse, equal to g (we checked it in the proof of Theorem 3) is also differentiable on \mathbf{R} (or $(-1, 1)$). Hence F is absolutely continuous.

Let us observe first that $(\mu_A)_{A \in [1/2, \infty]}$ is stochastically decreasing. To this end let us take any $B_1 = 1 - 1/(2A_1) < B_2 = 1 - 1/(2A_2)$, (i.e. $A_1 > A_2$). Assume that $g_{A_1}(x) < g_{A_2}(x)$ for some $x \in \mathbf{R}_+$. Then by (5)

$$\begin{aligned} g_{A_1}(x) &= x(1 - g_{A_1}^2(x))^{B_1} \geq x(1 - g_{A_2}^2(x))^{B_1} \\ &= x(1 - g_{A_2}(x))^{(B_1/B_2)B_2} \geq x(1 - g_{A_2}(x))^{B_2} = g_{A_2}(x), \end{aligned}$$

which is contradictory. Consequently $g_{A_1}(x) \geq g_{A_2}(x)$ for any $x \in \mathbf{R}_+$ and $F_{A_2}(x) \geq F_{A_1}(x)$ for any $\infty \geq A_1 > A_2 \geq \frac{1}{2}$.

Now to show the continuity of μ_A with respect to the weak convergence it is enough to show that at any point $x \in \mathbf{R}_+$ if $A_n \rightarrow A$ as $n \rightarrow \infty$, $A, A_1, A_2, \dots \in [\frac{1}{2}, \infty]$ one has $F_{A_n}(x) \rightarrow F_A(x)$. Take an increasing sequence $A_n \uparrow A$, then $F_{A_n}(x)$ is a decreasing sequence by the stochastic monotonicity property. Since it is bounded from below by 0, it has to converge. On the other hand respective $g_{A_n}(x)$'s fulfill (5) with $B = B_n = 1 - 1/(2A_n)$. And since they are convergent, the limit has also to fulfil (5) with $B = 1 - 1/(2A)$. Consequently $\lim_{n \rightarrow \infty} g_{A_n}(x) = g_A(x)$ for any $x \in \mathbf{R}$ by the uniqueness property of Theorem 3. A similar argument is valid for the decreasing sequence $A_n \downarrow A$. Since in both the cases the limits are equal the result follows.

The last assertion about the moments follows immediately if one rewrites (4) for positive x 's as:

$$x^{-\varepsilon}(2F(x) - 1)^{1/B} = K^{1/B} x^{1/B-\varepsilon} F(x)(1 - F(x)),$$

where ε is any number in $[0, 1/B)$. \square

Remark. Graphs of selected densities of X_A 's are presented in Fig. 1. The picture shows that the changes are quite visible with A moving first slowly from the lower boundary $A = \frac{1}{2}$ (the uniform $[-1, 1]$ distribution) to the right, but the changes become

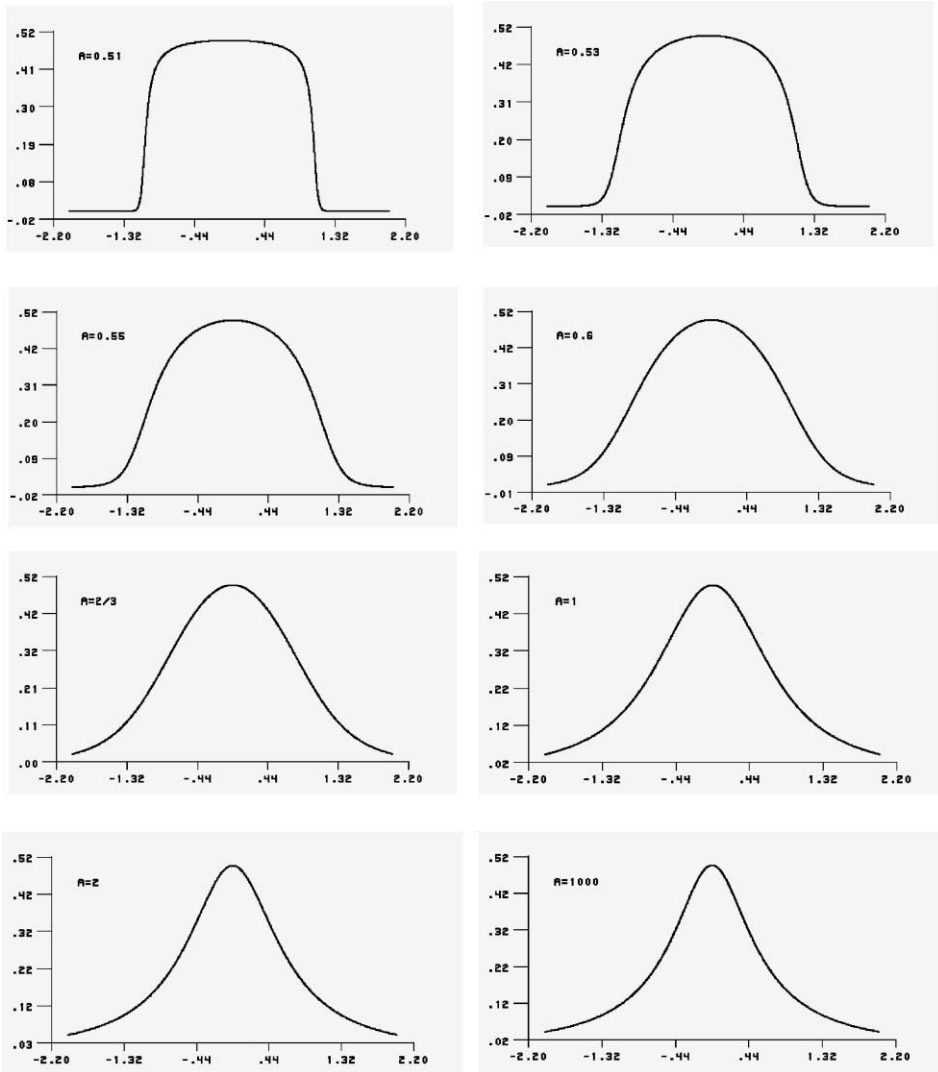


Fig. 1. The graphs of selected densities from the class M .

smaller and smaller as A grows up to 1 or 2, and almost invisible for large values of A —for instance the graphs for $A = 1000$ and any larger A , also $A = \infty$, are essentially indistinguishable, while different, of course.

Acknowledgements

Referee’s remarks helped a lot in improving the presentation—the one on moments of F_∞ was developed into the last assertion of Theorem 4.

References

- Arnold, B.C., Balakrishnan, N., Nagaraja, H.N., 1992. *A First Course in Order Statistics*, Wiley, New York.
- Dembińska, A., Wesołowski, J., 1998. Linearity of regression for non-adjacent order statistics. *Metrika* 48, 215–222.
- Ferguson, T.S., 1967. On characterizing distributions by properties of order statistics. *Sankhyā Ser. A* 29, 265–278.
- Galambos, J., Hagwood, C., 1992. The characterization of a distribution function by the second moment of the residual life. *Comm. Statist.—Theory Meth.* 21(5), 1463–1468.
- Galambos, J., Kotz, S., 1978. *Characterizations of Probability Distributions*, Lecture Notes in Mathematics, vol. 675. Springer, New York.
- Kotlarski, I., 1972. On a characterization of some distributions by conditional expectation. *Sankhyā Ser. A* 34, 461–467.
- Lillo, R.E., Martin, M., 1999. Characterization of Bayes estimators of the uniform scale parameter. *Statist. Decisions* 17, 31–47.
- López-Blázquez, F., Moreno-Rebollo, J.L., 1997. A characterization of distributions based on linearity of regression for order statistics and record values. *Sankhyā Ser. A* 59, 311–323.
- Nagaraja, H.N., 1988a. Some characterizations of discrete distributions based on linear regressions of adjacent order statistics. *J. Statist. Planning Inference* 20, 65–75.
- Nagaraja, H.N., 1988b. Some characterizations of continuous distributions based on regressions of adjacent order statistics and record values. *Sankhyā Ser. A* 50, 70–73.
- Nagaraja, H.N., Nevzerov, V.B., 1997. On characterizations based on record values and order statistics. *J. Statist. Planning Inference* 63, 271–284.
- Pakes, A.G., Fakhry, M.E., Mahmoud, M.R., Ahmad, A.A., 1996. Characterizations by regressions of order statistics. *J. Appl. Statist. Sci.* 3, 11–23.
- Wesołowski, J., Ahsanullah, M., 1997. On characterizing distributions via linearity of regression for order statistics. *Austral. J. Statist.* 39, 69–78.
- Zoroa, P., Ruiz, J.M., Marin, J., 1990. A characterization based on conditional expectations. *Comm. Statist.—Theory Meth.* 19(8), 3127–3135.