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STRONG LAWS OF LARGE NUMBERS FOR RANDOM PERMANENTS

BY

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Abstract. The strong laws of large numbers for random permanents of increasing order are derived. The method of proofs relies on the martingale decomposition of a random permanent function, similar to the one known for U-statistics.

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1. INTRODUCTION

Denote by $A = [a_{ij}]$ an $m \times n$ real matrix with $m \leqslant n$. Then a *permanent of the matrix A* is defined by

$$\operatorname{Per}(A) = \sum_{(i_1,\ldots,i_m):\{i_1,\ldots,i_m\} \subset \{1,\ldots,n\}} a_{1i_1} \ldots a_{mi_m}.$$

In this paper we study the almost sure asymptotic behavior of a permanent of an $m \times n$ ($m \leqslant n$) random matrix $X = [X_{ij}]$ of square integrable entries such that its columns are independent identically distributed random vectors of exchangeable components. For $i, k = 1, \ldots, m$ and $j = 1, \ldots, n$ we put $\mu = E(X_{ij})$, $\sigma^2 = \operatorname{Var}(X_{ij})$ and $\varrho = \operatorname{Corr}(X_{kj}, X_{ij})$. In the sequel we always assume that $\mu \neq 0$ and $0 < \sigma^2 < \infty$. We denote additionally by $\gamma = \sigma/\mu$ the variation coefficient. In this setting we are interested in finding the conditions under which

$$\frac{1}{\binom{n}{m} m!\, \mu^m} \operatorname{Per}(X) \to 1 \quad \text{a.s.} \tag{1}$$

as $n \to \infty$ and/or $m \to \infty$.

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2. THE BACKWARD MARTINGALE PROPERTY

It has been proved in Rempała and Wesołowski [6] that

$$\frac{\operatorname{Per}(X)}{\binom{n}{m} m!\,\mu^m} = 1 + \sum_{c=1}^{m} \binom{m}{c} U_c^{(m,n)},$$

where

$$U_c^{(m,n)} = \binom{n}{c}^{-1} \binom{m}{c}^{-1} (c!)^{-1} \sum_{1 \leqslant i_1 < \ldots < i_c \leqslant m} \ \sum_{1 \leqslant j_1 < \ldots < j_c \leqslant n} \operatorname{Per}[\tilde{X}_{i_u j_v}]_{\substack{u=1,\ldots,c \\ v=1,\ldots,c}}$$

for $\tilde{X}_{ij} = X_{ij}/\mu - 1$, $i = 1, \ldots, m$, $j = 1, \ldots, n$. Moreover, the $U_c^{(m,n)}$'s are orthogonal, i.e.,

$$\operatorname{Cov}(U_{c_1}^{(m,n)}, U_{c_2}^{(m,n)}) = 0 \quad \text{for } c_1 \neq c_2 \tag{2}$$

as well as

$$\operatorname{Var}(U_c^{(m,n)}) = \binom{n}{c}^{-1} \binom{m}{c}^{-1} \gamma^{2c} \sum_{r=0}^{c} \binom{m-r}{c-r} \frac{\varrho^{c-r}(1-\varrho)^r}{r!}. \tag{3}$$

Under our assumptions, when $\varrho = 1$, the above decomposition is simply the usual Hoeffding decomposition of the elementary symmetric polynomial statistic of increasing order (i.e., $m = m_n$ is a non-decreasing function of n), and then it is well known (cf., e.g., Rempała and Gupta [4]) that the sequence $(U_c^{(m_n,n)})_{n=n_0,n_0+1,\ldots}$ is a backward martingale for a natural sequence of σ-algebras.

Let us first show that this property is valid for any $\varrho \in [0, 1]$.

PROPOSITION 1. *Let $m = m_n$ be a non-decreasing sequence and for any fixed natural number c let us put $\mathcal{F}_c^{(n)} = \sigma\{U_c^{(m_n,n)}, U_c^{(m_{n+1},n+1)}, \ldots\}$. Then the sequence $(U_c^{(m_n,n)}, \mathcal{F}_c^{(n)})_{n=n_0,n_0+1,\ldots}$ is a backward martingale, i.e.,*

$$E(U_c^{(m_n,n)} \mid \mathcal{F}_c^{(n+1)}) = U_c^{(m_{n+1},n+1)} \quad \textit{for all } n = n_0, n_0+1, \ldots$$

Proof. Let us denote an element of the Hoeffding-like decomposition of an $m_n \times n$ matrix obtained from the $m_{n+1} \times n+1$ matrix by deleting $m_{n+1} - m_n$ rows: $l_1, \ldots, l_{m_{n+1}-m_n}$, and the k-th column by $U_c^{(m_n,n)}(l_1, \ldots, l_{m_{n+1}-m_n}; k)$, i.e.,

$$\begin{aligned} U_c^{(m_n,n)}&(l_1, \ldots, l_{m_{n+1}-m_n}; k) \\ &= \binom{m_n}{c}^{-1} \binom{n}{c}^{-1} (c!)^{-1} \\ &\quad \times \sum_{\substack{1 \leqslant i_1 < \ldots < i_c \leqslant m_{n+1} \\ \{i_1,\ldots,i_c\} \cap \{l_1,\ldots,l_{m_{n+1}-m_n}\} = \emptyset}} \ \sum_{\substack{1 \leqslant j_1 < \ldots < j_c \leqslant n+1 \\ k \notin \{j_1,\ldots,j_c\}}} \operatorname{Per}[\tilde{X}_{i_u j_v}]_{\substack{u=1,\ldots,c \\ v=1,\ldots,c}}. \end{aligned}$$

Observe that

$$\sum_{k=1}^{n+1} \sum_{1 \leq l_1 < \ldots < l_{m_{n+1}-m_n} \leq m_{n+1}} U_c^{(m_n,n)}(l_1, \ldots, l_{m_{n+1}-m_n}; k)$$

$$= \binom{m_{n+1}-c}{m_n-c} \frac{n+1-c}{\binom{m_n}{c}\binom{n}{c} c!} \sum_{1 \leq i_1 < \ldots < i_c \leq m_{n+1}} \sum_{1 \leq j_1 < \ldots < j_c \leq n+1} \operatorname{Per}[\tilde{X}_{i_u j_v}]_{\substack{u=1,\ldots,c \\ v=1,\ldots,c}}$$

since any given $\operatorname{Per}[\tilde{X}_{i_u j_v}]_{\substack{u=1,\ldots,c \\ v=1,\ldots,c}}$, i.e., a permanent with fixed rows $i_1, \ldots, i_c$ and columns $j_1, \ldots, j_c$, is contained in

$$(n+1-c)\binom{m_{n+1}-c}{m_{n+1}-m_n}$$

elements of the form $U_c^{(m_n,n)}(l_1, \ldots, l_{m_{n+1}-m_n}; k)$ such that

$$\{i_1, \ldots, i_c\} \cap \{l_1, \ldots, l_{m_{n+1}-m_n}\} = \emptyset, \quad k \notin \{j_1, \ldots, j_c\}.$$

Since

$$\binom{m_{n+1}-c}{m_{n+1}-m_n}\binom{m_n}{c}^{-1} = \binom{m_{n+1}}{m_n}\binom{m_{n+1}}{c}^{-1},$$

it follows that

$$\sum_{k=1}^{n+1} \sum_{1 \leq l_1 < \ldots < l_{m_{n+1}-m_n} \leq m_{n+1}} U_c^{(m_n,n)}(l_1, \ldots, l_{m_{n+1}-m_n}; k) = (n+1)\binom{m_{n+1}}{m_n} U_c^{(m_{n+1},n+1)}. \tag{4}$$

But due to the exchangeability of distribution within columns (treated as infinite sequences) and the assumption that columns are independent and identically distributed, it follows that the conditional distribution of $U_c^{(m_n,n)}(l_1, \ldots, l_{m_{n+1}-m_n}; k)$ given $\mathcal{F}_c^{(n+1)}$ is the same for any particular choice of $k \in \{1, \ldots, n+1\}$ and $\{l_1, \ldots, l_{m_{n+1}-m_n}\} \subset \{1, \ldots, m_{n+1}\}$. Consequently,

$$E\big(U_c^{(m_n,n)}(l_1, \ldots, l_{m_{n+1}-m_n}; k) \,|\, \mathcal{F}_c^{(n+1)}\big)$$
$$= E\big(U_c^{(m_n,n)}(m_n+1, m_n+2, \ldots, m_{n+1}; n+1) \,|\, \mathcal{F}_c^{(n+1)}\big) = E\big(U_c^{(m,n)} \,|\, \mathcal{F}_c^{(n+1)}\big).$$

Now, it follows that

$$E\Big(\sum_{k=1}^{n+1} \sum_{1 \leq l_1 < \ldots < l_{m_{n+1}-m_n} \leq m_{n+1}} U_c^{(m_n,n)}(l_1, \ldots, l_{m_{n+1}-m_n}; k) \,|\, \mathcal{F}_c^{(n+1)}\Big)$$
$$= (n+1)\binom{m_{n+1}}{m_n} E\big(U_c^{(m_n,n)} \,|\, \mathcal{F}_c^{(n+1)}\big),$$

since after changing the order of taking the expectation and summations we see that all the summands are equal, by what was observed above. But on the other hand, using the identity (4), we get

$$E\Big(\sum_{k=1}^{n+1}\sum_{1\leqslant l_1<\ldots<l_{m_{n+1}-m_n}\leqslant m_{n+1}} U_c^{(m_n,n)}(l_1,\ldots,l_{m_{n+1}-m_n};k)\,|\,\mathscr{F}_c^{(n+1)}\Big)$$
$$=(n+1)\binom{m_{n+1}}{m_n}E\big(U_c^{(m_{n+1},n+1)}\,|\,\mathscr{F}_c^{(n+1)}\big)=(n+1)\binom{m_{n+1}}{m_n}U_c^{(m_{n+1},n+1)}.$$

Consequently, combining the above two formulas, we get the final result. ■

3. LAWS OF LARGE NUMBERS

The first result which can be considered a version of the strong law of large numbers (SLLN) for random permanents was obtained in Halàsz and Szèkely [3]. The authors considered symmetric polynomials of increasing order for positive independent identically distributed random variables, which, for the matrix of positive entries, is equivalent to taking $\varrho = 1$ in the scheme we consider here. They have shown that under the condition $m/n \to \lambda \geqslant 0$

$$\left(\frac{1}{\binom{n}{m}m!}\operatorname{Per}(X)\right)^{1/m}\to S(\lambda)\ \text{a.s.},$$

where $S(\lambda)$ is uniquely determined and $S(0) = \mu$. The above result is implied by the relation (1) but only when $\lambda = 0$. For some further discussion of that and related issues see Szèkely [7] and, more recently, Rempała and Gupta [4]. In the latter paper it was argued that the relation (1) cannot hold if $m/\sqrt{n} \to \lambda > 0$ and a logarithmic version of the SLLN for elementary symmetric polynomials under even more restrictive technical assumptions was obtained.

Additionally, it was proved in van Es and Helmers [8] for $\varrho = 1$ in the scheme we consider here that a version of the CLT for random permanents holds if $m^2/n \to 0$. This result has been extended to any $\varrho \in (0, 1]$, again under the assumption that $m^2/n \to 0$, in Rempała and Wesołowski [6], but without the exchangeability assumption. Here we present the SLLN in the exchangeable scheme. It appears that in this setting a slightly more stringent condition on the rate of relative asymptotic behavior of m and n has to be imposed.

THEOREM 1. *Let $m = m_n$ be a non-decreasing sequence. If $\varrho \in (0, 1]$ and $m^p/n \to 0$ for some $p > 2$, then*

$$\frac{\operatorname{Per}(X)}{\binom{n}{m}m!\,\mu^m}\to 1\ \textit{a.s.}$$

Proof. We use the Hoeffding-like decomposition recalled briefly, after Rempała and Wesołowski [6], in Section 1. Hence it suffices to prove that

$$\sum_{c=1}^{m}\binom{m}{c}U_c^{(m,n)}\to 0 \text{ a.s.} \quad \text{as } n\to\infty.$$

Observe that for sufficiently large n, say $n>n_0$, we have $m=m_n\leqslant n^{1/p}$. Now, we shall prove that for any arbitrary but fixed $c\geqslant 1$

$$\binom{m}{c}U_c^{(m,n)}\to 0 \text{ a.s.}$$

To this end notice that for any $\varepsilon>0$

$$P\Big(\sup_{n\geqslant 2^{n_0}}\binom{m_n}{c}|U_c^{(m,n)}|>\varepsilon\Big)\leqslant\sum_{k\geqslant n_0}P\Big(\max_{2^k\leqslant n\leqslant 2^{k+1}}\binom{m_n}{c}|U_c^{(m,n)}|>\varepsilon\Big).$$

Since

$$\binom{m_n}{c}\leqslant\frac{m_n^c}{c!}\leqslant\frac{n^{c/p}}{c!},$$

it follows that

$$P\Big(\sup_{n\geqslant 2^{n_0}}\binom{m_n}{c}|U_c^{(m,n)}|>\varepsilon\Big)\leqslant\sum_{k\geqslant n_0}P\Big(\frac{2^{c(k+1)/p}}{c!}\max_{2^k\leqslant n\leqslant 2^{k+1}}|U_c^{(m,n)}|>\varepsilon\Big).$$

Now, by the maximal inequality for backward martingales (see, for instance, Chow and Teicher [1], Chapter 7), we get

$$P\Big(\sup_{n\geqslant 2^{n_0}}\binom{m_n}{c}|U_c^{(m,n)}|>\varepsilon\Big)\leqslant\sum_{k\geqslant n_0}\frac{\operatorname{Var}(U_c^{(m_{2^k},2^k)})\,2^{2c(k+1)/p}}{(c!)^2\varepsilon^2}\leqslant\sum_{k\geqslant n_0}\frac{(\gamma^2\varrho)^c e}{(c!)^2\varepsilon^2}\frac{2^{2c(k+1)/p}}{\binom{2^k}{c}},$$

since it follows by (3) for $\varrho>0$ that

$$\operatorname{Var}(U_c^{(m,n)})\leqslant\frac{(\gamma^2\varrho)^c e}{\binom{n}{c}}.$$

Let us note that for sufficiently large n we have

$$\binom{n}{c}\geqslant\frac{n^c}{2^c c!},$$

which entails

$$P\Big(\sup_{n\geqslant 2^{n_0}}\binom{m_n}{c}|U_c^{(m,n)}|>\varepsilon\Big)\leqslant\frac{(2\gamma^2\varrho)^c e}{c!\,\varepsilon^2}\sum_{k\geqslant n_0}\frac{2^{2c(k+1)/p}}{2^{kc}}=\frac{(2^{1+1/p}\gamma^2\varrho)^c e}{c!\,\varepsilon^2}\sum_{k\geqslant n_0}2^{ck(2/p-1)}$$

and the last series converges since $p > 2$, i.e., $2/p-1 < 0$. Thus, it follows that

$$\lim_{n_0\to\infty} P\left(\sup_{n\geq 2^{n_0}} \binom{m_n}{c} |U_c^{(m,n)}| > \varepsilon\right) = 0,$$

and, consequently,

$$\binom{m_n}{c} |U_c^{(m,n)}| \to 0 \text{ a.s.}$$

Now, let us put

$$R_{m,n}(k) = \sum_{c=k}^{m} \binom{m}{c} U_c^{(m,n)} \quad \text{for some } 2 \leqslant k \leqslant m.$$

We will prove that there exists k satisfying $2 \leqslant k \leqslant m$ such that $R_{m,n}(k)$ converges to zero completely, i.e., for all $\varepsilon > 0$, $\sum_{n=1}^{\infty} P(|R_{m,n}(k)| > \varepsilon) < \infty$. This will obviously hold if we can show that $\sum_{n=1}^{\infty} \mathrm{Var}(R_{m,n}(k)) < \infty$.

By (2) it follows that

$$\mathrm{Var}(R_{m,n}(k)) = \sum_{c=k}^{m} \binom{m}{c}^2 \mathrm{Var}(U_c^{(m,n)}).$$

Consequently, by (3),

$$\mathrm{Var}(R_{m,n}(k)) = \sum_{c=k}^{m} \frac{\binom{m}{c}}{\binom{n}{c}} \gamma^{2c} \sum_{r=0}^{c} \binom{m-r}{c-r} \frac{\varrho^{c-r}(1-\varrho)^r}{r!} \leqslant e \sum_{c=k}^{m} \frac{\binom{m}{c}^2}{\binom{n}{c}} \gamma^{2c},$$

since $\varrho^{c-r}(1-\varrho)^r \leqslant 1$ and

$$\binom{m-r}{c-r} \leqslant \binom{m}{c} \quad \text{for any } c = 1, \ldots, m \text{ and } r = 0, \ldots, c.$$

Observe now that the inequality $(m-r)^2/(n-r) \leqslant m^2/n$ implies that

$$c! \binom{m}{c}^2 \bigg/ \binom{n}{c} \leqslant (m^2/n)^c.$$

Applying this inequality to the relation above, we obtain

$$\mathrm{Var}(R_{m,n}(k)) \leqslant e \left(\frac{m^2}{n}\right)^k \sum_{c=k}^{m} \left(\frac{m^2}{n}\right)^{c-k} \frac{\gamma^{2c}}{c!}.$$

Now, for n large enough we have $m^2/n < 1$, and hence

$$\mathrm{Var}(R_{m,n}(k)) \leqslant \left(\frac{m^2}{n}\right)^k \exp(1+\gamma^2).$$

Since

$$\frac{m^2}{n} = \left(\frac{m^p}{n}\right)^{2/p} \frac{1}{n^{1-2/p}} \leqslant \frac{1}{n^{1-2/p}}$$

for sufficiently large n, it follows that for k such that $k(1-2/p) > 1$ the sequence $(R_{m,n}(k))$ converges completely to 0 as $n \to \infty$. ■

Now let us turn our attention to the case $\varrho = 0$. Extending the earlier results of Girko [2] it was proved in Rempała and Wesołowski [5] that in the case of independent identically distributed entries of the matrix $\boldsymbol{X}$ the CLT holds if $m/n \to 0$, as long as the coefficient of variation γ satisfies $0 < \gamma^2 < \infty$. However, in the case of uncorrelated (not necessarily independent) components of the column vector, in order to obtain asymptotic normality without the exchangeability assumption some additional restrictions on the rate of m/n were needed (cf. Rempała and Wesołowski [6]). Similarly, in the case of the permanent SLLN in the scheme we consider here, for uncorrelated (and thus possibly independent) within-column components, an additional technical condition on the behavior of the sequence $m = m_n$ is also needed.

THEOREM 2. *Let $\varrho = 0$ and let $m = m_n$ be a non-decreasing sequence. Assume that there exist $p > 1$ and $L > 0$ such that*

$$m^p/n \to 0 \tag{5}$$

and

$$m_{2n}^2 < L m_n n^{1/p}. \tag{6}$$

Then

$$\frac{\operatorname{Per}(\boldsymbol{X})}{\binom{n}{m} m!\, \mu^m} \to 1 \quad a.s.$$

Proof. Observe that in the case $\varrho = 0$ it follows from (3) that

$$\operatorname{Var}(U_c^{(m,n)}) = \frac{\gamma^{2c}}{c! \binom{m}{c}\binom{n}{c}}.$$

Consequently, using the same argument as in the proof of Theorem 1, we obtain

$$P\left(\sup_{n \geqslant 2^{n_0}} \binom{m}{c} |U_c^{(m,n)}| > \varepsilon\right) \leqslant \sum_{k \geqslant n_0} \frac{m_{2^{k+1}}^{2c}}{(c!)^2 \varepsilon^2} \operatorname{Var}(U_c^{(m_{2^k}, 2^k)}) \leqslant \frac{(2\gamma)^{2c}}{c!\, \varepsilon^2} \sum_{k \geqslant n_0} \left(\frac{m_{2^{k+1}}^2}{m_{2^k} 2^k}\right)^c.$$

Now by (6) it follows that

$$P\left(\sup_{n\geqslant 2^{n_0}}\binom{m}{c}|U_c^{(m,n)}|>\varepsilon\right)\leqslant\frac{L(2\gamma)^{2c}}{c!\,\varepsilon^2}\sum_{k\geqslant n_0}2^{kc(1/p-1)}$$

and the series converges, since $p>1$. Hence for any fixed c we have

$$\binom{m}{c}|U_c^{(m,n)}|\to 0 \text{ a.s.}$$

As in the proof of Theorem 1 we consider again

$$R_{m,n}(k)=\sum_{c=k}^{m}\binom{m}{c}U_c^{(m,n)}\quad\text{ for some } 2\leqslant k\leqslant m.$$

We shall prove that there exists k ($2\leqslant k\leqslant m$) such that $R_{m,n}(k)$ converges to 0 completely.

To this end it suffices to observe that

$$\operatorname{Var}(R_{m,n}(k))=\sum_{c=k}^{m}\frac{\binom{m}{c}}{\binom{n}{c}}\frac{\gamma^{2c}}{c!}\leqslant\sum_{c=k}^{m}\frac{m^c 2^c\gamma^{2c}}{n^c c!}$$

$$=\left(\frac{m}{n}\right)^k\sum_{c=k}^{m}\left(\frac{m}{n}\right)^{c-k}\frac{(2\gamma^2)^c}{c!}\leqslant\left(\frac{m}{n}\right)^k\exp(2\gamma^2).$$

But now it follows immediately that the series of variances converges if only the parameter k is chosen in such a way that $k(1-1/p)>1$, since

$$\frac{m}{n}=\left(\frac{m^p}{n}\right)^{1/p}\frac{1}{n^{1-1/p}}\leqslant\frac{1}{n^{1-1/p}}$$

for sufficiently large n's. ∎

Remark 1. Observe that if m_{2n}/m_n is bounded, then the condition (6) holds.

Remark 2. Instead of (6) one can require

$$\sum_n\frac{1}{m_n n^{2-2/p}}<\infty. \tag{6'}$$

To see this note that

$$P\left(\sup_{n\geqslant 2^{n_0}}\binom{m_n}{c}|U_c^{(m,n)}|>\varepsilon\right)\leqslant\frac{2^{2/p}(2\gamma)^{2c}}{c!\,\varepsilon^2}\sum_{k\geqslant n_0}\frac{2^k}{m_{2^k}2^{k(2-2/p)}},$$

which follows from the inequality $m_n<n^{1/p}$ holding true for sufficiently large n's. The fact that the last series converges follows in view of the condensation criterion and (6′).

The above implies in particular that if $p > 2$, then the assumption (6) may be dropped, since the condition (6′) is then always satisfied.

Let us conclude with the following simple example of our result to random graphs (see also Rempała and Wesołowski [5]).

EXAMPLE (*Counting matchings in a bipartite random graph*). Let $G_{m,n,p} = (V_1, V_2; E)$ be a bipartite random graph with $V_1 = \{r_1, r_2, \ldots, r_m\}$, $V_2 = \{c_1, c_2, \ldots, c_n\}$ $(m \leqslant n)$ and $E \subset V_1 \times V_2$. Assume that the edges occur independently with a fixed probability $0 < p < 1$. In this setting, the reduced adjacency matrix of $G_{m,n,p}$ is a random $m \times n$ matrix $X = [X_{ij}]$ of independent Bernoulli $B(p)$ random variables. Denoting the number of *fully saturating* matchings by $H(G_{m,n,p})$ we have $H(G_{m,n,p}) = \mathrm{Per}(X)$. Thus, in the notation of this section, we have $\mu = p > 0$. Therefore, by Theorem 2 under the provisos (5) and (6), we obtain

$$\frac{H(G_{m,n,p})}{\binom{n}{m} m!\, p^m} \to 1 \text{ a.s.}$$

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