Journal of Theoretical Probability, Vol. 15, No. 1, January 2002 (© 2002)

Central Limit Theorems for Random Permanents with Correlation Structure

Grzegorz A. Rempała[1] **and Jacek Wesołowski**[2]

Received May 23, 2000

Central limit theorems for permanents of random $m \times n$ matrices of iid columns with a common intercomponent correlation as $n-m \to \infty$ are derived. The results are obtained by introducing a Hoeffding-like orthogonal decomposition of a random permanent and deriving the variance formulae for a permanent with the homogeneous correlation structure.

KEY WORDS: Central limit theorem; orthogonal expansion; random permanent.

1. INTRODUCTION

Denote by $\mathbf{A} = [a_{ij}]$ an $m \times n$ real matrix with $m \leqslant n$. Then a permanent of the matrix A is defined by

$$\operatorname{Per}(\mathbf{A}) = \sum_{(i_1,\ldots,i_m):\ \{i_1,\ldots,i_m\} \subset \{1,\ldots,n\}} a_{1\,i_1} \cdots a_{m\,i_m}$$

In this paper we study asymptotic behavior of random permanents, and assume that $\mathbf{X} = [X_{ij}]$ is an $m \times n$ $(m \leqslant n)$ real random matrix of square integrable, identically distributed components and such that its columns are independent random vectors with fixed intercomponent correlation. For $i, k = 1,\ldots, m$ and $j = 1,\ldots, n$ we denote $\mu = EX_{ij}$, $\sigma^2 = Var\ X_{ij}$, and $\rho = Corr(X_{kj}, X_{ij})$. In the sequel we always assume that $\mu \neq 0$ and $0 < \sigma^2 < \infty$.

[1] Department of Mathematics, University of Louisville, Louisville, Kentucky 40292. E-mail: grzes@louisville.edu

[2] Wydział Matematyki i Nauk Informacyjnych, Politechnika Warszawska, Plac Politechniki 1, 00-661 Warszawa, Poland.

0894-9840/02/0100-0063/0 © 2002 Plenum Publishing Corporation

In this setting we are interested in finding the conditions under which the limiting law of

$$\frac{1}{\binom{n}{m}\, m!}\,\mathrm{Per}(\mathbf{X}) \tag{1}$$

as $n-m\to\infty$ is asymptotically normal.

Problems of this kind for permanents of one dimensional projection matrices ($\rho=1$, i.e., matrices with all rows identical) and some related themes have been studied by many authors; see, for instance, Székely,[11] van Es and Helmers,[13] Borovskikh and Korolyuk,[1] Korolyuk and Borovskikh,[6, 7] Kaneva,[4] and Kaneva and Korolyuk,[5] or Székely and Szeldi.[12] In general, it has been shown that when $\rho=1$ then the (appropriately normalized) statistic (1) is asymptotically normal or log-normal, depending on the rate of growth of m with respect to n. On the other hand, the case of $\rho=0$ with an additional assumption of independence of row vectors has been considered in the early papers of Girko (see, e.g., Girko[2] Chapters 2 and 7 and references therein), Rempała,[9] Janson,[3] and Rempała and Wesołowski.[10] In the latest paper it has been shown among others that for an appropriately normalized permanent of a matrix with square integrable, iid random entries of non-zero mean, CLT holds iff $m/n\to 0$, as $n-m\to\infty$. This finding has been in contrast with that in the case $\rho=1$ since then, as it had been shown by van Es and Helmers,[13] asymptotic normal law holds iff $m/\sqrt{n}\to 0$. This apparent difference in asymptotic behavior of random permanents has led some to conjecture that in the "intermediate" case of $0<\rho<1$ the dependence structure of the rows of the matrix $\mathbf{X}$ may force yet another, different from the above two, set of conditions on the rate of growth m relative to n, in order to ensure the asymptotic normality of (1).

In this paper we show that it is not so and that, in fact, under our assumptions on the structure of the matrix $\mathbf{X}$, the above two cases are the only possible ones. More precisely, we show herein that under appropriate moment conditions (i) when the columns entries of $\mathbf{X}$ are uncorrelated ($\rho=0$) then CLT holds if $m^{1+\varepsilon}/n\to 0$ and (ii) when the columns entries of $\mathbf{X}$ are correlated ($\rho>0$) then CLT holds if $m/\sqrt{n}\to 0$ (note that the case $\rho<0$ is not possible under our assumptions).

The idea of our proofs relies on the representation of a random permanent (1) as a sum of uncorrelated components and is closely related to the famous Hoeffding decomposition of a symmetric statistic, and, in some sense, may be viewed as its natural extension. In particular, using our orthogonal expansion method we show that in the both cases (i) and (ii) a random permanent (1) has the same asymptotic distribution as the normalized sum of all entries of **X**, since only the first term of the expansion contributes to the limiting behavior. It seems that the same or at least similar method shall also reveal the limiting distribution of (1) in the remaining cases, i.e., when $\rho = 0$ and $m/n \to \lambda > 0$, as well as when $\rho > 0$ and $m/\sqrt{n} \to \lambda > 0$. However, due to some technical complications and the fact that the limiting distributions will be non-normal (since now all terms of the expansion will contribute to the limit) they appear to warrant a separate treatment and are not considered here.

The paper is organized as follows. In the next section we derive the orthogonal expansion formula for a random permanent and then use it to find a general expression for the variance of (1). In the subsequent section we present the main results of the paper.

2. THE ORTHOGONAL DECOMPOSITION AND THE VARIANCE OF A RANDOM PERMANENT

In this section we introduce our main tools of investigating asymptotic behavior of a random permanent, namely its orthogonal expansion and the variance formula. First, let us consider the following

Proposition 1. Let $\mathbf{A} = [a_{ij}]$ be an $m \times n$ matrix with real entries for $m \leqslant n$. Denote $\tilde{a}_{ij} = a_{ij} - 1$ for any $i = 1, \ldots, m$, $j = 1, \ldots, n$. Then

$$\frac{\operatorname{Per} \mathbf{A}}{\binom{n}{m} m!} = 1 + \sum_{c=1}^{m} \binom{m}{c} U_c^{(m,n)}(\mathbf{A})$$

where

$$U_c^{(m,n)}(\mathbf{A}) = \binom{n}{c}^{-1} \binom{m}{c}^{-1} c!^{-1} \sum_{1 \leqslant i_1 < \cdots < i_c \leqslant m} \sum_{1 \leqslant j_1 < \cdots < j_c \leqslant n} \operatorname{Per}[\tilde{a}_{i_u j_v}]_{\substack{u = 1, \ldots, c \\ v = 1, \ldots, c}}$$

Proof. Observe that

$$\operatorname{Per} \mathbf{A} - \binom{n}{m} m! = \sum_{(j_1, \ldots, j_m):\, \{j_1, \ldots, j_m\} \subset \{1, \ldots, n\}} \left(\prod_{l=1}^{m} a_{l j_l} - 1 \right)$$

Apply now, for each of the summands above, the identity

$$\prod_{l=1}^{m} b_l - 1 = \sum_{c=1}^{m} \sum_{1 \leqslant i_1 < \cdots < i_c \leqslant m} \prod_{l=1}^{c} (b_{i_l} - 1)$$

which holds for any natural m and any real numbers $b_1,\dots, b_m$. Then

$$\begin{aligned}
\operatorname{Per}\mathbf{A} - \binom{n}{m} m! &= \sum_{(j_1,\dots,j_m):\,\{j_1,\dots,j_m\}\subset\{1,\dots,n\}} \sum_{c=1}^{m} \sum_{1\leqslant i_1<\cdots<i_c\leqslant m} \prod_{l=1}^{c} \tilde{a}_{i_l j_{i_l}} \\
&= \sum_{c=1}^{m} \sum_{1\leqslant i_1<\cdots<i_c\leqslant m} \sum_{(j_1,\dots,j_m):\,\{j_1,\dots,j_m\}\subset\{1,\dots,n\}} \prod_{l=1}^{c} \tilde{a}_{i_l j_{i_l}} \\
&= \sum_{c=1}^{m} \binom{n-c}{m-c}(m-c)! \sum_{1\leqslant i_1<\cdots<i_c\leqslant m} \sum_{(j_1,\dots,j_c):\,\{j_1,\dots,j_c\}\subset\{1,\dots,n\}} \prod_{l=1}^{c} \tilde{a}_{i_l j_{i_l}} \\
&= \sum_{c=1}^{m} \binom{n-c}{m-c}(m-c)! \sum_{1\leqslant i_1<\cdots<i_c\leqslant m} \operatorname{Per}[\tilde{a}_{i_u j}]_{\substack{u=1,\dots,c\\ j=1,\dots,n}}
\end{aligned}$$

where the one before last equality follows from the fact that only c column indices are present in the product, and the remaining $m-c$ indices may be chosen (in any order) from $n-c$ columns. Note that by the Laplace expansion formula for permanents (see, e.g., Minc[(8)]) we may rewrite the above expression as

$$\begin{aligned}
\operatorname{Per}\mathbf{A} - \binom{n}{m} m! &= \sum_{c=1}^{m} \binom{n-c}{m-c}(m-c)! \sum_{1\leqslant i_1<\cdots<i_c\leqslant m} \sum_{1\leqslant j_1<\cdots<j_c\leqslant n} \operatorname{Per}[\tilde{a}_{i_u j_v}]_{\substack{u=1,\dots,c\\ v=1,\dots,c}}
\end{aligned}$$

The final result follows by making a simple observation that

$$\binom{n-c}{m-c}(m-c)! = \binom{n}{m} m! \binom{n}{c}^{-1} (c!)^{-1} \qquad \square$$

The above formula applied to a permanent of a random matrix $\mathbf{X}$ provides us with a convenient way of investigating asymptotics of random permanents and can be viewed as a generalization of the Hoeffding decomposition for a permanent of one dimensional projection matrix (i.e., the case when $\rho = 1$).

Proposition 2. Assume $\mu = 1$, then

$$\frac{\operatorname{Per}\mathbf{X}}{\binom{n}{m}\, m!} = 1 + \sum_{c=1}^{m} \binom{m}{c} U_c^{(m,n)}$$

where $U_c^{(m,n)} = U_c^{(m,n)}(\mathbf{X})$ for $c = 1, \ldots, m$ and moreover,

$$Cov(U_{c_1}^{(m,n)}, U_{c_2}^{(m,n)}) = 0 \qquad \text{for} \quad c_1 \neq c_2 \tag{2}$$

Proof. The expansion formula follows immediately by Proposition 1. On the other hand, by the definition of $U_c^{(m,n)}$, in every product present in $Cov(U_{c_1}^{(m,n)}, U_{c_2}^{(m,n)})$ there is a single element (of the form $X_{ij}-1$) from at least one column. Thus, (2) follows from the independence of columns of $\mathbf{X}$. □

We are now in position to derive the general expression for the variance of a random permanent in our setting.

Theorem 1.

$$Var \frac{\operatorname{Per} \mathbf{X}}{\binom{n}{m}\, m!\, \mu^m} = \sum_{c=1}^{m} \frac{\binom{m}{c}\, \sigma^{2c}}{\binom{n}{c}\, \mu^{2c}} \sum_{r=0}^{c} \frac{1}{r!} \binom{m-r}{c-r} (1-\rho)^r \rho^{c-r}$$

Remark 1. Let us note that for $\rho = 1$ the above expression reduces to the well known formula for the variance of a permanent of a one dimensional projection matrix. On the other hand, if $\rho = 0$ we obtain in particular a formula for the variance of a permanent of iid random entries which was derived by a different method in Rempała and Wesołowski.[10]

Proof. First, let us note that without loss of generality we may assume $\mu = 1$. Then by Proposition 2 it follows that

$$\begin{aligned} Var \frac{\operatorname{Per} \mathbf{X}}{\binom{n}{m}\, m!} &= \sum_{c=1}^{m} \binom{m}{c}^2 Var\, U_c^{(m,n)} + \sum_{1 \leqslant c_1 \neq c_2 \leqslant m} \binom{m}{c_1}\binom{m}{c_2} Cov(U_{c_1}^{(m,n)}, U_{c_2}^{(m,n)}) \\ &= \sum_{c=1}^{m} \binom{m}{c}^2 Var\, U_c^{(m,n)} \end{aligned}$$

Now, for $c = 1, \ldots, m$

$$\begin{aligned} &\binom{n}{c}^2 \binom{m}{c}^2 c!^2\, Var\, U_c^{(m,n)} \\ &\quad = Var \left(\sum_{1 \leqslant i_1 < \cdots < i_c \leqslant m} \; \sum_{1 \leqslant j_1 < \cdots < j_c \leqslant n} \operatorname{Per}[\tilde{X}_{i_u j_v}]_{\substack{u=1,\ldots,c \\ v=1,\ldots,c}} \right) \\ &\quad = \sum_{1 \leqslant j_1 < \cdots < j_c \leqslant n} Var \left(\sum_{1 \leqslant i_1 < \cdots < i_c \leqslant m} \operatorname{Per}[\tilde{X}_{i_u j_v}]_{\substack{u=1,\ldots,c \\ v=1,\ldots,c}} \right) \end{aligned}$$

since, by independence of columns of $\mathbf{X}$, we have

$$Cov\left(\sum_{1\leqslant i_1<\cdots<i_c\leqslant m}\mathrm{Per}[\tilde{X}_{i_uj_v}]_{\substack{u=1,\dots,c\\ v=1,\dots,c}},\ \sum_{1\leqslant i_1<\cdots<i_c\leqslant m}\mathrm{Per}[\tilde{X}_{i_uk_v}]_{\substack{u=1,\dots,c\\ v=1,\dots,c}}\right)=0$$

if only $\{j_1,\dots,j_c\}\neq\{k_1,\dots,k_c\}$.

Since the columns of $\mathbf{X}$ are identically distributed, the variance formula above simplifies to

$$\binom{n}{c}\binom{m}{c}^2 c!^2\, Var\, U_c^{(m,n)}=Var\left(\sum_{1\leqslant i_1<\cdots<i_c\leqslant m}\mathrm{Per}[\tilde{X}_{i_uj}]_{\substack{u=1,\dots,c\\ j=1,\dots,c}}\right)$$

Let us note that the number of pairs of $c\times c$ submatrices of $\mathbf{X}$ having exactly k rows in common equals $\binom{m}{c}\binom{c}{k}\binom{m-c}{c-k}$, for $\max(0,2c-m)\leqslant k\leqslant c$, and each such pair has equal covariance (since the row vectors of $\mathbf{X}$ are identically distributed). Hence, for given c, the above right-hand side can be written as

$$\binom{m}{c}\sum_{k=\max(0,2c-m)}^{c}\binom{c}{k}\binom{m-c}{c-k}$$
$$\times Cov(\mathrm{Per}[\tilde{X}_{ij}]_{\substack{i=1,\dots,k,\,i_{k+1},\dots,i_c\\ j=1,\dots,c}},\ \mathrm{Per}[\tilde{X}_{ij}]_{\substack{i=1,\dots,k,\,l_{k+1},\dots,l_c\\ j=1,\dots,c}})$$

where $\{i_{k+1},\dots,i_c\}\cap\{l_{k+1},\dots,l_c\}=\varnothing$.

Observe that each term of the above sum is itself a sum of products of $2c$ factors (2 factors for each of given c columns). By the assumptions about the entries of the matrix $\mathbf{X}$ it follows that expectations of such products having exactly l $(0\leqslant l\leqslant k)$ elements in common are the same and equal to

$$E(\tilde{X}_{11}^2\cdots\tilde{X}_{ll}^2\tilde{X}_{i_{l+1},l+1}\tilde{X}_{j_{l+1},l+1}\cdots\tilde{X}_{i_c,c}\tilde{X}_{j_c,c})=\rho^{c-l}\sigma^{2c}$$

where $\{i_{l+1},\dots,i_c\}$ and $\{j_{l+1},\dots,j_c\}$ are fixed non-overlapping subsets of $\{l+1,\dots,m\}$.

Now, to compute the covariance of such $k\times c$ permanents it suffices to find the number of pairs of products with exactly l elements in common, $(0\leqslant l\leqslant k\leqslant c)$. Observe that it equals to the number of pairs of products having exactly l common elements in a permanent of the matrix $k\times c$, multiplied by $(c-k)!^2$—the number of all possible permutations of $i_{k+1},\dots,i_c$ and $l_{k+1},\dots,l_c$.

To compute the number of pairs of products with exactly l elements in common let us start with finding the number of products in Per $\mathbf{Y}[k,c]$, where $\mathbf{Y}[k, c]$ is a $k\times c$ matrix, having exactly l factors in common with the diagonal entries $y_{11}\cdots y_{kk}$. First, we fix l factors in $\binom{k}{l}$ ways. If we assume that $y_{11},\ldots, y_{ll}$ are fixed, then the remaining factors, in the products we are looking for, have to be of the form $y_{l+1, j_{l+1}},\ldots, y_{k, j_k}$, where $j_r\neq r$, $r=l+1,\ldots, k$. Finding the number of such products (say, $\pi_l(k, c)$) is equivalent to computing the number of summands in a permanent of the matrix of dimensions $(k-l)\times(c-l)$ which do not contain any diagonal entry. To this end, we subtract the number of all summands having at least one factor being the diagonal entry, from the total number of all summands in that permanent. Using the exclusion-inclusion formula we get that

$$\pi_l(k, c)=\binom{c-l}{k-l}(k-l)!-\sum_{j=1}^{k-l}(-1)^{j+1}\binom{k-l}{j}\binom{c-l-j}{k-l-j}(k-l-j)!$$

where the absolute value of the j-th member of the above sum denotes the number of products having exactly j factors being the diagonal entries (equal to the number of choices of j positions on the diagonal) multiplied by the number of products of $k-l-j$ factors from the outside of the diagonal (equal to the number of products in the permanent of the matrix of dimensions $(k-l-j)\times(c-l-j)$). Thus, in a slightly more compact form,

$$\pi_l(k, c)=\sum_{j=0}^{k-l}(-1)^{j}\binom{k-l}{j}\binom{c-l-j}{k-l-j}(k-l-j)! \tag{3}$$

Consequently, the number of pairs of products in Per $\mathbf{Y}[k, c]$ with exactly l factors in common equals to

$$\binom{c}{k}k!\binom{k}{l}\pi_l(k, c)$$

Hence, combining the above formula with an earlier one for the number of pairs of products with l identical factors we arrive at

$$Cov(\mathrm{Per}[\tilde{X}_{ij}]_{\substack{i=1,\ldots,k,\, i_{k+1},\ldots, i_c\\ j=1,\ldots,c}}, \mathrm{Per}[\tilde{X}_{ij}]_{\substack{i=1,\ldots,k,\, l_{k+1},\ldots, l_c\\ j=1,\ldots,c}})$$

$$=(c-k)!^2\,\sigma^{2c}\sum_{l=0}^{k}\binom{c}{k}k!\binom{k}{l}\rho^{c-l}\pi_l(k, c)$$

Now, returning to the formula for the variance of $U_c^{(m,n)}$ we obtain

$$\binom{n}{c}\binom{m}{c} Var\, U_c^{(m,n)}$$

$$= \frac{\sigma^{2c}}{c!^2} \sum_{k=\max(0,\,2c-m)}^{c} \binom{c}{k}^2 \binom{m-c}{c-k} k!(c-k)!^2 \sum_{l=0}^{k} \binom{k}{l} \rho^{c-l} \pi_l(k,c)$$

$$= \sigma^{2c} \sum_{k=\max(0,\,2c-m)}^{c} \binom{m-c}{c-k} \sum_{r=0}^{k} \binom{c-r}{k-r} \frac{1}{r!} \rho^{c-r}(1-\rho)^r \tag{4}$$

since

$$\sum_{l=0}^{k} \binom{k}{l} \rho^{c-l} \pi_l(k,c) = \sum_{l=0}^{k} \binom{k}{l} \rho^{c-l} \sum_{r=l}^{k} (-1)^{r-l} \binom{k-l}{r-l} \binom{c-r}{k-r} (k-r)!$$

$$= \sum_{r=0}^{k} \binom{c-r}{k-r} (k-r)!\, \rho^{c-r} \sum_{l=0}^{r} \binom{k}{l} \binom{k-l}{r-l} (-\rho)^{r-l}$$

$$= \sum_{r=0}^{k} \binom{c-r}{k-r} (k-r)!\, \rho^{c-r} \binom{k}{r} (1-\rho)^r$$

$$= \sum_{r=0}^{k} \binom{c-r}{k-r} \frac{k!}{r!} \rho^{c-r}(1-\rho)^r$$

Observe that

$$\sum_{k=\max(0,\,2c-m)}^{c} \binom{m-c}{c-k} \sum_{r=0}^{k} \binom{c-r}{k-r} \frac{r^{c-r}(1-\rho)^r}{r!}$$

$$= \sum_{r=0}^{c} \frac{\rho^{c-r}(1-\rho)^r}{r!} \sum_{k=\max(r,\,2c-m)}^{c} \binom{m-c}{c-k} \binom{c-r}{k-r}$$

$$= \sum_{r=0}^{c} \binom{m-r}{c-r} \frac{\rho^{c-r}(1-\rho)^r}{r!}$$

where the last equality follows by applying the hypergeometric summation rule for the inner sum. Applying the above to (4), we finally obtain

$$Var\, U_c^{(m,n)} = \binom{n}{c}^{-1} \binom{m}{c}^{-1} \sigma^{2c} \sum_{r=0}^{c} \binom{m-r}{c-r} \frac{\rho^{c-r}(1-\rho)^r}{r!} \tag{5}$$

which completes the proof. □

3. CENTRAL LIMIT THEOREMS

In this section we present the main results of the paper, namely two versions of permanent CLT dealing with the case when $\rho > 0$ and $\rho = 0$, respectively. First, in Theorem 1 below, we consider the case $\rho > 0$ which in a special case $\rho = 1$ reduces to the result of van Es and Helmers[13] obtained solely under the assumption of square integrability. Indeed, a brief inspection of our proof reveals that in the case if $\rho = 1$ or if m is a fixed constant independent of n the result holds as long as $0 < \sigma^2 < \infty$. Let us also note that the theorem remains valid if we assume only that $E\,|X_{11}|^{2+\delta} < \infty$ for some $0 < \delta \leqslant 1$ but strengthen the assumptions on the rates of m and n to $m/n^{\delta/2} \to 0$. It seems, however, that in general the assumption of the existence of a higher moment cannot be removed.

Theorem 2. Assume that $E\,|X_{11}|^3 < \infty$, $\mu \neq 0$ and $\rho \in (0, 1]$. If $m^2/n \to 0$ then

$$\frac{\sqrt{n}}{m}\left(\frac{\operatorname{Per} \mathbf{X}}{\binom{n}{m}\, m!\, \mu^m} - 1\right) \xrightarrow{D} \mathscr{N}(0, \tau^2)$$

where $\tau^2 = \rho\sigma^2/\mu^2$ if $m \to \infty$ and $\tau^2 = (\rho + (1-\rho)/m)\, \sigma^2/\mu^2$ for constant m.

Proof. Assuming (without loss of generality) that $\mu = 1$, by Proposition 2 we may write

$$\frac{\sqrt{n}}{m}\left(\frac{\operatorname{Per} \mathbf{X}}{\binom{n}{m}\, m!} - 1\right) = \frac{\sqrt{n}}{m} \sum_{c=1}^{m} \binom{m}{c} U_c^{(m,n)}$$

First, we will show that

$$R_{m,n} = \frac{\sqrt{n}}{m} \sum_{c=2}^{m} \binom{m}{c} U_c^{(m,n)} \to 0$$

in probability. To this end note that by (2) and (5) we have

$$Var\ R_{m,n} = \frac{n}{m^2} \sum_{c=2}^{m} \frac{\binom{m}{c} \sigma^{2c}}{\binom{n}{c}} \sum_{r=0}^{c} \frac{1}{r!} \binom{m-r}{c-r} (1-\rho)^r \rho^{c-r}$$

In view of the obvious inequality $\binom{m-r}{c-r} \leqslant \binom{m}{c}$ for $0 \leqslant r \leqslant c \leqslant m$, the inner sum above is majorized by

$$\binom{m}{c} \sum_{r=0}^{c} \frac{1}{r!} (1-\rho)^r \rho^{c-r} \leqslant \binom{m}{c} \exp(1)$$

since $0 < \rho \leqslant 1$. Thus,

$$Var\ R_{m,n} \leqslant \exp(1)\,\frac{n}{m^2}\sum_{c=2}^{m}\frac{\binom{m}{c}^2\sigma^{2c}}{\binom{n}{c}} \leqslant \exp(1)\,\frac{n}{m^2}\sum_{c=2}^{m}\left(\frac{m^2}{n}\right)^c\frac{\sigma^{2c}}{c!}$$

in view of

$$c!\,\frac{\binom{m}{c}^2}{\binom{n}{c}} \leqslant \left(\frac{m^2}{n}\right)^c$$

which follows from the inequality

$$\frac{(m-r)^2}{n-r} \leqslant \frac{m^2}{n}$$

valid for $0 \leqslant r \leqslant m \leqslant n$. Consequently,

$$Var\ R_{m,n} \leqslant \exp(1)\,\frac{m^2}{n}\sum_{c=2}^{m}\left(\frac{m^2}{n}\right)^{c-2}\frac{\sigma^{2c}}{c!}$$

Take n_0 large enough to have $m^2/n \leqslant 1$ for $n > n_0$ (recall that $m = m(n)$). For such n's we have

$$Var\ R_{m,n} \leqslant \exp(1)\,\frac{m^2}{n}\sum_{c=2}^{m}\frac{\sigma^{2c}}{c!} \leqslant \frac{m^2}{n}\exp(1+\sigma^2)$$

Hence $Var\ R_{m,n} \to 0$, as $m^2/n \to 0$ by the assumption, and the result follows via the Tchebychev inequality. In order to finish the proof we need only to show that

$$\frac{\sqrt{n}}{m}\binom{m}{1}U_1^{(m,n)} = \frac{1}{m\sqrt{n}}\sum_{i=1}^{m}\sum_{j=1}^{n}\tilde{X}_{ij} = \frac{1}{\sqrt{n}}\sum_{j=1}^{n}Y_j^{(m)} \xrightarrow{D} \mathcal{N}(0,\tau^2)$$

where $Y_j^{(m)} = \sum_{i=1}^{m}\tilde{X}_{ij}/m$, $j = 1,\ldots,n$. Let us consider an arbitrary sequence (m_n) such that $m_n^2/n \to 0$, as $n \to \infty$, and denote $Y_{nj} = Y_j^{(m_n)}/\sqrt{n}$, $j = 1,\ldots,n$. Due to the structure of the matrix $\mathbf{X}$ the triangular array (Y_{nj}) is rowwise independent and identically distributed. Furthermore, the entries of the array have zero means. Hence, by CLT for the rowwise independent triangular arrays, it suffices to show that

(i) $\sum_{j=1}^{n} Var\ Y_{nj} \to const > 0$,

(ii) $\sum_{j=1}^{n} EY_{nj}^2 I(|Y_{nj}| > \varepsilon) \to 0\ \forall \varepsilon > 0$.

Note that $\sum_{j=1}^{n} Var\, Y_{nj} = (\rho + (1-\rho)/m_n)\, \sigma^2 \to \tau^2$ and hence (i) follows immediately. In order to check the Lindeberg condition (ii) we use Lemma 1 (see below) obtaining for any $\varepsilon > 0$

$$\sum_{j=1}^{n} EY_{nj}^2 I(|Y_{nj}| > \varepsilon)$$

$$= m_n^{-2} E(\tilde{X}_{11} + \cdots + \tilde{X}_{m_n 1})^2\, I(|\tilde{X}_{11} + \cdots + \tilde{X}_{m1}| > m_n \sqrt{n}\, \varepsilon)$$

$$\leqslant \frac{1 + 4(m_n - 1)^2}{m_n}\, E\tilde{X}_{11}^2 I(|\tilde{X}_{11}| > \sqrt{n}\, \varepsilon)$$

$$\leqslant \frac{1 + 4(m_n - 1)^2}{m_n \sqrt{n}\, \varepsilon}\, E\, |\tilde{X}_{11}|^3 \to 0 \qquad \square$$

Our next result establishes CLT for random permanents in the case when the column entries are uncorrelated. Similarly as in Theorem 2 here also one can somewhat relax the moment assumptions and require only that $E\, |X_{11}|^{2+\delta} < \infty$ for some $\delta > 0$, as long as it is true that $m\, (m/n)^{\delta/2} \to 0$. It is perhaps somewhat surprising that apparently without assuming further structure of the joint distribution law of the row vectors of **X** one cannot eliminate these additional assumptions.

Theorem 3. Assume that $EX_{11}^4 < \infty$, $\mu \neq 0$ and $\rho = 0$. If $m^3/n \to 0$ then

$$\sqrt{\frac{n}{m}} \left(\frac{\text{Per}\, \mathbf{X}}{\binom{n}{m}\, m!\, \mu^m} - 1 \right) \xrightarrow{D} \mathcal{N}(0, \sigma^2/\mu^2)$$

Proof. As in the proof of Theorem 1 we assume, without loss of generality that $\mu = 1$. Now by Proposition 2 we have

$$\sqrt{\frac{n}{m}} \left(\frac{\text{Per}\, \mathbf{X}}{\binom{n}{m}\, m!} - 1 \right) = \sqrt{\frac{n}{m}} \binom{m}{1} U_1^{(m,\, n)} + R_{m,\, n}$$

where

$$R_{m,\, n} = \sqrt{\frac{n}{m}} \sum_{c=2}^{m} \binom{m}{c} U_c^{(m,\, n)}$$

Following the scheme of the proof of Theorem 1 we show first that $R_{m,n}$ tends to zero in probability. To this end observe that by (2), (5) and the fact that $\binom{m}{c}/\binom{n}{c} \leqslant (m/n)^c$, $(1 \leqslant c \leqslant m \leqslant n)$, it follows that

$$Var\, R_{m,n} = \frac{n}{m} \sum_{c=2}^{m} \frac{\binom{m}{c}\, \sigma^{2c}}{\binom{n}{c}\, c!} \leqslant \frac{m}{n} \exp(\sigma^2) \to 0$$

and hence $R_{m,n} \to 0$ in probability, in view of the Tchebychev inequality.

Secondly, we show that

$$\sqrt{\frac{n}{m}} \binom{m}{1} U_1^{(m,n)} = \frac{1}{\sqrt{mn}} \sum_{i=1}^{m} \sum_{j=1}^{n} \tilde{X}_{ij} = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} Y_j^{(m)} \xrightarrow{D} \mathcal{N}(0, \sigma^2)$$

where $Y_j^{(m)} = \sum_{i=1}^{m} \tilde{X}_{ij}/\sqrt{m}$, $j = 1,..., n$. As in the proof of Theorem 1 let us again consider an arbitrary sequence (m_n) such that $m_n^3/n \to 0$, as $n \to \infty$, and denote $Y_{nj} = Y_j^{(m_n)}/\sqrt{n}$, $j = 1,..., n$. Again, due to the structure of the matrix **X** the triangular array (Y_{nj}) is rowwise independent and identically distributed. Furthermore, the entries of the array have zero means. Hence to complete the proof we may again use the CLT for rowwise independent triangular arrays. Since now $\sum_{j=1}^{n} Var\, Y_{nj} = \sigma^2$ we need only to verify the Lindeberg condition. By Lemma 1 if follows that for any $\varepsilon > 0$

$$\begin{aligned} &\sum_{j=1}^{n} EY_{nj}^2 I(|Y_{nj}| > \varepsilon) \\ &\quad = m_n^{-1} E(\tilde{X}_{11} + \cdots + \tilde{X}_{m_n 1})^2\, I(|\tilde{X}_{11} + \cdots + \tilde{X}_{m_n 1}| > \sqrt{m_n n}\, \varepsilon) \\ &\quad \leqslant (1 + 4(m_n - 1)^2)\, E\tilde{X}_{11}^2 I(|\tilde{X}_{11}| > \sqrt{n/m_n}\, \varepsilon) \\ &\quad \leqslant \frac{(1 + 4(m_n - 1)^2)\, m_n}{n} E\, |\tilde{X}_{11}|^4 \to 0 \end{aligned}$$

since $m^3/n \to 0$. □

Lemma 1. Let $(X_1,..., X_n)$ be a random vector with square integrable, identically distributed components. Then for any $a > 0$

$$\begin{aligned} &E(X_1 + \cdots + X_n)^2\, I(|X_1 + \cdots + X_n| > a) \\ &\quad \leqslant n(1 + 4(n-1)^2)\, EX_1^2 I(|X_1| > a/n) \end{aligned}$$

Proof. Observe that

$$
\begin{aligned}
E(X_1+\cdots+X_n)^2\, I(|X_1+\cdots+X_n|>a) \\
&\leqslant \sum_{i=1}^{n} E(X_1+\cdots+X_n)^2\, I(|X_i|>a/n) \\
&= \sum_{i=1}^{n} \Bigg(EX_i^2 I(|X_i|>a/n) \\
&\quad + \sum_{j\neq i}^{n} EX_j^2 I(|X_i|>a/n) + \sum_{j\neq i}^{n} EX_i X_j I(|X_i|>a/n) \\
&\quad + \sum_{\substack{1\leqslant j<k\leqslant n \\ j\neq i\neq k}} EX_j X_k I(|X_i|>a/n) \Bigg) \qquad (6)
\end{aligned}
$$

But for any X, Y, Z identically distributed, square integrable random variables and any positive b we have

$$
\begin{aligned}
EX^2 I(|Z|>b) &= EX^2 I(|X|>b)\, I(|Z|>b) + EX^2 I(|X|\leqslant b)\, I(|Z|>b) \\
&\leqslant EX^2 I(|X|>b) + EZ^2 I(|Z|>b) \\
&= 2EX^2 I(|X|>b)
\end{aligned}
$$

Similarly, but additionally using the Cauchy Schwartz, inequality we get

$$
\begin{aligned}
E\,|XZ|\, I(|Z|>b) \\
&= E\,|XZ|\, I(|X|>b)\, I(|Z|>b) + E\,|XZ|\, I(|X|\leqslant b)\, I(|Z|>b) \\
&\leqslant \sqrt{EX^2 I(|X|>b)\; E\,Z^2 I(|Z|>b)} + EZ^2 I(|Z|>b) \\
&= 2EX^2 I(|X|>b).
\end{aligned}
$$

Now, by the above inequality, it follows also that

$$
\begin{aligned}
E\,|XY|\, I(|Z|>b) \\
&= E\,|XY|\, I(|X|>b)\, I(|Z|>b) + E\,|XY|\, I(|X|\leqslant b)\, I(|Z|>b) \\
&\leqslant E\,|XY|\, I(|X|>b) + E\,|ZY|\, I(|Z|>b) \\
&\leqslant 4EX^2 I(|X|>b)
\end{aligned}
$$

Finally, applying the above three inequalities to the right hand side of (6) we get

$$\begin{aligned}
&E(X_1+\cdots+X_n)^2\, I(|X_1+\cdots+X_n|>a)\\
&\quad\leqslant n[EX_1^2 I(|X_1|>a/n)+2(n-1)\, EX_1^2 I(|X_1|>a/n)\\
&\qquad +2(n-1)\, EX_1^2 I(|X_1|>a/n)\\
&\qquad +4(n-1)(n-2)\, EX_1^2 I(|X_1|>a/n)]\\
&\quad= n(1+4(n-1)^2)\, EX_1^2 I(|X_1|>a/n)
\end{aligned}$$

which completes the proof. □

REFERENCES

1. Borovskikh, Y. V., and Korolyuk, V. S. (1994). Random permanents and symmetric statistics. *Acta Appl. Math.* **36**, 227–288.
2. Girko, V. L. (1990). Theory of random determinants. In *Mathematics and Its Applications (Soviet Series)*, Vol. 45, Kluwer Academic Publishers Group, Dordrecht. [Translated from Russian.]
3. Janson, S. (1994). The numbers of spanning trees, Hamilton cycles and perfect matchings in a random graph. *Combin. Probab. Comput.* **3**, 97–126.
4. Kaneva, E. Y. (1995). Random permanents of mixed multisample matrices. *Ukraïn. Mat. Zh.* **47**, 1002–1005.
5. Kaneva, E. Y. and Korolyuk, V. S. (1996). Random permanents of mixed sample matrices, *Ukraïn. Mat. Zh.* **48**(1), 44–49.
6. Korolyuk, V. S., and Borovskikh, Y. V. (1992). Random permanents and symmetric statistics. In *Probability Theory and Mathematical Statistics (Kiev, 1991)*, pp. 176–187, World Sci. Publishing, River Edge, NJ.
7. Korolyuk, V. S., and Borovskikh, Y. V. (1995). Normal approximation of random permanents. *Ukraïn. Mat. Zh.* **47**, 922–927.
8. Minc, H. (1978). *Permanents*, Addison-Wesley, Reading, Massachusetts. With a foreword by Marvin Marcus, Encyclopedia of Mathematics and Its Applications, Vol. 6.
9. Rempała, G. (1996). Asymptotic behavior of random permanents. *Random Oper. Stochastic Equations* **4**, 33–42.
10. Rempała, G., and Wesołowski, J. (1999). Limiting behavior of random permanents. *Statist. Probab. Lett.* **45**, 149–158.
11. Székely, G. J. (1982). A limit theorem for elementary symmetric polynomials of independent random variables. *Z. Wahrsch. Verw. Gebiete* **59**, 355–359.
12. Székely, G. J., and Szeidl, L. (1995). On the limit distribution of random permanents, In *Exploring Stochastic Laws*, pp. 443–455. VSP, Utrecht.
13. van Es, A. J., and Helmers, R. (1988). Elementary symmetric polynomials of increasing order. *Probab. Theory Related Fields* **80**, 21–35.