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The Matsumoto–Yor independence property for GIG and Gamma laws, revisited

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Abstract

Matsumoto and Yor have recently discovered an interesting invariance property of a product of the generalized inverse Gaussian and gamma distributions. In this paper we obtain: (1) a complete regression version of its converse; (2) a converse to the matrix variate Matsumoto–Yor property which extends an earlier result. Of independent interest is a functional equation for matrix valued functions, which has been solved in the course of investigation of the second problem.

1. *Introduction*

Matsumoto and Yor [**11**] have recently described an interesting invariance property of the product of the generalized inverse Gaussian and gamma distributions. This paper is devoted to study converses of this property, and thus investigates conditions under which it uniquely determines both the distributions.

To state the property let us first recall that the generalized inverse Gaussian (GIG) distribution $\mu_{p,a,b}$, where $p \in \mathbf{R}$ and $a,\ b \in (0,\infty)$ is defined by

$$\mu_{p,a,b}(dx) = \frac{(a/b)^{p/2}}{2K_p(2\sqrt{ab})} x^{p-1} \exp\left(-ax - b/x\right) I_{(0,\infty)}(x)\, dx,$$

where K_p is the modified Bessel function of the third kind, and the gamma distribution $\gamma_{q,c}$, where $q,\ c \in (0,\infty)$ is defined by

$$\gamma_{q,c}(dy) = \frac{c^q}{\Gamma(q)} y^{q-1} \exp\left(-cy\right) I_{(0,\infty)}(y)\, dy.$$

Consider a one-to-one map $\psi\colon (0,\infty)^2 \to (0,\infty)^2$ defined by $\psi(x,y) = (u(x,y), v(x,y)) = (1/(x+y), 1/x - 1/(x+y))$, $x, y > 0$. Matsumoto and Yor [**11**] proved that the measure $\mu_{-p,a,a} \otimes \gamma_{p,a}$ on $(0,\infty)^2$, $p > 0$, is invariant under ψ and asked about converses to the property they discovered. A kind of a partial converse was known earlier and is due to [**7**], who proved for independent random variables X and Y, by continuous fraction technique, that if X has the same distribution as

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$U = 1/(X+Y)$ then it follows that X is GIG if Y is assumed to be gamma. This result was generalized to the matrix variate GIG and Wishart-gamma distribution in [**1**].

The property derived in [**11**] can be extended in the following way: assume that X and Y are independent random variables with respective distributions $\mu_{-p,a,b}$ and $\gamma_{p,a}$, $(p > 0)$ and define a random vector $(U, V) = \psi(X, Y)$. Then it follows that U and V are also independent and their distributions are $\mu_{-p,b,a}$ and $\gamma_{p,b}$, respectively.

Here we will concentrate on converse statements based on the independence of U and V. Assume then that X, Y are non-negative independent random variables and assume also that $U = 1/(X+Y)$, $V = 1/X - 1/(X+Y)$ are independent. Is it true that $(X, Y) \sim \mu_{-p,a,b} \otimes \gamma_{p,a}$ (and consequently $(U, V) \sim \mu_{-p.b,a} \otimes \gamma_{p,b}$) for some positive numbers a, b and p?

A complete solution to this problem (in the univariate case) was obtained in [**9**], with the proof based on application of the Laplace transform technique – see also [**17**] for an approach based on densities.

A regression version of the problem was partially solved in [**14**], where the constancy of regression of V or V^{-1} on U, separately, was considered. However relaxing the independence assumption was possible only under the cost of assuming one law to obtain the other, i.e. the characterization was not simultaneous. In Section 2 we consider jointly the constancy of regression conditions which leads to a characterization of both distributions – extending in a way the result of [**9**] with a natural restriction of existence of the inverse moment for X.

Another question posed in [**11**] was related to the matrix variate version of the property they discovered. This problem was settled in [**9**] for the GIG and Wishart-gamma distributions on the cone of symmetric postive definite matrices. Also in that paper a partial converse was obtained under the assumption that densities of random matrices X and Y exist and are strictly positive C^2 real functions. Here in Section 3 we give a straightforward extension of this result by weakening the smoothness assumption, imposed on densities, to differentiability. A core of the proof is a solution of some functional equation for matrix functions in matrix variables, which seems to be of independent interest.

2. *Constancy of regression*

It has been recently proved in [**14**] that, under suitable conditions on existence of moments, the constancy of regression

$$E(V|U) = c, \tag{1}$$

characterizes the distribution of X as GIG if the distribution of Y is assumed to be gamma, and in the opposite direction, it determines the distribution of Y to be gamma if X is assumed to have a GIG distribution. Here and in the sequel the equations between random variables are understood to hold almost surely.

Similar results follow if the constancy of the regression of the reciprocal of V, i.e.

$$E(V^{-1}|U) = d, \tag{2}$$

is considered.

In both cases the proofs were based on identification of probabilistic solutions of second-order differential equations for Laplace transforms of measures related to distributions of X and Y.

As pointed out in [**14**], since

$$V = \frac{Y}{X(X+Y)},$$

then (1) is equivalent to

$$E(Y/X|X+Y) = c(X+Y)$$

and (2) is equivalent to

$$E(X/Y|X+Y) = d/(X+Y).$$

Here we develop further the approach by considering both the regression conditions (1) and (2) simultaneously. Then the complete characterization of the distribution of X and Y is obtained. Consequently the result is a regression version of the characterization through independence obtained in [**9**].

It is worthy to recall at this moment that, in past studies, characterizations based on independence were often accompanied by their regression counterparts. The celebrated Darmois–Skitovitch theorem on characterization of the normal law by independence of linear forms in independent random variables was converted to a regression version in [**5**]. The regression analogues of the famous Lukacs theorem on characterization of the gamma distribution by independence of the sum and the quotient of two independent non-negative random variables were obtained in [**15**], [**16**] or [**10**], with a recent contribution by [**6**] for positive definite random matrices (even Jordan algebras, as a matter of fact). For the inverse Gaussian distribution the constancy of regression counterpart of the converse to the Tweedie theorem, obtained in [**4**], was proved by [**8**]. So the result we formulate now can be seen as a new contribution within such a scheme.

THEOREM 1. *Let X and Y be positive independent random variables and $E(1/X) < \infty$. For $U = 1/(X+Y)$ and $V = 1/X - 1/(X+Y)$ assume that the conditions* (1) *and* (2) *hold. Then $cd > 1$ and there exist $a > 0$ such that $X \sim \mu_{-p,a,b}$ and $Y \sim \gamma_{p,a}$, where $p = cd/(cd-1) > 1$, and $b = d/(cd-1) > 0$.*

Proof. All the expressions beneath are considered for any $s < 0$.

Observe that (1) is equivalent to

$$EX^{-1}e^{s(X+Y)} - E(X+Y)^{-1}e^{s(X+Y)} = cEe^{s(X+Y)} \tag{3}$$

and, by the observation above, (2) is equivalent to

$$EXY^{-1}e^{s(X+Y)} = dE(X+Y)^{-1}e^{s(X+Y)}. \tag{4}$$

Combining (3) and (4) we get

$$EX^{-1}e^{s(X+Y)} - d^{-1}EXY^{-1}e^{s(X+Y)} = cEe^{s(X+Y)}. \tag{5}$$

Now we introduce two Laplace transforms:

$$h_X(s) = EX^{-1}\exp(sX)\,, \quad h_Y(s) = EY^{-1}\exp(sY).$$

Then (5) can be rewritten in the form

$$dh_Y'(h_X - ch_X') = h_X'' h_Y. \tag{6}$$

On the other hand after taking derivatives in (3), which for $s < 0$ is always allowed, we get

$$h''_Y(h_X - ch'_X) = ch''_X h'_Y. \tag{7}$$

Observe that h'_X and h'_Y are, respectively, the Laplace transforms of X and Y. Hence it follows that neither h''_X nor h''_Y can be identically zero in any open interval contained in $(-\infty, 0)$, since then the Laplace transforms would be affine functions, which is contradictory to their boundedness property. Consequently from (6) and (7) it follows that

$$cd(h'_Y)^2 = h''_Y h_Y.$$

Exactly such an equation was solved in [**16**] – see equation (3) in that paper. Following that solution we get

$$E(\exp(sY)) = (1 - s/a)^{-cd/(cd-1)},$$

which implies that $cd > 1$ and that $Y \sim \gamma_{p,a}$, where $p = cd/(cd-1) > 1$ and $a > 0$ is a free parameter.

Consequently, substituting h'_Y and h''_Y in (7) we arrive at the second-order differential equation for h_X

$$(a - s)h''_X(s) + ph'_X(s) - bh_X(s) = 0$$

at least for $s < 0$. Then following the proof of Theorem 1 of [**14**], where the same equation was encountered, we conclude that its only probabilistic solution is the modified Bessel function of the third kind $K_{(p+1)}$, and consequently $X \sim \mu_{-p,a,b}$ with $b = d/(cd-1)$. Note that the assumption $EX < \infty$, imposed there, is not necessary since for $s < 0$ the random variable $X \exp(sX)$ is always bounded. So we can solve the equation for $s < 0$ and then use the analytical extension principle to define the respective Laplace transform for any $s < a$.

3. *A refinement of the converse for random matrices*

Denote by $\mathscr{V}$ the Euclidean space of symmetric real $r \times r$ matrices with the inner product $(a, b) = \mathrm{trace}(ab)$, $a, b \in V$. The Lebesgue measure dx on $\mathscr{V}$ assigns unit mass to the unit cube of $\mathscr{V}$. By $\mathscr{V}_+ \subset \mathscr{V}$ we denote the cone of positive definite symmetric matrices.

The GIG $\mu_{p,a,b}$, where $p \in \mathbf{R}$ and $a, b \in \mathscr{V}_+$, is defined by the density

$$\mu_{p,a,b}(dx) = \frac{1}{K_p(a,b)}(\det x)^{p-\frac{r+1}{2}} \exp(-(a,x) - (b, x^{-1}))I_{\mathscr{V}_+}(x)\, dx\,,$$

where K_p is a version of the matrix Bessel function (see [**3**] or [**9**]), and the Wishart-gamma $\gamma_{q,c}$ distribution on $\mathscr{V}_+$ for $q > (r-1)/2$ and $c \in \mathscr{V}_+$ is defined by

$$\gamma_{q,c}(dy) = \frac{(\det(c))^q}{\Gamma_r(q)}(\det y)^{q-\frac{r+1}{2}} \exp(-(c, y))I_{\mathscr{V}_+}(y)\, dy\,,$$

where Γ_r is the multivariate Gamma function (see, for instance [**12**, p. 61]).

Some basic relations between Wishart-gamma and GIG distributions were studied in [**1**] and [**2**]. In [**9**] it was proved that if the random matrices X and Y are independent and have, respectively, the distributions $\mu_{-p,a,b}$ and $\gamma_{p,a}$, where $p > (r-1)/2$, $a, b \in \mathscr{V}_+$, then $U = (X+Y)^{-1}$ and $V = X^{-1} - (X+Y)^{-1}$ are also independent with respective distributions $\mu_{-p,b,a}$ and $\gamma_{p,b}$. The result holds also true for the singular Wishart-gamma distribution. The proof in this direction was based on the Laplace transforms technique.

A partial converse obtained in [**9**] relied on a restrictive smoothness C^2 condition imposed on the densities. The aim of this section is to weaken the smoothness assumption. A result in this direction, but in the univariate case, has been recently obtained in [**17**], where a functional equation for densities was solved under the assumption of local integrability of logarithms of the densities. However it is unknown how to adapt the argument developed in that paper to the matrix variate case. The proof of the converse to the Matsumoto–Yor property we offer, relies on differentiability of densities, and though has the same starting point as the proof in [**9**], is rather different, and essentially is based on a solution of a functional equation for functions of matrix variate arguments, which seems to be also of independent interest. The solution is rather easy in the univariate case, but for matrices, due to the non-commutativity of multiplication, the argument becomes much more involved.

For any $u \in \mathscr{V}_+$ we will consider two linear operators $\mathbb{P}$ and $\mathbb{L}$ defined on $\mathscr{V}$ by

$$\mathbb{P}(u)h = uhu\ , \quad \mathbb{L}(u)h = uh + hu\ , \quad \forall\, h \in \mathscr{V}.$$

Observe that they are related by

$$\mathbb{P}(u^{-1})\mathbb{L}(u) = \mathbb{L}(u)\mathbb{P}(u^{-1}) = \mathbb{L}(u^{-1}).$$

Also $\mathbb{P}^{-1}$ and $\mathbb{L}^{-1}$ exist; for $\mathbb{P}^{-1}$ we simply have $\mathbb{P}^{-1}(u) = \mathbb{P}(u^{-1})$ and for $\mathbb{L}^{-1}$ it follows for instance from [**13**, theorem 5·1]. Observe that the two inverse operators are related by

$$\mathbb{L}^{-1}(u)\mathbb{P}(u) = \mathbb{P}(u)\mathbb{L}^{-1}(u) = \mathbb{L}^{-1}(u^{-1}). \tag{8}$$

Now we are ready to present the result on the solution of a matrix variate functional equation.

Theorem 2. *Let A, $B\colon \mathscr{V}_+ \to \mathscr{V}$ and $C\colon \mathscr{V}_+^2 \to \mathscr{V}$ be some functions and C is symmetric, i.e. $C(u,v) = C(v,u)$ for any u, $v \in \mathscr{V}_+$. Assume that*

$$A(u) + [\mathbb{P}(u+v) - \mathbb{P}(u)]B(v) = C(u,v) \tag{9}$$

holds for any $u, v \in \mathscr{V}_+$. Then there exist a, $b \in \mathscr{V}$ and $\lambda \in \mathbf{R}$ such that for any u, $v \in \mathscr{V}_+$

$$A(u) = b - \lambda u + \mathbb{P}(u)a,$$

$$B(u) = a + \lambda u^{-1},$$

$$C(u,v) = b + \lambda(u+v) + \mathbb{P}(u+v)a.$$

Proof. Denote by $\mathrm{id}_{\mathscr{V}}$ the identity matrix in $\mathscr{V}$ and by Id the identity operator on $\mathscr{V}$. Observe that for any $u \in \mathscr{V}_+$ and any real number β

$$\mathbb{P}(u + \beta\, \mathrm{id}_{\mathscr{V}}) - \mathbb{P}(u) = \beta\mathbb{L}(u) + \beta^2\mathrm{Id}, \tag{10}$$

$$\mathbb{P}(u + \beta\, \mathrm{id}_{\mathscr{V}}) - \mathbb{P}(\beta\, \mathrm{id}_{\mathscr{V}}) = \mathbb{P}(u) + \beta\mathbb{L}(u). \tag{11}$$

By symmetry of C it follows from (9) that

$$A(u) + [\mathbb{P}(u+v) - \mathbb{P}(u)]B(v) = A(v) + [\mathbb{P}(u+v) - \mathbb{P}(v)]B(u). \tag{12}$$

Now, using (10) and (11) for $\beta = 1$ and $\beta = 2$, we obtain two versions of (12) by substituting $v = \mathrm{id}_{\mathscr{V}}$ and $v = 2\mathrm{id}_{\mathscr{V}}$:

$$A(u) + \mathbb{L}(u)B(\mathrm{id}_{\mathscr{V}}) + B(\mathrm{id}_{\mathscr{V}}) = A(\mathrm{id}_{\mathscr{V}}) + \mathbb{P}(u)B(u) + \mathbb{L}(u)B(u), \tag{13}$$

$$A(u) + 2\mathbb{L}(u)B(2\mathrm{id}_{\mathscr{V}}) + 4B(2\mathrm{id}_{\mathscr{V}}) = A(2\mathrm{id}_{\mathscr{V}}) + \mathbb{P}(u)B(u) + 2\mathbb{L}(u)B(u).$$

Subtracting the first from the second we get

$$\mathbb{L}(u)a + c = \mathbb{L}(u)B(u),$$

where

$$a = 2B(2\mathrm{id}_{\mathscr{V}}) - B(\mathrm{id}_{\mathscr{V}}) \in \mathscr{V},$$

$$c = 4B(2\mathrm{id}_{\mathscr{V}}) - B(\mathrm{id}_{\mathscr{V}}) - A(2\mathrm{id}_{\mathscr{V}}) + A(\mathrm{id}_{\mathscr{V}}) \in \mathscr{V}.$$

Hence

$$B(u) = a + \mathbb{L}^{-1}(u)c\,, \quad \forall\, u \in \mathscr{V}_+. \tag{14}$$

Since $\mathbb{L}^{-1}(\mathrm{id}_{\mathscr{V}})c = \frac{1}{2}c$, then $B(\mathrm{id}_{\mathscr{V}}) = a + \frac{1}{2}c$, and plugging (14) into (13) we arrive at

$$A(u) + \mathbb{L}(u)a + \tfrac{1}{2}\mathbb{L}(u)c + a + \tfrac{1}{2}c = A(\mathrm{id}_{\mathscr{V}}) + \mathbb{P}(u)a + \mathbb{P}(u)\mathbb{L}^{-1}(u) + \mathbb{L}(u)a + c.$$

Finally it follows by (8) that

$$A(u) = b + \mathbb{P}(u)a - \tfrac{1}{2}\mathbb{L}(u)c + \mathbb{L}^{-1}(u^{-1})c\,, \quad u \in \mathscr{V}_+, \tag{15}$$

where $b = A(\mathrm{id}_{\mathscr{V}}) - a + \frac{1}{2}c \in \mathscr{V}$.

Now our aim is to show that c is a multiple of $\mathrm{id}_{\mathscr{V}}$. To this end let us insert (14) and (15) into (12), which gives

$$\begin{aligned} -\tfrac{1}{2}\mathbb{L}(u)c + \mathbb{L}^{-1}(u^{-1})c + [\mathbb{P}(u+v) - \mathbb{P}(u)]\mathbb{L}^{-1}(v)c \\ = -\tfrac{1}{2}\mathbb{L}(v)c + \mathbb{L}^{-1}(v^{-1})c + [\mathbb{P}(u+v) - \mathbb{P}(v)]\mathbb{L}^{-1}(u)c. \end{aligned}$$

Since for any $h \in \mathscr{V}$

$$[\mathbb{P}(u+v) - \mathbb{P}(u) - \mathbb{P}(v)]h = uhv + vhu,$$

then the above equation can be rewritten as

$$\begin{aligned} -\tfrac{1}{2}\mathbb{L}(u)c + \mathbb{L}^{-1}(u^{-1})c + u\mathbb{L}^{-1}(v)cv + v\mathbb{L}^{-1}(v)cu + \mathbb{P}(v)\mathbb{L}^{-1}(v)c \\ = -\tfrac{1}{2}\mathbb{L}(v)c + \mathbb{L}^{-1}(v^{-1})c + v\mathbb{L}^{-1}(u)cu + u\mathbb{L}^{-1}(u)cv + \mathbb{P}(u)\mathbb{L}^{-1}(u)c. \end{aligned}$$

Again using (8) the above identity simplifies to

$$-\tfrac{1}{2}\mathbb{L}(u)c + u\mathbb{L}^{-1}(v)cv + v\mathbb{L}^{-1}(v)cu = -\tfrac{1}{2}\mathbb{L}(v)c + v\mathbb{L}^{-1}(u)cu + u\mathbb{L}^{-1}(u)cv.$$

For any positive α and β, change in the above equation u into αu and v into βv. Observing that $\mathbb{L}(\alpha u) = \alpha\mathbb{L}(u)$ and $\alpha\mathbb{L}^{-1}(\alpha u) = \mathbb{L}^{-1}(u)$, we get for any α, $\beta > 0$ that

$$\alpha(-\tfrac{1}{2}\mathbb{L}(u) + u\mathbb{L}^{-1}(v)cv + v\mathbb{L}^{-1}(v)cu) = \beta(-\tfrac{1}{2}\mathbb{L}(v) + v\mathbb{L}^{-1}(u)cu + u\mathbb{L}^{-1}(u)cv).$$

Consequently

$$\tfrac{1}{2}\mathbb{L}(u)c = u\mathbb{L}^{-1}(v)cv + v\mathbb{L}^{-1}(v)cu \tag{16}$$

for any u, $v \in \mathscr{V}_+$. Now insert $u = v$ in (16). Then

$$\tfrac{1}{2}\mathbb{L}(v)c = 2v\mathbb{L}^{-1}(v)cv = 2\mathbb{P}(v)\mathbb{L}^{-1}(v)c.$$

Hence

$$\mathbb{L}^{-1}(v)c = \tfrac{1}{4}\mathbb{P}(v^{-1})\mathbb{L}(v)c = \tfrac{1}{4}\mathbb{L}(v^{-1})c.$$

Substituting it back into (16) we obtain

$$2\mathbb{L}(u)c = u\mathbb{L}(v^{-1})cv + v\mathbb{L}(v^{-1})cu.$$

For $u = v^2$ the right-hand side of the above identity has the form:

$$\mathbb{P}(v)[v\mathbb{L}(v^{-1})c + \mathbb{L}(v^{-1})cv] = \mathbb{P}(v)(2c + vcv^{-1} + v^{-1}cv) = 2\mathbb{P}(v)c + \mathbb{L}(v^2)c.$$

Then we conclude that

$$\mathbb{L}(v^2)c = 2\mathbb{P}(v)c$$

for any $v \in \mathscr{V}_+$. Consequently, using the basic properties of the inner product, we have the following sequence of identities

$$\begin{aligned} 0 &= (\mathbb{L}(v^2)c - 2\mathbb{P}(v)c, c) = (v^2c, c) - (vcv, c) + (cv^2, c) - (vcv, c) \\ &= (cv, vc) - (vc, vc) + (cv, vc) - (cv, cv) \\ &= (cv - vc, vc) + (cv, vc - cv) = (cv - vc, vc) - (cv - vc, cv) \\ &= (cv - vc, vc - cv) = (cv - vc, (cv - vc)^{\mathsf{T}}) = ||cv - vc||^2, \end{aligned}$$

where $^{\mathsf{T}}$ denotes transpose, and $||\cdot||$ denotes the norm defined by the inner product.

Then it follows that for any $v \in \mathscr{V}_+$ we have $cv = vc$. Now insert in this identity $v = v_{ij} = \mathrm{id}_{\mathscr{V}} + \varepsilon e_{ij}$ for any $i, j \in \{1, \dots, r\}$, where e_{ij} is an $r \times r$ matrix with zero entries, except (i, j) and (j, i) entries which are equal 1 and ε is chosen sufficiently small to have $v_{ij} \in \mathscr{V}_+$. Then it follows immediately that $c_{ik} = c_{jk} = 0$ for any $k \neq i, j$ and $c_{ii} = c_{jj}$ for any $i, j = 1, \dots, r$. Thus $c = 2\lambda\,\mathrm{id}$ for some real number λ.

Since $\mathbb{L}^{-1}(u)2\lambda\,\mathrm{id} = 2\lambda u^{-1}$ then from (14) and (15) we get the assertion of the theorem for A and B. The formula for C follows just by inserting the final expressions for A and B into (9).

The above result is the core of the proof of our main result in this section, which is a converse to the Matsumoto-Yor property in the matrix variate case under mild smoothness conditions for densities.

THEOREM 3. *Let X and Y be independent random variables valued in $\mathscr{V}_+$ with strictly positive differentiable densities. Assume that $U = (X + Y)^{-1}$ and $V = X^{-1} - (X + Y)^{-1}$ are independent. Then there exist $p > (r - 1)/2$ and $a, b \in \mathscr{V}_+$ such that $X \sim \mu_{-p,a,b}$ and $Y \sim \gamma_{p,a}$.*

Proof. Define the map $\psi: \mathscr{V}_+^2 \to \mathscr{V}_+^2$ by $\psi(x, y) = ((x + y)^{-1}, x^{-1} - (x + y)^{-1})$, $x, y \in \mathscr{V}_+$. It follows that ψ is a bijection and $\psi = \psi^{-1}$. Obviously $(U, V) = \psi(X, Y)$. Now to find the joint density the essential computation is involved with finding the Jacobian of ψ which equals $(\det(x)\det(x + y))^{-(r+1)}$ (for details see [**9**]). Then the independence assumption leads to the equation

$$f_U(u)f_V(v) = (\det(u)\det(u + v))^{-(r+1)} f_X((u + v)^{-1}) f_Y(y(u, v)), \tag{17}$$

for any $u, v \in \mathcal{V}_+$, where $y(u,v) = u^{-1} - (u+v)^{-1}$ and f_Z denotes the density of the random variable Z, $Z \in \{U, V, X, Y\}$. Now upon taking natural logarithms in (17) we get

$$\phi_1(u) + \phi_2(v) = \phi_3(u+v) + \phi_4(y(u,v)), \tag{18}$$

where

$$\phi_1(u) = (r+1)\log(\det(u)) + \log(f_U(u)),$$

$$\phi_3(u) = -(r+1)\log(\det(u)) + \log(f_X(u^{-1})),$$

$$\phi_2 = \log(f_V), \quad \phi_4 = \log(f_Y).$$

All these functions are differentiable, since f_X and f_Y are differentiable by assumption.

Before we differentiate (18) let us observe that $(x^{-1})' = -\mathbb{P}(x^{-1})$. Now we differentiate (18), first with respect to u getting

$$\phi_1'(u) = \phi_3'(u+v) + [\mathbb{P}((u+v)^{-1}) - \mathbb{P}(u^{-1}]\phi_4'(y(u,v)).$$

Differentiation of (18) with respect to v gives

$$\phi_2'(v) = \phi_3'(u+v) + \mathbb{P}((u+v)^{-1})\phi_4'(y(u,v)). \tag{19}$$

Now in order to eliminate from the above two equations $\phi_4'(y(u,v))$ we compose each side of (19) with $\mathbb{P}(u+v)$, and then we arrive at

$$\phi_1'(u) + \mathbb{P}(u^{-1})\mathbb{P}(u+v)\phi_2'(v) - \phi_2'(v) = \mathbb{P}(u^{-1})\mathbb{P}(u+v)\phi_3'(u+v).$$

Again composing each side of the above equation with $\mathbb{P}(u)$ from the left-hand side we arrive at

$$\mathbb{P}(u)\phi_1'(u) - [\mathbb{P}(u) - \mathbb{P}(u+v)]\phi_2'(v) = \mathbb{P}(u+v)\phi_3'(u+v).$$

Now by Theorem 2 it follows that

$$\phi_1'(u) = \mathbb{P}(u^{-1})a - \lambda u^{-1} - b,$$

for some $a, b \in \mathcal{V}$ and $\lambda \in \mathbf{R}$. Consequently

$$\phi_1(u) = K_1 - \lambda \log(\det(u)) - (b,u) - (a,u^{-1}),$$

where K_1 is a constant; this is deduced from the following formulas for derivatives of matrix variate functions with respect to the matrix argument: $(\log(\det(u))' = u^{-1}$, $(b,u)' = b$ and $(a,u^{-1})' = -\mathbb{P}(u^{-1})a$. Now writing $\lambda = p - (r+1)/2$ we get, by the definition of ϕ_1, that

$$\begin{aligned}\log(f_U(u)) &= K_1 - (p-(r+1)/2)\log(\det(u)) - (b,u) - (a,u^{-1}) - (r+1)\log(\det(u)) \\ &= K_1 - (p+(r+1)/2)\log(\det(u)) - (b,u) - (a,u^{-1}).\end{aligned}$$

Since f_U is a density then necessarily $a, b \in \mathcal{V}_+$ and $p - (r-1)/2 > 0$, and thus $U \sim \mu_{-p,b,a}$.

Similarly

$$\phi_3'(u) = \mathbb{P}(u^{-1})a + \lambda u^{-1} - b,$$

which results in

$$\phi_3(u) = K_3 + \lambda \log(\det(u)) - (b, u) - (a, u^{-1}),$$

where K_3 is a constant. Hence

$$\begin{aligned} \log(f_X(u)) &= \phi_3(u^{-1}) - (r+1)\log(\det(u)) \\ &= K_3 - (p + (r-1)/2)\log(\det(u)) - (a, u) - (b, u^{-1}) \end{aligned}$$

and thus $X \sim \mu_{p,a,b}$.

Also by Theorem 2 we get

$$\phi_2'(v) = -b + (p - (r+1)/2)v^{-1}.$$

Hence

$$\log(f_V(v)) = K_2 - (b, v) + (p - (r+1)/2)\log(\det(v)),$$

where K_2 is a constant, and consequently $V \sim \gamma_{p,b}$.

Finally the adequate formula for the density f_Y follows by inserting what we have just obtained into (17).

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