Aequationes Math. 63 (2002) 245–250
0001-9054/02/030245-6 \$ 1.50+0.20/0

Aequationes Mathematicae

On a functional equation related to the Matsumoto–Yor property

Jacek Wesołowski

Summary. A functional equation arising from the independence properties of some transformations of independent generalized inverse Gaussian and gamma variables is completely solved under the local integrability assumption.

Mathematics Subject Classification (2000). 39B40, 39B22, 60E05, 62E10.

Keywords. Functional equations, gamma distribution, generalized inverse Gaussian distribution, independence, local integrability.

1. Introduction

Matsumoto and Yor (2001) have recently discovered that the map ψ, defined by $\psi(x, y) = ((x + y)^{-1}, x^{-1} - (x + y)^{-1})$, acting on $(0,\infty)^2$, preserves a bivariate probability measure which is a product of the generalized inverse Gaussian and the gamma distributions. Recall, that the generalized inverse Gaussian (GIG) distribution $\mu_{-p,a,b}$, where $p \in \mathbb{R}$, $a, b \in (0,\infty)$, are the parameters, is defined by

$$\mu_{-p,a,b}(dx) = K_1 x^{-p-1} \exp(-(ax + b/x)/2) I_{(0,\infty)}(x)\, dx,$$

and the gamma distribution $\gamma_{q,c/2}$ is defined by

$$\gamma_{q,c/2}(dy) = K_2 y^{q-1} \exp(-cy/2) I_{(0,\infty)}(y)\, dy,$$

where $q, c \in (0,\infty)$ are parameters; K_1 and K_2 are norming constants. Matsumoto and Yor (2001) observed that if random variables X and Y are independent, X has the GIG distribution $\mu_{-p,a,a}$ and Y has the gamma distribution $\gamma_{p,a/2}$ $(p > 0)$, then the random vector $(U, V) = \psi(X, Y) = ((X+Y)^{-1}, X^{-1}-(X+Y)^{-1})$ has the same distribution as (X, Y), hence, in particular, U and V are independent. As observed in Letac and Wesołowski (2000), the following extension of the Matsumoto–Yor property holds: if (X, Y) has the distribution $\mu_{-p,a,b} \otimes \gamma_{p,a/2}$ ($\otimes$ denotes the product measure) then (U, V) is distributed according to $\mu_{-p,b,a} \otimes \gamma_{p,b/2}$. This follows by an elementary computation involving the Jacobian of the map ψ^{-1}, which is equal to $u^{-2}(u + v)^{-2}$. Consequently, denoting by f_Z the density of a random variable Z, we get

$$f_{(U,V)}(u,v) = u^{-2}(u+v)^{-2} f_X((u+v)^{-1}) f_Y(u^{-1} - (u+v)^{-1}), \quad \forall\, u,v \in (0,\infty).$$

Applying, to the right-hand side above, the formulas for the densities of the GIG and gamma distributions, we find out easily that the left-hand side of this identity can be factored into a function of u and a function of v, both functions being densities of the respective distributions. Hence we conclude also that U and V are independent.

Matsumoto and Yor (2001) asked a question about the converse to their observation. As a matter of fact, to some extent such a problem was investigated earlier in Letac and Seshadri (1983), where it was proved that if U and X have the same distribution and Y is gamma, then X has the GIG law. But, especially remarkable in the present setting is restricting the attention only to the independence property. Assume that X and Y are independent, and that the random vector $(U,V) = \psi(X,Y)$ has independent components. Is it true then that (X,Y) has the distribution $\mu_{-p,a,b} \otimes \gamma_{p,a/2}$ (and consequently (U,V) is distributed according to $\mu_{-p,b,a} \otimes \gamma_{p,b/2}$)? This question has been answered in the affirmative in a recent paper by Letac and Wesołowski (2000) (a related problem involving constancy of regression of V or V^{-1} on U has been considered also recently in Seshadri and Wesołowski (2001)), by using techniques related to Laplace transforms. Also in that paper the authors dealt with an important general case of distributions on the cone of positive definite symmetric matrices. The characterization obtained in this case however was restricted to distributions having strictly positive densities of the class C_2. Recently the smoothness assumptions were reduced only to differentiability in Wesołowski (2001). Since the proof in the univariate case, relying, as mentioned above, on Laplace transforms techniques, is hard to adopt in the matrix case, a possible way of dealing with the subject could be solving the real case by using densities, but with less strict smoothness conditions, and then trying to translate the argument to the matrix variables.

While attacking the question along these lines (see below) we are faced up with an intriguing functional equation:

$$g(x(x+y) - g(y(x+y)) = \alpha(x) - \alpha(y), \quad \forall\, x,y \in (0,\infty) \tag{1}$$

with unknown functions g and α. A search through the literature on functional equations including the classical positions as Aczél (1966), Kuczma (1968) or more recent, as Ramachandran and Lau (1992), Sahoo and Riedel (1998), revealed that equations of this type have not been considered yet. The present paper is devoted to study solutions of this equation under a mild regularity assumption. It remains still an open problem how to translate the argument developed here into the matrix variate case. But first we will explain how to arrive at the equation (1) starting from the independence property described above.

Since the random vectors (X,Y) and (U,V) have independent components, the following identity holds true for the densities:

$$f_U(u)f_V(v) = u^{-2}(u+v)^{-2}f_X((u+v)^{-1})f_Y(u^{-1}-(u+v)^{-1}), \quad \forall u, v \in (0,\infty). \tag{2}$$

Changing now the role of u and v in (2) one gets

$$f_U(v)f_V(u) = v^{-2}(u+v)^{-2}f_X((u+v)^{-1})f_Y(v^{-1}-(u+v)^{-1}), \quad \forall u, v \in (0,\infty). \tag{3}$$

Now combining (2) and (3), under the assumption that the densities are always strictly positive, it follows that

$$\frac{h(u)}{h(v)} = \frac{f_Y(u^{-1}-(u+v)^{-1})}{f_Y(v^{-1}-(u+v)^{-1})}, \quad \forall\, u, v \in (0,\infty),$$

where the function h is defined by $h(x) = x^2 f_U(x)/f_V(x)$, $x > 0$. Substituting $x^2 = u^{-1} - (u+v)^{-1}$ and $y^2 = v^{-1} - (u+v)^{-1}$ one gets

$$\frac{h(x^{-1}(x+y)^{-1})}{h(y^{-1}(x+y)^{-1})} = \frac{f_Y(x^2)}{f_Y(y^2)}, \quad \forall\, x, y \in (0,\infty).$$

Finally introduce new functions g and α by the formulas: $g(x) = \log(h(x^{-1}))$, $\alpha(x) = \log(f_Y(x^2))$, $x > 0$.

2. Solution of the functional equation

Now we are ready to formulate the main result of the paper, which gives the solution to a more general version of the equation (1).

Theorem 1. *Let g_1, g_2, α_1 and α_2 be locally integrable real functions defined on $(0,\infty)$ satisfying the equation*

$$g_1(x(x+y)) + g_2(y(x+y)) = \alpha_1(x) + \alpha_2(y), \quad \forall\, x, y \in (0,\infty). \tag{4}$$

Then there exist real numbers A, B, C and D such that $\forall\, x \in (0,\infty)$

$$g_1(x) = Ax + B\log(x) + C = -g_2(x), \quad \alpha_1(x) = Ax^2 + B\log(x) + D = -\alpha_2(x).$$

Proof. Since the functions are locally integrable we can take any $x_0,\ x_1 \in (0,\infty)$ such that $x_0 < x_1$, and integrate both sides of the equation (4) with respect to x. Then

$$\int_{x_0}^{x_1} g_1(x(x+y))\,dx + \int_{x_0}^{x_1} g_2(y(x+y))\,dx = \int_{x_0}^{x_1} \alpha_1(x)\,dx + (x_1 - x_0)\alpha_2(y),$$
$$\forall\, y \in (0,\infty).$$

Then substituting in the first integral on the left-hand side $s = x(x+y)$, i.e. $dx = ds/\sqrt{y^2+s}$, and in the second $t = y(x+y)$, i.e. $dx = dt/y$, one gets

$$\int_{x_0(x_0+y)}^{x_1(x_1+y)} \frac{g_1(s)}{\sqrt{y^2+s}}\,ds + \int_{y(x_0+y)}^{y(x_1+y)} \frac{g_2(t)}{y}\,dt = \int_{x_0}^{x_1} \alpha_1(x)\,dx + (x_1 - x_0)\alpha_2(y),$$
$$\forall\, y \in (0, \infty). \tag{5}$$

Dually, integrating (4) with respect to y from y_0 to y_1, $0 < y_0 < y_1$, we get

$$\int_{x(x+y_0)}^{x(x+y_1)} \frac{g_1(s)}{x}\,ds + \int_{y_0(x+y_0)}^{y_1(x+y_1)} \frac{g_2(t)}{\sqrt{x^2+t}}\,dt = (y_1 - y_0)\alpha_1(x) + \int_{y_0}^{y_1} \alpha_2(y)\,dy, \tag{6}$$
$$\forall\, x \in (0, \infty).$$

Observe that the left-hand side of (5) is a continuous function in y. Consequently α_2 is a continuous function. Similarly, by (6) it follows that α_1 is continuous.

Now insert in (4) $u = x(x+y)$ and $v = y(x+y)$. Consequently $x = u/\sqrt{u+v}$, $y = v/\sqrt{u+v}$ and (4) assumes the form

$$g_1(u) + g_2(v) = \alpha_1(u/\sqrt{u+v}) + \alpha_2(v/\sqrt{u+v}), \quad \forall\, u, v \in (0, \infty). \tag{7}$$

Since α_1 and α_2 are continuous it follows by (7) that g_1 and g_2 are also continuous. But for continuous g_i's the left-hand side of (5) is a C^1 function in y. Hence α_2 is also C^1. Also (6) implies, analogously, that α_1 is a C^1 function. Using again (7) we conclude that g_1 and g_2 are also C^1 functions.

Let us now differentiate (4) with respect to x. Then

$$(2x+y)g_1'(x(x+y)) + yg_2'(y(x+y)) = \alpha_1'(x), \quad \forall\, x, y \in (0, \infty).$$

Inserting in the above equation $x = y$ we get immediately that

$$\alpha_1'(x) = 3xg_1'(2x^2) + xg_2'(2x^2), \quad x > 0.$$

Then the equation takes the shape

$$(2x+y)g_1'(x(x+y)) + yg_2'(y(x+y)) = x(3g_1'(2x^2) + g_2'(2x^2)), \quad \forall\, x, y \in (0, \infty).$$

Plugging here u and v defined earlier we arrive at

$$(2u+v)g_1'(u) + vg_2'(v) = u(3g_1'(2u^2/(u+v)) + g_2'(2u^2/(u+v))), \quad \forall u, v \in (0, \infty). \tag{8}$$

Observe that $\lim_{v\to 0}(2u+v)g_1'(u) = 2ug_1'(u)$ and by C^1 property $\lim_{v\to 0} g_i'(2u^2/(u+v)) = g_i'(2u)$, $i = 1, 2$. Hence (8) implies that $B_2 \stackrel{\text{def}}{=} \lim_{v\to 0} vg_2(v)$ exists. Then

$$2ug_1'(u) + B_2 = u(3g_1'(2u) + g_2'(2u)), \quad u > 0. \tag{9}$$

Dually, differentiating (4) with respect to y we arrive at

$$\alpha_2'(y) = yg_1'(2y^2) + 3yg_2'(2y^2), \quad y > 0$$

and

$$ug_1'(u) + (u+2v)g_2'(u) = v(g_1'(2v^2/(u+v)) + 3g_2'(2v^2/(u+v))), \; \forall u, v \in (0, \infty). \tag{10}$$

Consequently, as above, it follows that $B_1 \stackrel{\text{def}}{=} \lim_{u\to 0} u g_1'(u)$ exists.

Then passing to the limit as $u \to 0$ in (9) we get $2B_1 + B_2 = (3B_1 + B_2)/2$ and consequently $B_1 = -B_2 = B$, say.

On the other hand let us divide both sides of (8) by v. Then it takes the form

$$\frac{2u+v}{v} g_1'(u) + g_2'(v) = \frac{u}{v}\left[3g_1'\left(\frac{2u^2}{u+v}\right) + g_2'\left(\frac{2u^2}{u+v}\right)\right], \quad \forall\, u,v \in (0,\infty). \quad (11)$$

Now observe that, as $v \to \infty$ then

$$\frac{2u+v}{v} g_1'(u) \to g_1'(u)$$

and by the C_1 property

$$\frac{u}{v} g_i'\left(\frac{2u^2}{u+v}\right) = \frac{2u^2}{u+v} g_i'\left(\frac{2u^2}{u+v}\right)\frac{u+v}{2uv} \to \frac{B_i}{2u}, \quad i = 1,2.$$

Consequently $A_2 \stackrel{\text{def}}{=} \lim_{v\to\infty} g_2'(v)$ exists. Dually, starting from (10), we conclude that $A_1 \stackrel{\text{def}}{=} \lim_{u\to\infty} g_1'(u)$ exists also. Now we divide both sides of (9) by u, and then take the limit as $u \to \infty$. Then it follows that $2A_1 = 3A_1 + A_2$, and consequently, $A_1 = -A_2 = A$, say.

Finally we get, by passing to the limit in (11) as $v \to \infty$, that $g_1'(u) = A + B/u$, $\forall\, u \in (0,\infty)$. By taking the limit as $u \to \infty$ in the dual to (11):

$$g_1'(u) + \frac{u+2v}{u} g_2'(v) = \frac{v}{u}\left[g_1'\left(\frac{2v^2}{u+v}\right) + 3g_2'\left(\frac{2v^2}{u+v}\right)\right], \quad \forall\, u,v \in (0,\infty)$$

one gets $g_2'(v) = -A - B/v\ \forall\, v > 0$. Hence using the expressions for α_i', $i = 1,2$, (in terms of g_i', $i = 1,2$) derived earlier we obtain that $\alpha_1'(u) = 2Au + B/u$, $u > 0$, and $\alpha_2'(v) = -2Av - B/v$, $v > 0$. Integrate now g_i' and α_i', $i = 1,2$, to arrive at the desired results. □

Remark 1. Observe that without any conditions on the behaviour of g_i's and α_i's a possible solution could be of the form: $g_1(x) = f(x) + h(x) + C = -g_2$, where f is any additive function, i.e. a function satisfying the classical Cauchy equation: $f(x+y) = f(x) + f(y)$, $x, y > 0$, which in general is not necessarily linear, see for instance Sahoo and Riedel (1998), and h is any logarithmic function, i.e. a function satisfying the equation $h(xy) = h(x) + h(y)$, $x, y > 0$.

Remark 2. Observe that in our original problem of characterizing the GIG and gamma distributions by the independence property, as described in Section 1, we arrived at the equation (1) which is a version of (4) with $g_1 = -g_2 = g$ and $\alpha_1 = -\alpha_2 = \alpha$. In such a setting the results of Theorem 1 hold true even under the assumption that at least one of the functions g or α is locally integrable. Hence if we assume that $\log(f_Y)$ is locally integrable on $(0,\infty)$, then it follows from Theorem 1 that the density function f_Y has the form

$$f_Y(x) = \exp(\alpha(\sqrt{x})) = e^D x^{A/2} e^{Bx}, \quad x > 0.$$

Consequently f_Y has to be the density of a gamma distribution, say $\gamma_{p_1,a_1/2}$. Observe now that since $\psi = \psi^{-1}$ then we can write down the equation dual to (2) by simply changing (U,V) into (X,Y). Again using the result of Theorem 1, this time with $f_V(x) = \exp(\alpha(\sqrt{x}))$, $x > 0$, we conclude that V is also a gamma random variable, say $\gamma_{p_2,b_2/2}$. On the other hand in the original setting, again by Theorem 1, we have

$$x^2 f_U(x)/f_V(x) = \exp(g(1/x)) = e^C e^{A/x} x^{-B}, \quad x > 0.$$

Since V has the gamma distribution one gets

$$f_U(x) = Kx^{-p_3-1}e^{-(b_3x+a_3/x)/2}, \quad x > 0,$$

which is the density of the GIG distribution μ_{-p_3,b_3,a_3}. Dually X is also a GIG random variable with a distribution $\mu_{-p,a,b}$, say. Inserting now respective densities into (2) we conclude that $p_1 = p_2 = p_3 = p > 0$, $a_1 = a_2 = a_3 = a$, $b_1 = b_2 = b_3 = b$.

Acknowledgement. The author is indebted to Prof. R. Ger for comments, which led to Remark 1.

References

[1] J. Aczél, *Lectures on Functional Equations and Their Applications*, Academic Press, New York, 1966.
[2] M. Kuczma, *Functional Equations in a Single Variable*, Polish Scientific Publishers, Warszawa, 1968.
[3] B. Ramachandran and K.-L. Lau, *Functional Equations in Probability Theory*, Academic Press, Boston, 1991.
[4] G. Letac and V. Seshadri, *A characterization of the generalized inverse Gaussian distribution by continued fraction*, Z. Wahrsch. Verw. Gebiete *62* (1983), 485–489.
[5] G. Letac and J. Wesołowski, *An independence property for the product of GIG and gamma laws*, Ann. Probab. *28* (2000), 1371–1383.
[6] H. Matsumoto and M. Yor, *An analogue of Pitman's $2M - X$ theorem for exponential Wiener functionals. II. The role of the generalized inverse Gaussian laws*, Nagoya Math. J. *162* (2001), 65–86.
[7] P. K. Sahoo and T. Riedel, *Mean Value Theorems and Functional Equations*, World Scientific, River Edge, 1998.
[8] V. Seshadri and J. Wesołowski, *Mutual characterizations of the gamma and the generalized inverse Gaussian laws by constancy of regression*, Sankhyā, A *63* (2001), 107–112.
[9] J. Wesołowski, *The Matsumoto–Yor independence property for GIG and gamma laws, revisited*, Math. Proc. Cambridge Philos. Soc., to appear.

J. Wesołowski
Wydział Matematyki i Nauk Informacyjnych
Politechnika Warszawska
Warszawa
Poland
e-mail: wesolo@alpha.mini.pw.edu.pl

Manuscript received: March 3, 2000 and, in final form, May 30, 2000.