ON THE LUKACS CHARACTERIZATION OF THE GAMMA DISTRIBUTION - REVIEW AND NEW RESULTS

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1. Introduction. The gamma distribution $\gamma_{p,b}$ is defined by the density $\gamma_{p,b}(dx) = \frac{b^p}{\Gamma(p)} x^{p-1} e^{-bx} I_{(0,\infty)}(x)dx$, $p, b > 0$. Let $\psi : (0,\infty)^2 \to (0,1)\times(0,\infty)$ be defined by $\psi(x,y) = \left(\frac{x}{x+y}, x+y\right)$. Let $(X,Y) \sim \gamma_{p,b} \otimes \gamma_{q,b}$. Then $(U,V) = \psi(X,Y) \sim \beta_{p,q} \otimes \gamma_{p+q,b}$, where $\beta_{p,q}$ denotes the beta distribution defined by the density $\beta_{p,q}(dx) = \frac{x^{p-1}(1-x)^{q-1}}{B(p,q)} I_{(0,1)}(x)\, dx$.

One of the most beautiful results in characterizations of probability distributions gives a converse to the above stated observation:

Theorem 1. (Lukacs, 1955). *Let (X,Y) be a random vector with nondegenerate, positive components. If both (X,Y) and $(U,V) = \psi(X,Y)$ have independent components then $(X,Y) \sim \gamma_{p,b} \otimes \gamma_{q,b}$ for some $p,q,b > 0$*

The original proof is based on the characteristic functions technique, though an approach through the Laplace transforms seems to be more natural and leads to a simpler proof. If densities are assumed then the following functional equation holds

$$f_U(u)f_V(v) = vf_X(uv)f_Y((1-u)v)\ ,$$

$0 < u < 1$, $v > 0$, which can be solved under some smoothness conditions. The functions f_X, f_Y, f_U, f_V are densties, i.e. nonnegative and integrable over their domains (to 1).

The Lukacs theorem was a starting point of investigations of numerous related problems.

2. Constancy of regressions. One of the approachess was through considering conditions of constancy of regressions (below we assume that X and Y are positive and nondegenerate rv's and U and V are defined as above):

Ordinary regressions: Assume that X, Y are independent. Additionally pairs of conditions of constancy of regressions for U and V are imposed: (1) or (2) or (3) below. In each of the settings we ask: Does it follow that X and Y are gamma rv's?

$$E(U^i|V) = c_i\ , \quad i = r, s\ . \tag{1}$$

The answer is "yes" if: $i = 1, 2$ - Bolger, Harkness (1965); $i = \pm 1$- Wesołowski (1990); $i = -1, -2$ - Li, Huang, Huang (1994); and the answer is "no" if $i = 1, 3$ - Hall, Simons (1969).

$$E(U^i|V) = c\ , \quad E((1-U)^i|V) = d. \tag{2}$$

The answer is "yes" if: $i = 2$ - Hall, Simons (1969); $i = -1$ - Huang, Su (1997). It is trivial to check that if $i = 1$ then the answer is "no".

$$E(U|V) = c\ , \quad E(V|U) = d. \tag{3}$$

The answer is "no" - Bobecka and Wesołowski (2002a).

Dual regressions: Assume that U, V are independent. Again conditions of constancy of regressions, this time for X and Y are imposed: one of (4), or (5) below. Again we ask: Does it follow that X and Y are gamma rv's?

$$E(X^i|Y) = c_i \ , \quad \textit{or} \quad E(Y^i|X) = d_i \ , \quad i = r, s \ . \tag{4}$$

The answer is "yes" in both situations if: $i = 1, 2$ or $i = \pm 1$ or $i = -1, -2$ - Bobecka, Wesołowski (2002a). These results have been recently generalized by Chou, Huang (2002).

$$E(X|Y) = c \ , \quad E(Y|X) = d \ . \tag{5}$$

The answer is "no" - Bobecka and Wesołowski (2002a).

3. Random matrices. Let $\mathcal{V}$ denote the Euclidean space of symmetric real $n \times n$ matrices with the inner product $(a, b) = tr(ab)$, $\forall\, a, b \in \mathcal{V}$ and the identity matrix e. The Lebesque measure assigns a unit mass to the unit cube. Let $\mathcal{V}_+ \subset \mathcal{V}$ denote the cone of symmetric positive definite matrices, and $\overline{\mathcal{V}}_+$ its closure.

The Wishart (matrix variate gamma) distribution $\gamma_{p,a}$ in $\overline{\mathcal{V}}_+$ is defined by its Laplace transform:

$$\int_{\overline{\mathcal{V}}_+} e^{-(\theta, y)} \gamma_{p,a}(dy) = [\det(e + \theta a^{-1})]^{-p},$$

$a \in \mathcal{V}_+$, $p \in \Lambda_n = \{\frac{1}{2}, \frac{2}{2}, \ldots, \frac{n-1}{2}\} \cup (\frac{n-1}{2}, \infty)$, for any $\theta \in \mathcal{V}_+$.

If $p > \frac{n-1}{2}$ then it is absolutely continuous with the density

$$\gamma_{p,a}(dy) = \frac{(\det a)^p}{\Gamma_n(p)} (\det y)^{p - \frac{n+1}{2}} e^{-(a,y)} I_{\mathcal{V}_+}(y)\, dy \ .$$

An analogue of the Lukacs theorem was proved in Olkin, Rubin (1962). Quite recently the argument was enhanced in Casalis, Letac (1996).

Theorem 2. (Lukacs-Olkin-Rubin) *Let X, Y be independent rv's in $\overline{\mathcal{V}}_+$ such that $X + Y$ is in $\mathcal{V}_+$ a.s. Let $w : \mathcal{V}_+ \to \mathcal{M}$ be a measurable function such that*

$$w(y)[w(y)]^T = y \ , \quad y \in \mathcal{V}_+ \ .$$

Define

$$V = X + Y \ , \quad U = (w(V))^{-1} X ([w(V)]^T)^{-1} \ .$$

Assume that for any orthogonal matrix $\mathcal{O}$

$$U \stackrel{d}{=} \mathcal{O} U \mathcal{O}^T \ . \tag{6}$$

If U and V are independent then there exist $a \in \mathcal{V}_+$ and $p, q \in \Lambda_n$ such that $p + q > \frac{n-1}{2}$ and $(X, Y) \sim \gamma_{p,a} \otimes \gamma_{q,a}$.

An important role in the known proofs of the Lukacs-Olkin-Rubin Theorem was played by the condition of invariance of the "quotient" (6). Therefore through last fourty years it was believed that this invariance property is somehow deeply rooted into the nature of the problem in the matrix variate case. However, only recently, Bobecka and Wesołowski (2002b) proved that the result (under rather technical assumptions) holds true without assuming invariance of the "quotient". They considered a natural choice for w: $w(y) = y^{1/2}$.

Theorem 3. *Let X and Y be independent rv's valued in $\mathcal{V}_+$ with strictly positive, twice differentiable densities. Denote $V = X+Y$, $U = V^{-1/2}XV^{-1/2}$. Let U and V be independent.*
Then $\exists\, p, q > \frac{n-1}{2}$ and $a \in \mathcal{V}_+$ such that $(X,Y) \sim \gamma_{p,a} \otimes \gamma_{q,a}$.

Instead of using the Laplace transform technique the proof is based on investigations of densities. The basic equation has the form:

$$f_U(u)f_V(v) = (\det v)^{\frac{n+1}{2}} f_X(v^{\frac{1}{2}}uv^{\frac{1}{2}})f_Y(v^{\frac{1}{2}}(e-u)v^{\frac{1}{2}}) ,$$

$u \in \mathcal{D} = \{x \in \mathcal{V}_+ : \; e - x \in \mathcal{V}_+\}$, $v \in \mathcal{V}_+$. Essentially the proof is based on solutions of two functional equations:

Proposition 1. *Let $a : \mathcal{D} \to \mathbf{R}$, $g : \mathcal{V}_+ \to \mathbf{R}$ be functions such that*

$$a(x) = g(yxy) - g(y(e-x)y)$$

holds for any $x \in \mathcal{D}$, $y \in \mathcal{V}_+$. Assume that g is differentiable.
Then there exist $\lambda, \beta \in \mathbf{R}$ such that $a(x) = \lambda \log(\det x(e-x)^{-1})$, $x \in \mathcal{D}$, $g(y) = \lambda \log(\det y) + \beta$, $y \in \mathcal{V}_+$.

Proposition 2. *Let $a_1 : \mathcal{D} \to \mathbf{R}$ and $a_2, g : \mathcal{V}_+ \to \mathbf{R}$ be functions satisfying*

$$a_1(x) + a_2(y) = g(yxy) + g(y(e-x)y)$$

for any $x \in \mathcal{D}$ and $y \in \mathcal{V}_+$. Assume that g is twice differentiable.
Then there exist $\delta \in \mathcal{V}$ and λ, σ_1, $\sigma_2 \in \mathbf{R}$ such that $a_1(x) = \lambda \log(\det x(e-x)) + \sigma_1$, $x \in \mathcal{D}$, $a_2(y) = 4\lambda \log(\det y) + (\delta, y^2) + \sigma_2$, $y \in \mathcal{V}_+$, $g(z) = \lambda \log(\det z) + (\delta, z) + \sigma_1 + \sigma_2$, $z \in \mathcal{V}_+$.

In the matrix variate case conditions of constancy of regressions have been considered rather recently in Letac, Massam (1998):

For any $x \in \mathcal{V}_+$ define two linear operators $x \otimes x$, $\mathbf{P}(x) : \mathcal{V} \to \mathcal{V}$ by: $x \otimes x(h) = (x,h)x$, $\mathbf{P}(x)(h) = xhx$, $h \in \mathcal{V}$.

Let X, Y be independent and $E(X|X+Y) = \alpha(X+Y)$. If

$$E(X \otimes X|X+Y) = a(X+Y) \otimes (X+Y) + b\mathbf{P}(X+Y) ,$$

$$E(\mathbf{P}(X)|X+Y) = c(X+Y) \otimes (X+Y) + d\mathbf{P}(X+Y) ,$$

or

$$E(X^{-1}|X+Y) = a(X+Y)^{-1} ,$$

$$E(X^{-1} \otimes X|X+Y) = b\mathbf{Id}_{\mathcal{V}} + c(X+Y)^{-1} \otimes (X+Y) ,$$

then X and Y have the matrix variate gamma (Wishart) distributions.

4. Random vectors. Characterizations of the bivariate gamma distribution through Lukacs type regression conditions were obtained in Wang (1981) and Bobecka (2002). Unexpectedly the n-variate version of the Lukacs independence conditon characterizes only gamma random vectors with a very special structure (Bobecka and Wesołowski (2002c)):

Theorem 4. *Let $\underline{X} = (X_1, \ldots, X_n)$ and $\underline{Y} = (Y_1, \ldots, Y_n)$ be independent, nondegenerate n-variate random vectors with positive components. Assume that the random vectors $\underline{U} = (U_1, \ldots, U_n) = (\frac{X_1}{X_1+Y_1}, \ldots, \frac{X_n}{X_n+Y_n})$ and $\underline{V} = (V_1, \ldots, V_n) = \underline{X} + \underline{Y}$ are independent.*

Then $\exists\, r \in \{1,\ldots,n\}$ *and sets* $\exists\, A_1,\ldots,A_r$ *such that* $\bigcup_{i=1}^r A_i = \{1,\ldots,n\}$, $A_i \cap A_j = \emptyset$, $i \neq j$ *and the Laplace transform of* $\underline{X}$ *is of the form*

$$L_{\underline{X}}(\underline{s}) = \prod_{i=1}^{r}\left(1 - \sum_{j\in A_i} \lambda_j s_j\right)^{-p_i},$$

where λ_j*'s and* p_i*'s are positive, i.e. the random vectors* $\underline{Z}_i = (X_j)_{j\in A_i}$, $i = 1,\ldots,r$, *are independent and* $\forall\, i\ \exists\, k \in A_i$ *such that* $\forall\, j \in A_i\ X_j = \alpha_j X_k$ *for some constants* α_j*'s* (X_k*'s are gammas). For* $\underline{Y}$ *change* p_i*'s into* q_i*'s.*

A similar result holds true if constancy of n variate versions of dual regressions is assumed. It is not known what happens if the constancy of n variate versions of ordinary regressions is considered instead. The main problem here is related to infinite divisibility of n variate gamma distribution, which, except the case $n = 2$, does not hold in general - see for instance Vere-Jones (1967), Moran, Vere-Jones (1969) or Griffiths (1979, 1984),

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