BIULETYN WAT
VOL. LI, NR 4 (2002)

Bivariate Pareto conditionals distributions revisited

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Abstract. A new bivariate Pareto conditional distribution is discovered by considering a common constant scale parameter in conditional distributions. Also regression specifications of this measure involving one Pareto conditional are discussed. Linear scale parameters lead to a new characterizations of the Mardia bivariate Pareto distribution.
Keywords: conditional specification, conditional distribution, bivariate probability distribution, Pareto conditional distribution, Mardia bivariate Pareto distribution, bivariate Pareto conditionals distribution, regression function
AMS 2000 subject classification: 62H05, 62E10

1. Introduction

Let (X, Y) be a random vector with normal conditionals, i.e. $\mu_{X|Y} = \mathcal{N}(m_1(Y), \sigma_1(Y))$ and $\mu_{Y|X} = \mathcal{N}(m_2(X), \sigma_2(X))$, for some functions m_1, m_2, $\sigma_1 > 0, \sigma_2 > 0$. Then the form of the joint density was determined by Castillo and Galambos (1989) (see also Gelman and Meng (1991)). Many results of this kind involving other conditional distributions, scattered in numerous papers, are gathered in a recent monograph by Arnold et al. (1992). In this paper we study Pareto conditionals case.

Denote by $\mathcal{P}(\sigma, \alpha)$ the Pareto distribution with the density function

$$f(x) = \begin{cases} \frac{\alpha\sigma^\alpha}{(\sigma+x)^{\alpha+1}} & x > 0 \\ 0 & x \leq 0. \end{cases}$$

We say that a random vector (X, Y) has a bivariate Pareto conditionals distribution if

$$\mu_{X|Y} = \mathcal{P}(\sigma_1(Y), \alpha_1(Y)), \qquad \mu_{Y|X} = \mathcal{P}(\sigma_2(X), \alpha_2(X)), \tag{1}$$

where α's and σ's are some positive functions.

An example of such a measure is the Mardia bivariate Pareto distribution with the density

$$f(x,y) = \begin{cases} \frac{p(p+1)ab}{(ax+by+1)^{p+2}}, & x > 0,\ y > 0, \\ 0 & \text{otherwise} \end{cases}$$

where a, b, p are some positive numbers. This distribution was introduced in Mardia (1962) and proved to be very useful in many applications. It is not difficult to check that in this case

$$\mu_{X|Y} = \mathcal{P}\left(\frac{bY+d}{a}, p+1\right)$$

and

$$\mu_{Y|X} = \mathcal{P}\left(\frac{aX+d}{b}, p+1\right).$$

Other bivariate Pareto conditionals distributions are also known. However there are no satisfactory characterizations of the whole family. In Arnold (1987), with a slight generalization by Castillo and Sarabia (1990), it was proved that any bivariate distribution fulfilling (1) with constant α's ($\alpha_1(Y) = \alpha_2(X) = \alpha$) has the density of the form

$$f(x,y) = \frac{K}{(ax+by+cxy+d)^{\alpha+1}}, \quad x > 0,\ y > 0 \tag{2}$$

$f(x,y) = 0$ otherwise, K is a normalizing constant, $a > 0$, $b > 0$, $c \geq 0$, $d \geq 0$; if $d = 0$ then $\alpha \in (0,1)$, if $c = 0$ then $\alpha \in (1, \infty)$. Then

$$\sigma_1(Y) = \frac{d+bY}{a+cY}, \qquad \sigma_2(X) = \frac{d+aX}{b+cX}.$$

Observe that for $c = 0$ we obtain the Mardia distribution. This distribution was also characterized in the book Arnold et al. (1992), Section 7.2, by (1) and linearity of regressions $E(X|Y)$ and $E(Y|X)$.

Other conditional specification involving Paretian conditional structure are the following: In Arnold et al (1992), Section 8.5, a multivariate distribution with all univariate Pareto conditionals was considered while multivariate measures with all bivariate Pareto conditionals has been recently treated in Arnold et al. (1993). Another kind of conditional specification featuring one univariate Pareto conditional and equidistribution of some marginals was given in Arnold and Pourahmadi (1988), developed recently by Wesolowski and Ahsanullah (1995). In Wesolowski (1995) specifications of bivariate measures involving one Pareto conditional and a regression function were studied.

Castillo and Sarabia (1990) (Corollary 4.1) claimed that (2) is a density of the most general bivariate probability measure with Pareto conditionals. However it is not the case. In Section 2 studying bivariate Pareto conditonals distributions with constant common σ's we introduce a new bivariate distribution with Paretian conditionals whose density is not of the form (2). In Section 3 regression conditions together with one Pareto conditional are considered. A new characterization of the Mardia bivariate Pareto distribuion is given in Section 4. An extension of all the results to the bivariate second kind beta conditionals distribution seems to be possible.

2. Pareto conditionals with constant common scale

In Arnold (1987) bivariate measures fulfilling (1) with any positive functions σ's and constant common α's were considered. In this section we investigate a parellel problem assuming common constant σ's while α's are any positive functions.

Theorem 1. *Let (X,Y) be a random vector fulfilling (1) with*

$$\sigma_1(Y) = \sigma_2(X) = \sigma,$$

where σ is a positive constant. Then the joint density f of (X,Y) has the form

$$f(x,y) = \frac{K\exp[-b\ln(\sigma+x)\ln(\sigma+y)]}{(\sigma+x)^{a_1}(\sigma+y)^{a_2}} \tag{3}$$

for $x>0$, $y>0$, where K is a normalizing constant, $b\geq 0$, $a_i + b\ln(\sigma) > 1$, $i=1,2$.

Proof. By the elementary property of conditional distributions

$$f(x,y) = f_{X|Y=y}(x)f_1(y) = f_{Y|X=x}(y)f_2(x), \quad x>0,\ y>0,$$

where $f_{X|Y}$, f_1, $f_{Y|X}$, f_2 are densities of the conditional distribution $\mu_{X|Y}$, marginal distribution of Y, conditional distribution $\mu_{Y|X}$, marginal distribution of X, respectively. Hence

$$\frac{\alpha_1(y)\sigma^{\alpha_1(y)}}{(\sigma+x)^{\alpha_1(y)+1}}f_1(y) = \frac{\alpha_2(x)\sigma^{\alpha_2(x)}}{(\sigma+y)^{\alpha_2(x)+1}}f_2(x), \quad x>0,\ y>0. \tag{4}$$

By (4) f_1 and f_2 are positive functions. Consequently we can take logarithms of both sides of this formula. Denote

$$\beta_i = \alpha_i \sigma^{\alpha_i} f_i, \quad \gamma_i = \alpha_i + 1, \quad i = 1, 2.$$

Then (4) yields

$$\beta_1(y) - \gamma_1(y) \ln(\sigma + x) = \beta_2(x) - \gamma_2(x) \ln(\sigma + y), \quad x > 0, \; y > 0. \tag{5}$$

Fix an arbitrary y_0 and compute β_2. From (5) we get

$$\gamma_2(x) \ln\left(\frac{\sigma + y_0}{\sigma + y}\right) = \beta_1(y) - \beta_1(y_0) - [\gamma_1(y) - \gamma_1(y_0)] \ln(\sigma + x), \quad x > 0, \; y > 0. \tag{6}$$

Now put in (6) an arbitrary fixed x_0 and compute β_1. Coming back to (5) we obtain

$$[\gamma_1(y) - \gamma_1(y_0)] \ln\left(\frac{\sigma + y_0}{\sigma + y}\right) = [\gamma_2(x) - \gamma_2(x_0)] \ln\left(\frac{\sigma + x_0}{\sigma + x}\right), \quad x > 0, \; y > 0.$$

Hence

$$\gamma_i(x) = a_i + 1 + b_i \ln(\sigma + x), \quad x > 0, \; i = 1, 2$$

and from (6) and then (5) we get

$$\beta_i(x) = A_i + B_i \ln(\sigma + x), \quad x > 0, \; i = 1, 2,$$

where A_i, B_i, a_i, b_i are some real constants. Consequently (4) yields

$$\frac{e^{A_1}(\sigma + y)^{B_1}}{(\sigma + x)^{a_1 + b_1 \ln(\sigma + x)}} = \frac{e^{A_2}(\sigma + x)^{B_2}}{(\sigma + y)^{a_2 + b_2 \ln(\sigma + y)}}, \quad x > 0, \; y > 0.$$

Hence $A_1 = A_2$, $B_1 = -a_1$, $B_2 = -a_2$ and $b_1 = b_2 = -b$. Now the integrability of f implies $b > 0$ and $a_i + b \ln \sigma > 1$, $i = 1, 2$. □

In Theorem 1 we introduced a new version of bivariate Poisson conditionals distribution with the density (3). The conditional distributions are

$$\mu_{X|Y} = \mathcal{P}(\sigma, a_1 + b \ln(\sigma + Y))$$

and

$$\mu_{Y|X} = \mathcal{P}(\sigma, a_2 + b \ln(\sigma + X)).$$

The marginal densities have the form

$$f_i(x) = \frac{K_i}{[a_i - 1 + b\ln(\sigma + x)](\sigma + x)^{a_j + b\ln(\sigma)}}, \quad x > 0,$$

where $K_i = K\sigma^{1-a_i}$ and $i,j = 1,2,\ i \neq j$. In the constants involved a non-elementery function — integral exponent Ei — appears:

$$K = \begin{cases} b\exp\left(\frac{(a_1-1)(a_2-1)}{b}\right)/c(a_1,a_2,b,\sigma) & b > 0 \\ (a_1-1)(a_2-1)\sigma^{a_1+a_2-2} & b = 0 \end{cases},$$

where

$$c(a_1,a_2,b,\sigma) = -Ei\left(-\frac{(a_1 + b\ln(\sigma) - 1)(a_2 + \ln(\sigma) - 1)}{b}\right).$$

Observe that for the random vector (X,Y) with the density (3) we have

$$E(X|Y) = \frac{\sigma}{a_1 - 1 + b\ln(\sigma + Y)}$$

and

$$E(Y|X) = \frac{\sigma}{a_2 - 1 + b\ln(\sigma + X)}.$$

Hence we can easily identify the case of independence by regression functions.

Corollary 1. *Let (X,Y) be a random vector with bivariate Pareto conditionals (1) with constant common σ's. If $E(X|Y)$ or $E(Y|X)$ are constant then X and Y are independent Pareto r. v's.*

Proof. By Theorem 1 the joint density is given in (3) and the regressions are given above. Hence the constancy of any of regressions implies $b = 0$ and the result follows immediately from (3). □

3. One Pareto conditional and regressions

Now we consider a random vector (X, Y) with one conditional as in the case of the bivariate Pareto conditionals distribution introduced in Theorem 1, i.e.

$$f_{Y|X=x}(y) = \frac{[a_2 + b\ln(\sigma + x)]\sigma^{a_2+b\ln(\sigma+x)}}{(\sigma + y)^{a_2+1+b\ln(\sigma+x)}}, \quad y > 0,\ x \in S_X \tag{7}$$

or equivalently

$$\mu_{Y|X} = \mathcal{P}(\sigma, a_2 + b\ln(\sigma + X)), \tag{8}$$

where σ, b are some positive numbers and $a_2 + b\ln(\sigma) > 0$, $S_X = supp(X) \subset [0, \infty)$. Our aim is to study the uniqueness of the bivariate distribution in the case when $E(X|Y)$ is known. The same question for $\mu_{Y|X}$ as in the Arnold bivariate Pareto conditionals distribution was investigated in Wesolowski (1994), where also a short review of numerous results on characterization of bivariate distributions by a conditional distribution and a regression function is given.

Theorem 2. *Let (X, Y) be a random vector fulfilling (8). Then its distribution is uniquely determined by $E(X|Y)$.*

Proof. Denote by m the regression function of X given Y:

$$m(y) = E(X|Y = y) = \int_{S_X} x dF_{X|Y=y}(x), \quad y > 0,$$

where $F_{X|Y=y}$ is the conditional distribution function. From (7) it follows that Y has a density, say f_Y. Then an elementary identity

$$f_Y(y)dF_{X|Y=y}(x) = f_{Y|X=x}(y)dF_X(x),$$

where F_X denotes the distribution function of X, yields

$$m(y)\int_{S_X} f_{Y|X=x}(y)dF_X(x) = \int_{S_X} x f_{Y|X=x}(y)dF_X(x), \quad y > 0. \tag{9}$$

Hence for any $y > 0$

$$m(y)\int_{S_X} \frac{a_2 + b\ln(\sigma + x)}{(\sigma + x)^{b\ln(\frac{\sigma+y}{\sigma})}} dF_X(x) = \int_{S_X} x\frac{a_2 + b\ln(\sigma + x)}{(\sigma + x)^{b\ln(\frac{\sigma+y}{\sigma})}} dF_X(x). \tag{10}$$

Define a distribution function H by the formula

$$dH(x) = c^{-1}[a_2 + b\ln(\sigma + x)]dF_X(x), \quad x \in S_X,$$

where

$$c = \int_{S_X} [a_2 + b\ln(\sigma + x)] dF_X(x).$$

Let $Z = (\sigma + U)^{-1}$, where U is a r.v. with the d.f. H. Then (9) implies

$$[m(\sigma e^{t/b} - \sigma) + \sigma]E(Z^t) = E(Z^{t-1}), \quad t > 0. \tag{11}$$

Consequently for any $k = 1, 2, \ldots$

$$E(Z^k) = \prod_{i=1}^{k} \frac{1}{m(\sigma e^{i/b} - \sigma) + \sigma}.$$

Since $0 < Z < 1$ a.s. then its distribution is uniquely determined by the sequence $(E(Z^k))_{k=1,2,\ldots}$. Hence the d.f. H is uniquely determined by the regression function m. Now to obtain F_X it suffices to compute the constant c:

$$c = \int_{S_X} \frac{dH(x)}{a_2 + b\ln(\sigma + x)}.$$

□

By Theorem 2 bivariate distribution can be specified by the conditional Pareto distribution of the form (8) and a regression function. Now we use this result to characterize the bivariate Pareto conditionals distribution introduced in the beginning of this section.

Corollary 2. *Let (X, Y) be a random vector fulfilling* (8) *and*

$$E(X|Y) = \frac{\sigma}{a_1 - 1 + b\ln(\sigma + Y)},$$

where b, σ are positive numbers, $a_1 + b\ln(\sigma) > 1$ and $a_2 + b\ln(\sigma) > 0$. Then (X, Y) has the bivariate Pareto conditionals distribtution with the density (3).

Proof. It is an immediate consequence of Theorem 2. □

Now we are going to study the regression of the inverse of a shift of X instead of the conditional mean. It appears that using quite similar method to that applied in the proof of Theorem 2 we can also obtain a uniqueness of the bivariate distribution.

Theorem 3. *Let (X,Y) be a random vector fulfilling* (8). *Then its distribution is uniquely determined by* $E[(\sigma+X)^{-1}|Y]$.

Proof. Adopting the notations introduced in the course of the proof of Theorem 2 and additionally denoting

$$n(y)=E[(\sigma+X)^{-1}|Y=y),\quad y>0,$$

we have

$$n(\sigma e^{t/b}-\sigma)E(Z^t)=E(Z^{t+1}),\quad t\geq 0.$$

Consequently for any $k=1,2,\ldots$

$$E(Z^k)=\prod_{i=0}^{k-1} n(\sigma e^{i/b}-\sigma).$$

Now similarly as in the proof of Theorem 2 we conclude that F_X is uniquely determined by the function n and (8). □

Observe that for a r.v. X with the Pareto $\mathcal{P}(\sigma,\alpha)$ distribution $E(\frac{\sigma}{\sigma+X})=\frac{\alpha}{\alpha+1}$. This yields for the random vector (X,Y) with the density (3) the following formula

$$E((\sigma+X)^{-1}|Y)=\frac{a_1+b\ln(\sigma+Y)}{\sigma(a_1+1+\ln(\sigma+Y)}. \tag{12}$$

Hence Theorem 3 gives the following chracterization:

Corollary 3. *Let (X,Y) be a random vector fulfilling* (8) *and* (11), *where b, σ are positive numbers, $a_1+1+\ln\sigma>0$ and $a_1+\ln(\sigma)>0$.*
Then it has the bivariate Pareto conditionals distribution with the density (3). □

4. Mardia bivariate Pareto distribution

In the case of the Mardia bivariate Pareto distribution both the conditionals are Paretian with linear σ's and common constant α's — see Section 1. Conversely, if we assume that both the conditionals are Pareto with linear σ's and common constant α's then the joint distribution is of the Mardia type. However if we impose the condition only for α's (to be common and constant) allowing σ's to be any positive functions then the joint distribution belongs to a wider family of bivariate Pareto conditionals measures. These results are given in Arnold et al. (1992). In this section we treat a complementary question of bivariate Pareto conditionals distribution with linear σ's and α's being any positive functions. This problem leads to a new characterization of the Mardia bivariate Pareto distribution.

Theorem 4. *Let (X, Y) be a random vector such that*

$$\mu_{X|Y} = \mathcal{P}((bY+1)/a, \alpha_1(Y)), \quad \mu_{Y|X} = \mathcal{P}((aX+1)/b, \alpha_2(X)), \tag{13}$$

where a and b are some positive numbers.
Then the joint density of (X, Y) is Mardia bivariate Pareto, i.e. $\alpha_1(Y) = \alpha_2(X) = \alpha$ a.s. for some positive number α.

Proof. We use the same notations as in the proof of Theorem 1. Similarly as there we have to solve the following equation

$$\frac{a\alpha_1(y)(by+1)^{\alpha_1(y)}}{(ax+by+1)^{\alpha_1(y)+1}} f_1(y) = \frac{b\alpha_2(x)(ax+1)^{\alpha_2(x)}}{(ax+by+1)^{\alpha_2(x)+1}} f_2(x), \quad x > 0, \ y > 0. \tag{14}$$

Denote

$$\beta_1(y) = a\alpha_1(y)(by+1)^{\alpha_1(y)} f_1(y), \quad y > 0,$$

$$\beta_2(x) = b\alpha_2(x)(ax+1)^{\alpha_2(x)} f_2(x), \quad x > 0,$$

$$\gamma_i = \alpha_i + 1, \quad i = 1, 2.$$

Then (12) implies

$$\delta_1(y) - \gamma_1(y)\ln(ax+by+1) = \delta_2(x) - \gamma_2(x)\ln(ax+by+1), \quad x > 0, \ y > 0. \tag{15}$$

Fixing two arbitrary different x's in (13) and subtracting the eqautions we conclude that γ_1 is differentiable infinitely many times. Hence the same fact holds for all the other functions involved in (13). We differentiate (13) with respect to x and then y obtaining

$$ab\gamma_1(y) - a(ax+by+1)\gamma_1'(y) = ab\gamma_2(x) - b(ax+by+1)\gamma_2'(x), \quad x > 0, \ y > 0. \tag{16}$$

Again differentiate with respect to x and then y the above equation. Hence

$$a^2\gamma_1''(y) = b^2\gamma_2''(x), \quad x > 0, \ y > 0$$

and thus

$$\gamma_i(x) = A_i x^2 + B_i x + C_i, \quad x > 0, \ i = 1, 2.$$

Now (14) implies $A_i = B_i = 0$, $i = 1, 2$ and $C_1 = C_2 = p+1$, say. Consequently (X, Y) hast the Mardia bivariate Pareto distribution with the density given in Section 1. □

In the above proof we used some ideas from James (1975), where bivariate measures with beta conditionals were investigated. Similar method may be applied in studying bivariate second kind beta conditionals distribution.

Characterizations of the Mardia bivariate Pareto distribution by one Pareto conditional of the form (12) with constant α and linearity of regression is given in Wesołowski (1995).

Received June 26, 2000; revised December 31, 2001.

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J. ПУШ J. ВЕСОЛОВСКИ

Характеризации двумерных распределений Парето

Резюме. В работе определяется двумерное распределение Парето посредством условных характеристик. Это касается обоих условных распределений или одного условного распределения и функции регресии. Двумерное распределение Парето типа Мардии характеризуется посредством линейности параметров масштаба в условных распределениях.

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Charakteryzacje dwuwymiarowych rozkładów Pareto

Streszczenie. W pracy określono nowy dwuwymiarowy rozkład Pareto za pomocą charakterystyk warunkowych. Dotyczy to obu rozkładów warunkowych lub jednego rozkładu warunkowego i funkcji regresji. Dwuwymiarowy rozkład Pareto typu Mardii scharakteryzowany jest poprzez liniowość parametrów skali w rozkładach warunkowych.

Słowa kluczowe: rozkład Pareto, rozkład warunkowy
Symbole UKD: 519.2