Journal of Statistical Research
2003, Vol. 37, No. 1, pp. 57-81
Bangladesh

ISSN 0256 - 422 X

MULTIVARIATE DUAL REGRESSION SCHEMES FOR THE LUKACS THEOREM

KONSTANCJA BOBECKA AND JACEK WESOŁOWSKI
Wydział Matematyki i Nauk Informacyjnych,
Politechnika Warszawska, Plac Politechniki 1, 00-661 Warszawa, POLAND
Email: bobecka@mini.pw.edu.pl
Email: wesolo@mini.pw.edu.pl

SUMMARY

Regression versions of the Lukacs independence condition (characterizing the gamma law) are considered in the multivariate framework. It is proved that, unexpectedly, constancy of dual regressions is a characteristic property only of a rather special multivariate gamma vectors: decomposable into independent subvectors of linearly dependent univariate gamma components. The method of moments is exploited in the proofs.

Keywords and phrases: gamma distribution, constancy of regression, Lukacs theorem, method of moments

AMS Classification: 62H05, 62E10, 60E05.

1 Introduction

Let X and Y be nondegenerate, positive, independent random variables. If $U=X/(X+Y)$ and $V=X+Y$ are also independent, then the distributions of X and Y are gamma with the common scale parameter. This famous result was obtained in Lukacs (1955). It was followed by other studies which led to matrix variate versions proved in Olkin and Rubin (1962), and more recently by Casalis and Letac (1996), Letac and Massam (1998) and Bobecka and Wesołowski (2002a). However the only result for n - dimensional random vectors is given in Bobecka and Wesołowski (2001). It appears that n - variate version of the Lukacs independence property does not characterize the classical multivariate gamma distribution, which is defined by the Laplace transform of the form

$$L(t_1, ..., t_n) = [\det (I + VT)]^{-\alpha},$$

where $\alpha > \frac{n-1}{2}$, V is a positive definite matrix, $T = diag(t_1, ..., t_n)$ and I is the identity matrix (see Wang (1981)), but implies that the random vectors can be decomposed into independent subvectors with linearly dependent gamma components.

Another direction of studies was concerned with regression versions of the Lukacs property, see for instance Bolger and Harkness (1965), Hall and Simons (1969), Wesołowski

(1990), Li, Huang and Huang (1994), Huang and Su (1997), Bobecka and Wesołowski (2002b), Chou and Huang (2002a) for the univariate case, Letac and Massam (1998) for the matrix variables. For random vectors only the bivariate case was considered in the literature, see Wang (1981), Bobecka (2002) and Chou and Huang (2002b). In contrast to the independence condition in both these papers the classical bivariate gamma distribution is characterized by some constancy of regression properties. However an extension of these results to higher dimensions seems to be difficult due to problems related to infinite divisibility of the classical multivariate gamma distribution.

In Bobecka and Wesołowski (2002b) a new regression scheme (called dual) was proposed: Instead of independence of X and Y it was assumed that U and V are independent and instead of constancy of regressions of U given V it was assumed that regressions of Y given X are constant:

$$E(Y^r|X) = c, \qquad E(Y^s|X) = d,$$

or that regressions of X given Y are constant:

$$E(X^r|Y) = c, \qquad E(X^s|Y) = d,$$

for some real constants c, d, and some fixed pairs of integers $(r, s) = (1, 2)$ or $(1, -1)$ or $(-1, -2)$. The results obtained there are dual, respectively, to characterizations due to Bolger and Harkness (1965), Wesołowski (1990), Li, Huang and Huang (1994). Further results for the dual regression scheme have been obtained recently in Chou and Huang (2002a)

The present paper is concerned with the multivariate dual regression scheme. It appears that in this setting not only bivariate, but general n - dimensional distributions can be characterized. The reason is twofold: first, the method is based on moments computation, second, the n - dimensional random vectors characterized are decomposable into independent subvectors of linearly dependent components (exactly as in the case of the n - variate Lukacs theorem given in Bobecka and Wesołowski (2001)).

As mentioned above, the constancy of dual regressions is a characteristic property of a very special multivariate gamma distribution: We say that an n - dimensional random vector $\bar{X}$ with positive components has $MG^*(\bar{A}, \bar{\alpha}, \bar{p})$ distribution, where $\bar{A} = (A_1, \dots, A_r)$ is such that $\bigcup_{i=1}^{r} A_i = \{1, \dots, n\}$, $A_i \cap A_j = \emptyset$, $i \neq j$, $\bar{\alpha} = (\alpha_1, \dots, \alpha_n)$ and $\bar{p} = (p_1, \dots, p_r)$, for $\alpha_j, p_i > 0$, if the Laplace transform of $\bar{X}$ is of the form

$$L_{\bar{X}}(t_1, \dots, t_n) = \prod_{i=1}^{r} (1 + \sum_{j \in A_i} \alpha_j t_j)^{-p_i},$$

i.e. the random vectors $\bar{Z}_1 = (X_l)_{l \in A_1}, \dots, \bar{Z}_r = (X_l)_{l \in A_r}$ are independent and have linearly dependent components with univariate gamma distributions: $\forall i\ \exists k \in A_i$ such that $X_j = \frac{\alpha_j}{\alpha_k} X_k \quad \forall j \in A_i$ and $X_k \sim \gamma_{\alpha_k, p_i}$ (observe that in the case $r = n$ all the components of the random vector $\bar{X}$ are independent gamma: $X_j \sim \gamma_{\alpha_j, p_j}$, $j = 1, \dots, n$).

The main results are presented in Section 3. They include three dual regression characterizations, which are multivariate versions of univariate results from Bobecka and Wesołowski (2002b). Here similarly as in the univariate case the method of moments is used, however this time it leads to much more involved functional equations. They seem to be also of a separate interest and are studied first in Section 2.

2 Functional equations

Let $\bar{\mathbf{N}} = \mathbf{N} \cup \{0\}$. For any $j = 1, \ldots, n$ let us denote

$$\bar{k} = (k_1, ..., k_n) \in \mathbf{R}^n, \qquad \bar{k}^{(j)} = (k_1, ..., k_{j-1}, 0, k_{j+1}, ..., k_n) \in \mathbf{R}^n,$$
$$\bar{k}_{(j)} = (k_1, ..., k_{j-1}, k_{j+1}, ..., k_n) \in \mathbf{R}^{n-1}.$$

Lemma 2.2, presented below, will be used in the proofs of n - variate versions of the dual regression schemes for the Lukacs theorem. First we consider the bivariate version of this lemma which enables us to carry out the induction proof in the n - variate case.

Lemma 2.1. *Let $f, g : \bar{\mathbf{N}} \to (0, +\infty)$. Suppose that*

$$\prod_{j=1}^{l} \frac{f(k)+j}{f(0)+j} = \prod_{i=1}^{k} \frac{g(l)+i}{g(0)+i}, \tag{2.1}$$

holds for every $k, l = 0, 1, \ldots$ (we adopt the convention that $\prod_{i=1}^{0} \equiv 1$). Then either

1. $f(k) = f(0)$ *and* $g(k) = g(0) \quad \forall k = 0, 1, ...,$

   or

2. $f(k) = g(k) = k + f(0) \quad \forall k = 0, 1, \ldots$

Proof. For $l = 1$ and any $k = 0, 1, ...,$ (2.1) takes the form

$$\frac{f(k)+1}{f(0)+1} = \prod_{i=1}^{k} \frac{g(1)+i}{g(0)+i}. \tag{2.2}$$

Replacing k by $k-1$ in (2.2) we obtain

$$\frac{f(k-1)+1}{f(0)+1} = \prod_{i=1}^{k-1} \frac{g(1)+i}{g(0)+i}, \qquad k = 1, 2, \ldots \tag{2.3}$$

Dividing (2.2) by (2.3) we get

$$\frac{f(k)+1}{f(k-1)+1} = \frac{g(1)+k}{g(0)+k}, \qquad k = 1, 2, \ldots \tag{2.4}$$

Similarly we can show that

$$\frac{g(l)+1}{g(l-1)+1}=\frac{f(1)+l}{f(0)+l}, \quad l=1,2,\ldots \tag{2.5}$$

For $k=l$ (2.1) is of the form

$$\prod_{j=1}^{k}\frac{f(k)+j}{f(0)+j}=\prod_{i=1}^{k}\frac{g(k)+i}{g(0)+i}, \quad k=0,1,\ldots. \tag{2.6}$$

and for $l=k-1$ we can rewrite (2.1) as

$$\prod_{j=1}^{k-1}\frac{f(k)+j}{f(0)+j}=\prod_{i=1}^{k}\frac{g(k-1)+i}{g(0)+i}, \quad k=1,2\ldots. \tag{2.7}$$

Combinig (2.6) with (2.7) gives

$$\frac{f(k)+k}{f(0)+k}=\prod_{i=1}^{k}\frac{g(k)+i}{g(k-1)+i}, \quad k=1,2,\ldots. \tag{2.8}$$

Replacing k by $k-1$ in (2.6) leads to

$$\prod_{j=1}^{k-1}\frac{f(k-1)+j}{f(0)+j}=\prod_{i=1}^{k-1}\frac{g(k-1)+i}{g(0)+i}, \quad k=1,2,\ldots. \tag{2.9}$$

Write now (2.1) for $k-1$ and $l=k$

$$\prod_{j=1}^{k}\frac{f(k-1)+j}{f(0)+j}=\prod_{i=1}^{k-1}\frac{g(k)+i}{g(0)+i}, \quad k=1,2,\ldots. \tag{2.10}$$

From (2.9) and (2.10) it follows that

$$\frac{f(k-1)+k}{f(0)+k}=\prod_{i=1}^{k-1}\frac{g(k)+i}{g(k-1)+i}, \quad k=1,2\ldots. \tag{2.11}$$

Dividing (2.8) by (2.11) we obtain

$$\frac{f(k)+k}{f(k-1)+k}=\frac{g(k)+k}{g(k-1)+k} \quad k=1,2,\ldots. \tag{2.12}$$

Consider (2.6) for $k=1$ and $k=2$. We get, respectively,

$$\frac{f(1)+1}{f(0)+1}=\frac{g(1)+1}{g(0)+1}, \tag{2.13}$$

$$\frac{f(2)+1}{f(0)+1}\frac{f(2)+2}{f(0)+2}=\frac{g(2)+1}{g(0)+1}\frac{g(2)+2}{g(0)+2}. \tag{2.14}$$

The equations (2.4) and (2.5) imply

$$f(2) = \frac{g(1)+2}{g(0)+2}\,[f(1)+1] - 1, \tag{2.15}$$

$$g(2) = \frac{f(1)+2}{f(0)+2}\,[g(1)+1] - 1. \tag{2.16}$$

Inserting (2.15), (2.16) into (2.14) and using (2.13) we get

$$[g(1)+2]\left[\frac{g(1)+2}{g(0)+2}\,[f(1)+1]+1\right] = [f(1)+2]\left[\frac{f(1)+2}{f(0)+2}\,[g(1)+1]+1\right],$$

which implies

$$[g(1)+1]\,[f(0)+1]\,[f(0)-g(0)]\,[g(1)-g(0)]^2 = 0.$$

Hence $f(0) = g(0)$ or $g(0) = g(1)$.

If $g(0) = g(1)$ then, by (2.2), $f(k) = f(0)$ and, by (2.5), $g(l) = g(0)$, $k, l = 0, 1, \ldots$

If $f(0) = g(0)$ then by (2.12) we have

$$f(k) = g(k), \quad k = 0, 1, \ldots$$

We will show that in this case either $f(k) = f(0)$, or $f(k) = k + f(0)$, $k = 0, 1, \ldots$. Taking $k = 1$, $l = 2$ in (2.1) and replacing g by f we obtain

$$[f(1)+1]\,\frac{f(1)+2}{f(0)+2} = f(2)+1. \tag{2.17}$$

Similarly, for $k = 1$, $l = 3$ we get

$$[f(1)+1]\,\frac{f(1)+2}{f(0)+2}\,\frac{f(1)+3}{f(0)+3} = f(3)+1 \tag{2.18}$$

and for $k = 2$, $l = 3$ we have

$$[f(2)+1]\,[f(2)+2]\,\frac{f(2)+3}{f(0)+3} = [f(3)+1]\,[f(3)+2]. \tag{2.19}$$

From (2.17) and (2.18) it follows that

$$f(2) = \frac{[f(1)+1]\,[f(1)+2]}{f(0)+2} - 1,$$

$$f(3) = \frac{[f(1)+1]\,[f(1)+2]\,[f(1)+3]}{[f(0)+2]\,[[f(0)+3]]} - 1.$$

Inserting the above equations into (2.19) gives

$$\left[\frac{[f(1)+1]\,[f(1)+2]}{f(0)+2}+1\right]\left[\frac{[f(1)+1]\,[f(1)+2]}{f(0)+2}+2\right] = [f(1)+3]\left[\frac{[f(1)+1]\,[f(1)+2]\,[f(1)+3]}{[f(0)+2]\,[[f(0)+3]]}+1\right].$$

which implies

$$[f(1)+1][f(1)-f(0)-1][f(1)-f(0)]^2=0.$$

Hence $f(1)=f(0)$ or $f(1)=f(0)+1$.

If $f(1)=f(0)$ then, as before, by (2.2) we get $f(k)=f(0),\ \ k=0,1,...$

If $f(1)=f(0)+1$ then, by (2.4) and by the induction argument, we conclude that $f(k)=k+f(0),\ \ k=0,1,...$ □

The n - variate analogue of the above lemma is as follows

Lemma 2.2. *Let $h_i:\bar{\mathbb{N}}^{n-1}\to(0,+\infty),\ \ i=1,2,...,n$. Suppose that*

$$W\left(\bar{k}\right)=\prod_{i_{\sigma_1}=1}^{k_{\sigma_1}}\left[h_{\sigma_1}\left(\bar{k}_{(\sigma_1)}\right)+i_{\sigma_1}\right]\prod_{i_{\sigma_2}=1}^{k_{\sigma_2}}\left[h_{\sigma_2}\left(\bar{k}_{(\sigma_2)}^{(\sigma_1)}\right)+i_{\sigma_2}\right]\cdots \tag{2.20}$$

$$\cdots\prod_{i_{\sigma_n}=1}^{k_{\sigma_n}}\left[h_{\sigma_n}\left(\bar{k}_{(\sigma_n)}^{(\sigma_1,\sigma_2,\dots,\sigma_{n-1})}\right)+i_{\sigma_n}\right]\qquad\forall\bar{k}\in\bar{\mathbb{N}}^n,$$

holds for all permutations $\sigma:j\to\sigma_j$ of the set $\{1,2,...,n\}$ (we adopt the convention that $\prod_{i=1}^{0}\equiv 1$). Then $\exists\ r\in\{1,2,...,n\}$ and $A_1,...,A_r$ such that $\bigcup_{i=1}^{r}A_i=\{1,2,...,n\}$, $A_i\cap A_j=\emptyset$, $i\neq j$ and $\forall j\in A_i$

$$h_j\left(\bar{k}_{(j)}\right)=\sum_{l\in A_i\setminus\{j\}}k_l+\delta_i,$$

where $\delta_i=h_j\left(\bar{0}_{(j)}\right),\ j\in A_i,\ i=1,...,r$.

Proof. The proof is by induction on n. For $n=2$ lemma holds true by Lemma 2.1. Assume that it is true for $n-1\geq 2$.

Dividing both sides of (2.20) by

$$\prod_{i_{\sigma_n}=1}^{k_{\sigma_n}}\left[h_{\sigma_n}\left(\bar{k}_{(\sigma_n)}^{(\sigma_1,\sigma_2,\dots,\sigma_{n-1})}\right)+i_{\sigma_n}\right]$$

and observing that

$$\bar{k}_{(\sigma_n)}^{(\sigma_1,\sigma_2,\dots,\sigma_{n-1})}=\bar{0}_{(\sigma_n)}$$

we obtain

$$\frac{W\left(\bar{k}\right)}{\prod_{i_{\sigma_n}=1}^{k_{\sigma_n}}\left[h_{\sigma_n}\left(\bar{0}_{(\sigma_n)}\right)+i_{\sigma_n}\right]}=$$

$$=\prod_{i_{\sigma_1}=1}^{k_{\sigma_1}}\left[h_{\sigma_1}\left(\bar{k}_{(\sigma_1)}\right)+i_{\sigma_1}\right]\prod_{i_{\sigma_2}=1}^{k_{\sigma_2}}\left[h_{\sigma_2}\left(\bar{k}_{(\sigma_2)}^{(\sigma_1)}\right)+i_{\sigma_2}\right]\cdots\prod_{i_{\sigma_{n-1}}=1}^{k_{\sigma_{n-1}}}\left[h_{\sigma_{n-1}}\left(\bar{k}_{(\sigma_{n-1})}^{(\sigma_1,\sigma_2,\dots,\sigma_{n-2})}\right)+i_{\sigma_{n-1}}\right].$$

Thus for every $m = \sigma_n$ we have

$$W^{(k_m)}\left(\bar{k}\right) = \prod_{i_{\sigma_1}=1}^{k_{\sigma_1}} \left[h_{\sigma_1}^{(k_m)}\left(\bar{k}_{(\sigma_1,m)}\right) + i_{\sigma_1}\right] \prod_{i_{\sigma_2}=1}^{k_{\sigma_2}} \left[h_{\sigma_2}^{(k_m)}\left(\bar{k}_{(\sigma_2,m)}^{(\sigma_1)}\right) + i_{\sigma_2}\right] \cdots$$
$$\cdots \prod_{i_{\sigma_{n-1}}=1}^{k_{\sigma_{n-1}}} \left[h_{\sigma_{n-1}}^{(k_m)}\left(\bar{k}_{(\sigma_{n-1},m)}^{(\sigma_1,\sigma_2,\dots,\sigma_{n-2})}\right) + i_{\sigma_{n-1}}\right],$$

where

$$W^{(k_m)}\left(\bar{k}\right) = \frac{W\left(\bar{k}\right)}{\prod\limits_{i_m=1}^{k_m}\left[h_m\left(\bar{0}_{(m)}\right) + i_m\right]}, \qquad h_{\sigma_j}^{(k_m)}\left(\bar{k}_{(\sigma_j,m)}\right) = h_{\sigma_j}\left(\bar{k}_{(\sigma_j)}\right). \tag{2.21}$$

Therefore, by the induction assumption, we get, for any m

$$k_j + h_j^{(k_m)}\left(\bar{k}_{(j,m)}\right) = \sum_{l\in A_i^{(m)}} k_l + \delta_i^{(m)}\left(k_m\right) \quad \forall j \in A_i^{(m)}, \tag{2.22}$$

where $\bigcup\limits_i A_i^{(m)} = \{1,2,\dots,n\}\backslash\{m\}$. Similarly, for any $\tilde{m} \neq m$ we have

$$k_j + h_j^{(k_{\tilde{m}})}\left(\bar{k}_{(j,\tilde{m})}\right) = \sum_{l\in A_i^{(\tilde{m})}} k_l + \delta_i^{(\tilde{m})}\left(k_{\tilde{m}}\right) \quad \forall j \in A_i^{(\tilde{m})},$$

where $\bigcup\limits_i A_i^{(\tilde{m})} = \{1,2,\dots,n\}\backslash\{\tilde{m}\}$. But

$$h_j^{(k_m)}\left(\bar{k}_{(j,m)}\right) = h_j\left(\bar{k}_{(j)}\right) \quad \forall m = 1,2,\dots,n. \tag{2.23}$$

Thus $\forall j \in A_i^{(m)} \cap A_i^{(\tilde{m})}$ we obtain

$$k_{\tilde{m}} I_{A_i^{(m)}\cap\{\tilde{m}\}} + \delta_i^{(m)}\left(k_m\right) = k_m I_{A_i^{(\tilde{m})}\cap\{m\}} + \delta_i^{(\tilde{m})}\left(k_{\tilde{m}}\right) \quad \forall m \neq \tilde{m}.$$

Let $A_i = \bigcup\limits_m A_i^{(m)}$. Observe that $A_i^{(m)} \cap \{\tilde{m}\} = A_i \cap \{\tilde{m}\}$ and $A_i^{(\tilde{m})} \cap \{m\} = A_i \cap \{m\}$. Hence

$$\delta_i^{(m)}\left(k_m\right) - k_m I_{A_i\cap\{m\}} = \delta_i^{(\tilde{m})}\left(k_{\tilde{m}}\right) - k_{\tilde{m}} I_{A_i\cap\{\tilde{m}\}} = \delta_i$$

and we get

$$\delta_i^{(m)}\left(k_m\right) = k_m I_{A_i\cap\{m\}} + \delta_i.$$

Therefore, by (2.22) and (2.23), for every m we obtain

$$k_j + h_j\left(\bar{k}_{(j)}\right) = \sum_{l\in A_i^{(m)}} k_l + k_m I_{A_i\cap\{m\}} + \delta_i = \sum_{l\in A_i} k_l + \delta_i \quad \forall j \in A_i,$$

where $\delta_i = h_j\left(\bar{0}_{(j)}\right)$ (because $A_i^{(m)} \cup \left(A_i \cap \{m\}\right) = A_i$). □

3 Multivariate dual regression theorems

Below we present dual regression theorems for the n - variate random vectors.

We will use the following notation

$$\begin{aligned} \bar{X}^s &= (X_1^s, ..., X_n^s), \quad \bar{Y}^s = (Y_1^s, ..., Y_n^s), \quad \text{for} \quad s = -2, -1, 1, 2, \\ \bar{U} &= (U_1, ..., U_n) = \left(\frac{X_1}{X_1 + Y_1}, ..., \frac{X_n}{X_n + Y_n}\right), \\ \bar{V} &= (V_1, ..., V_n) = (X_1 + Y_1, ..., X_n + Y_n), \\ \bar{c} &= (c_1, ..., c_n), \qquad \bar{d} = (d_1, ..., d_n). \end{aligned}$$

First result is a multivariate analogue of Theorem 1 in Bobecka and Wesołowski (2002b), i.e. it is the multivariate version of the dual theorem to the characterization of the gamma law obtained in Bolger and Harkness (1965).

Theorem 1. *Let $\bar{U}$ and $\bar{V}$ be independent random vectors. Assume that*

$$E(\bar{Y}|\bar{X}) = \bar{c}, \qquad E(\bar{Y}^2|\bar{X}) = \bar{d}, \tag{3.1}$$

for some real vectors $\bar{c}$, $\bar{d}$. Then $d_j > c_j^2$, $j = 1, 2, ..., n$, and $\bar{X}$, $\bar{Y}$ have distributions $MG^*(\bar{A}, \bar{\alpha}, \bar{q})$, $MG^*(\bar{A}, \bar{\alpha}, \bar{t})$, respectively, for some $\bar{A}$, $\bar{\alpha}$, $\bar{q}$ and $\bar{t}$, where

$$\alpha_j = \frac{d_j - c_j^2}{c_j} > 0, \quad t_i = \frac{c_j}{\alpha_j} > 0, \quad j \in A_i, \quad i = 1, ..., r.$$

Proof. Conditions (3.1) are equivalent to

$$\begin{aligned} E\left(V_j(1 - U_j) \mid V_1U_1, ..., V_nU_n\right) &= c_j, \\ E\left(V_j^2(1 - U_j)^2 \mid V_1U_1, ..., V_nU_n\right) &= d_j, \quad j = 1, 2, ..., n. \end{aligned} \tag{3.2}$$

Let us note that $\bar{U}$ has values in $(0, 1)^n$. Thus we have $E(U_1^{k_1} ... U_n^{k_n}) < \infty$ for $k_j = 1, 2, ...$, $j = 1, 2, ..., n$. To show that also $E(V_1^{k_1} ... V_n^{k_n}) < \infty$ for $k_j = 1, 2, ...$ it suffices to prove that for $j = 1, 2, ..., n$ all moments $E(V_j^{k_j})$ are finite. To this end let us fix j (without loss of generality we can take $j = 1$) and apply induction with respect to $k_1 = k$. For $k = 1, 2$ we have $E(V_1^k) < \infty$ by the conditions (3.2). Now, let us assume that $E(V_1^k) < \infty$ for some k and use the following lemma (Wesołowski (1993)): if random variables A, B $(B \geq 0)$, $C = AE(B|A)$ are integrable then the product AB is also integrable. Take

$$\begin{aligned} A &= (V_1U_1)^k, \quad B = V_1(1 - U_1), \\ C = AE(B|A) &= (V_1U_1)^k E\left[V_1(1 - U_1) | (V_1U_1)^k\right] = \\ &= (V_1U_1)^k E\left[E\left[V_1(1 - U_1) | V_1U_1, ..., V_nU_n\right] | (V_1U_1)^k\right] = \\ &= (V_1U_1)^k E\left[c_1 | (V_1U_1)^k\right] = (V_1U_1)^k c_1. \end{aligned}$$

Thus $V_1^{k+1}U_1^k(1-U_1)$ is integrable. Hence $E(V_1^{k+1}) < \infty$ and by the induction argument we conclude that all the moments of V_1 are finite.

From (3.2) it follows that

$$E\left(V_j(1-U_j)\prod_{l=1}^{n}(V_lU_l)^{k_l}\right) = c_jE\left(\prod_{l=1}^{n}(V_lU_l)^{k_l}\right), \tag{3.3}$$
$$E\left(V_j^2(1-U_j)^2\prod_{l=1}^{n}(V_lU_l)^{k_l}\right) = d_jE\left(\prod_{l=1}^{n}(V_lU_l)^{k_l}\right),$$
$$k_j = 0,1,..., \quad j=1,2,...,n.$$

Since $\bar{U}$ and $\bar{V}$ are independent, by (3.3) we get

$$E\left(V_j^{k_j+1}\prod_{l\neq j}V_l^{k_l}\right)E\left(U_j^{k_j}\prod_{l\neq j}U_l^{k_l}\right) - E\left(V_j^{k_j+1}\prod_{l\neq j}V_l^{k_l}\right)E\left(U_j^{k_j+1}\prod_{l\neq j}U_l^{k_l}\right) =$$
$$= c_jE\left(V_j^{k_j}\prod_{l\neq j}V_l^{k_l}\right)E\left(U_j^{k_j}\prod_{l\neq j}U_l^{k_l}\right),$$

$$E\left(V_j^{k_j+2}\prod_{l\neq j}V_l^{k_l}\right)E\left(U_j^{k_j}\prod_{l\neq j}U_l^{k_l}\right) - 2E\left(V_j^{k_j+2}\prod_{l\neq j}V_l^{k_l}\right)E\left(U_j^{k_j+1}\prod_{l\neq j}U_l^{k_l}\right)$$
$$+E\left(V_j^{k_j+2}\prod_{l\neq j}V_l^{k_l}\right)E\left(U_j^{k_j+2}\prod_{l\neq j}U_l^{k_l}\right) = d_jE\left(V_j^{k_j}\prod_{l\neq j}V_l^{k_l}\right)E\left(U_j^{k_j}\prod_{l\neq j}U_l^{k_l}\right),$$
$$k_j = 0,1,..., \quad j=1,2,...,n,$$

which can be written as, respectively,

$$h_j(\bar{k}) = c_j + g_j(\bar{k})h_j(\bar{k}), \tag{3.4}$$

$$h_j(k_1,...,k_{j-1},k_j+1,k_{j+1},...,k_n)h_j(\bar{k}) - 2g_j(\bar{k})h_i(k_1,...,k_{j-1},k_j+1,k_{j+1},...,k_n)h_j(\bar{k}) + \tag{3.5}$$
$$+g_j(k_1,...,k_{j-1},k_j+1,k_{j+1},...,k_n)g_j(\bar{k})h_j(k_1,...,k_{j-1},k_j+1,k_{j+1},...,k_n)h_j(\bar{k}) = d_j,$$

where

$$h_j(\bar{k}) = \frac{E\left(V_j^{k_j+1}\prod_{l\neq j}V_l^{k_l}\right)}{E\left(V_j^{k_j}\prod_{l\neq j}V_l^{k_l}\right)}, \quad g_j(\bar{k}) = \frac{E\left(U_j^{k_j+1}\prod_{l\neq j}U_l^{k_l}\right)}{E\left(U_j^{k_j}\prod_{l\neq j}U_l^{k_l}\right)}, \quad k_j = 0,1,..., \quad j=1,2,...,n.$$

Substituting $g_j(\bar{k})h_j(\bar{k})$ and $g_j(k_1,...,k_{j-1},k_j+1,k_{j+1},...,k_n)h_j(k_1,...,k_{j-1},k_j+1,k_{j+1},...,k_n)$ from (3.4) into (3.5) we obtain

$$h_j(k_1,...,k_{j-1},k_j+1,k_{j+1},...,k_n) - h_j(\bar{k}) = d_j/c_j - c_j, \quad k_j = 0,1,..., \quad j = 1,2,...,n \quad (3.6)$$

Observe that since $c_j = EY_j$, $d_j = EY_j^2$, we have $c_j > 0$ and $VarY_j = d_j - c_j^2 > 0$, $j = 1,2,...,n$. Let us denote

$$\alpha_j = d_j/c_j - c_j > 0, \quad j = 1,2,...,n.$$

Then, by (3.6), we get, iterating,

$$h_j(\bar{k}) = \alpha_j k_j + h_j(\bar{k}^{(j)}), \quad k_j = 0,1..., \quad j = 1,2,..,n. \quad (3.7)$$

Inserting $k_i = 0$ for $i \neq j$ *we* obtain

$$h_j(0,...,0,k_j,0,...,0) = \alpha_j k_j + h_j(\bar{0}), \quad k_j = 0,1..., \quad j = 1,2,..,n. \quad (3.8)$$

Let us define $p_j = \frac{1}{\alpha_j}h_j(\bar{0}) = \frac{1}{\alpha_j}EV_j > 0$. Then (3.8) leads to

$$E\left(V_j^{k_j}\right) = \alpha_j^{k_j}\frac{\Gamma(k_j+p_j)}{\Gamma(p_j)}, \quad k_j = 1,2,..., \quad j = 1,2,...,n. \quad (3.9)$$

Hence, by the uniqueness of the moments sequence for the gamma distribution, we get $V_j \sim \gamma_{\alpha_j,p_j}$ $\forall j = 1,2,...,n$. Denote

$$m(\bar{k}) = E\left(\prod_{j=1}^{n} V_j^{k_j}\right), \quad k_j = 0,1,..., \quad j = 1,2,...,n.$$

From (3.7) it follows that

$$m(k_1,..,k_{j-1},k_j+1,k_{j+1},...,k_n) = m(\bar{k})\left[\alpha_j k_j + h_j(\bar{k}^{(j)})\right], \quad (3.10)$$
$$k_j = 0,1,..., \quad j = 1,2,..,n.$$

Replacing k_j by $k_j - 1$ in (3.10), and iterating this operation, we obtain

$$m(\bar{k}) = m(\bar{k}^{(j)})\prod_{l_j=0}^{k_j-1}\left[h_j(\bar{k}^{(j)}) + \alpha_j l_j\right], \quad k_j = 0,1,..., \quad j = 1,2,...,n, \quad (3.11)$$

(we adopt the convention that $\prod_{l=0}^{-1} \equiv 1$). Repeating the above argumentation for the succeeding $j = 1,2,...,n$, eventually we get the equation

$$m(\bar{k}) = \prod_{i_1=0}^{k_1-1}\left[h_1\left(\bar{k}^{(1)}\right) + \alpha_1 i_1\right]\prod_{i_2=0}^{k_2-1}\left[h_2\left(\bar{k}^{(1,2)}\right) + \alpha_2 i_2\right] \cdots \prod_{i_n=0}^{k_n-1}\left[h_n\left(\bar{k}^{(1,2,...,n-1,n)}\right) + \alpha_n i_n\right], \quad (3.12)$$

$$k_j = 0,1,..., \quad j = 1,2,...,n.$$

Since the order of choosing the indices j in the succeeding iterations of (3.11) can be optional, we have

$$m(\bar{k}) = \prod_{i_{\sigma_1}=0}^{k_{\sigma_1}-1}\left[h_{\sigma_1}\left(\bar{k}^{(\sigma_1)}\right)+\alpha_{\sigma_1}i_{\sigma_1}\right]\prod_{i_{\sigma_2}=0}^{k_{\sigma_2}-1}\left[h_{\sigma_2}\left(\bar{k}^{(\sigma_1,\sigma_2)}\right)+\alpha_{\sigma_2}i_{\sigma_2}\right]\cdots \tag{3.13}$$
$$\cdots\prod_{i_{\sigma_n}=0}^{k_{\sigma_n}-1}\left[h_{\sigma_n}\left(\bar{k}^{(\sigma_1,\sigma_2,\ldots,\sigma_{n-1},\sigma_n)}\right)+\alpha_{\sigma_n}i_{\sigma_n}\right],\quad k_{\sigma_j}=0,1,\ldots,$$

for all permutations $\sigma : j \to \sigma_j$ of the set $\{1,2,...,n\}$ (observe that $\bar{k}^{(\sigma_1,\sigma_2,\ldots,\sigma_{n-1},\sigma_n)} = \bar{0}$). Let

$$\tilde{h}_j(\bar{k}_{(j)}) = \frac{h_j(\bar{k}^{(j)})}{\alpha_j} - 1, \quad \tilde{m}(\bar{k}) = \frac{m(\bar{k})}{\prod_{l=1}^{n}\alpha_l^{k_l}}, \quad j = 1,2,...,n. \tag{3.14}$$

Then (3.13) can be rewritten as

$$\tilde{m}(\bar{k}) = \prod_{i_{\sigma_1}=1}^{k_{\sigma_1}}\left[\tilde{h}_{\sigma_1}(\bar{k}_{(\sigma_1)})+i_{\sigma_1}\right]\prod_{i_{\sigma_2}=1}^{k_{\sigma_2}}\left[\tilde{h}_{\sigma_2}(\bar{k}_{(\sigma_2)}^{(\sigma_1)})+i_{\sigma_2}\right]\cdots\prod_{i_{\sigma_n}=1}^{k_{\sigma_n}}\left[\tilde{h}_{\sigma_n}\left(\bar{k}_{(\sigma_n)}^{(\sigma_1,\sigma_2,\ldots,\sigma_{n-1})}\right)+i_{\sigma_n}\right], \tag{3.15}$$
$$k_{\sigma_j}=0,1,\ldots,\quad j=1,2,\ldots,n,$$

(we adopt the convention that $\prod_{i=1}^{0} \equiv 1$). Thus, by Lemma 2.2, we obtain that $\exists\, r \in \{1,2,...,n\}$ and $A_1,...,A_r$ such that $\bigcup_{i=1}^{r} A_i = \{1,2,...,n\}$, $A_i \cap A_j = \emptyset$, $i \neq j$ and $\forall j \in A_i$

$$\tilde{h}_j\left(\bar{k}_{(j)}\right) = \sum_{l\in A_i\setminus\{j\}} k_l + \delta_i, \quad k_l = 0,1,....$$

where $\delta_i = \tilde{h}_j\left(\bar{0}_{(j)}\right)$. Therefore, by (3.14), we obtain

$$h_j\left(\bar{k}^{(j)}\right) = \alpha_j\left[\sum_{l\in A_i\setminus\{j\}} k_l + \delta_i + 1\right] \quad \forall j \in A_i, \quad k_j = 0,1,..... \tag{3.16}$$

Observe that without loss of generality we can assume that $A_1 = \{1,2,...,m_1\}$, $A_2 = \{m_1+1,...,m_2\}$, ..., $A_r = \{m_{r-1}+1,....,m_r\}$, where $m_r = n$. Then, inserting (3.16) into (3.12), gives

$$m\left(\bar{k}\right) = \prod_{j=1}^{n}\alpha_j^{k_j}\prod_{i=1}^{r}\frac{\Gamma\left(\sum_{l\in A_i}k_l+\delta_i+1\right)}{\Gamma\left(\delta_i+1\right)}, \quad k_l = 0,1,... \tag{3.17}$$

Combining (3.17) with (3.9) we obtain $\delta_i + 1 = p_j = p_{m_i}$ $\forall j \in A_i$, $i = 1, ..., r$. Thus (3.17) takes the form

$$m(\bar{k}) = \prod_{j=1}^{n} \alpha_j^{k_j} \prod_{i=1}^{r} \frac{\Gamma\left(\sum_{l \in A_i} k_l + p_{m_i}\right)}{\Gamma(p_{m_i})}, \quad k_l = 0, 1, ..., \tag{3.18}$$

which can be written as

$$E\left(\prod_{j=1}^{n} \left(\frac{V_j}{\alpha_j}\right)^{k_j}\right) = \prod_{i=1}^{r} \frac{\Gamma\left(\sum_{l \in A_i} k_l + p_{m_i}\right)}{\Gamma(p_{m_i})}, \quad k_l = 0, 1, ... \tag{3.19}$$

Note that if $r < n$ then there exists a set A_i such that $\sharp A_i \geq 2$. Without loss of generality we can assume that $1, 2 \in A_1$. Inserting $k_l = 0$ into (3.19) for $l \neq 1, 2 \in A_1$ gives

$$E\left(\left(\frac{V_1}{\alpha_1}\right)^{k_1} \left(\frac{V_2}{\alpha_2}\right)^{k_2}\right) = \frac{\Gamma(k_1 + k_2 + p_{m_1})}{\Gamma(p_{m_1})}, \tag{3.20}$$

which imply

$$E\left[\left(\frac{V_1}{\alpha_1} - \frac{V_2}{\alpha_2}\right)^2 \prod_{j=1}^{n} \left(\frac{V_j}{\alpha_j}\right)\right] = 0.$$

Since $V_j > 0$, $j = 1, ..., n$, we have $\frac{V_1}{\alpha_1} = \frac{V_2}{\alpha_2}$. Similarly we can show that $\forall i$ $V_l = \frac{\alpha_l}{\alpha_{m_i}} V_{m_i}$ $\forall l \in A_i$, $i = 1, 2, ..., r$. Inserting this into (3.18) we obtain

$$E\left(\prod_{i=1}^{r} V_{m_i}^{k_i}\right) = \prod_{i=1}^{r} \alpha_{m_i}^{k_i} \frac{\Gamma(k_i + p_{m_i})}{\Gamma(p_{m_i})}, \quad k_i = 0, 1, ... \tag{3.21}$$

The formula (3.21) for $r = n$ is an obvious consequence of (3.19). Thus the vector $(V_{m_1}, ..., V_{m_r})$ has independent components. Denote $\bar{D}_1 = (V_l)_{l \in A_1}, ..., \bar{D}_r = (V_l)_{l \in A_r}$. From the above considerations it follows that the vectors $\bar{D}_1, ..., \bar{D}_r$ are independent and have linearly dependent components having gamma distributions, namely $\forall i$ $V_j = \frac{\alpha_j}{\alpha_{m_i}} V_{m_i}$ $\forall j \in A_i$ and $V_{m_i} \sim \gamma_{\alpha_{m_i}, p_{m_i}}$. Hence, by the definition of the function h_j and (3.7), we obtain

$$h_j(\bar{k}) = \frac{\alpha_j}{\alpha_{m_i}} \frac{E\left(V_{m_i} \prod_{l \in A_i} V_{m_i}^{k_l}\right)}{E\left(\prod_{l \in A_i} V_{m_i}^{k_l}\right)} = \frac{\alpha_j}{\alpha_{m_i}} \left(\alpha_{m_i} \sum_{l \in A_i} k_l + h_{m_i}(\bar{0})\right) =$$

$$= \alpha_j \left(\sum_{l \in A_i} k_l + p_{m_i}\right) \quad \forall j \in A_i, \quad k_l = 0, 1, ...$$

Thus by (3.4) we get

$$g_j\left(\bar{k}\right) = \frac{h_j\left(\bar{k}\right) - c_j}{h_j\left(\bar{k}\right)} = \frac{\alpha_j\left(\sum\limits_{l\in A_i} k_l + p_{m_i}\right) - c_j}{\alpha_j\left(\sum\limits_{l\in A_i} k_l + p_{m_i}\right)}, \quad \forall j \in A_i, \quad k_l = 0,1,... \tag{3.22}$$

Inserting $k_l = 0$ for $l \neq j$ into the above equation gives

$$g_j\left(0,...,0,k_j,0,...,0\right) = \frac{\alpha_j\left(k_j + p_{m_i}\right) - c_j}{\alpha_j\left(k_j + p_{m_i}\right)}, \quad k_j = 0,1,... \tag{3.23}$$

Let

$$t_j = c_j/\alpha_j > 0, \quad q_j = p_{m_i} - t_j = \frac{1}{\alpha_j}\left[\alpha_j p_{m_i}\left(1 - \frac{c_j}{\alpha_j p_{m_i}}\right)\right] = \frac{1}{\alpha_j} EV_j EU_j > 0.$$

Then by (3.23)and the definition of g_j we have

$$E\left(U_j^{k_j}\right) = \frac{k_j + q_j - 1}{k_j + q_j + t_j - 1} E\left(U_j^{k_j - 1}\right) =$$
$$= \frac{\Gamma\left(k_j + q_j\right)}{\Gamma\left(q_j\right)} \frac{\Gamma\left(q_j + t_j\right)}{\Gamma\left(k_j + q_j + t_j\right)} \quad \forall j \in A_i, \quad k_j = 0,1,....$$

The uniqueness of the moments sequence for the beta distribution implies that $U_j \sim \beta_{q_j,t_j}$ $\forall j \in A_i$, $i = 1,...,r$.

Let

$$M\left(\bar{k}\right) = E\left(\prod_{j=1}^{n} U_j^{kj}\right), \quad k_j = 0,1,..., \quad j = 1,2,...,n.$$

By (3.22), iterating, we obtain

$$M\left(\bar{k}\right) = \frac{\sum\limits_{l\in A_i} k_l + q_j - 1}{\sum\limits_{l\in A_i} k_l + q_j + t_j - 1} E\left(U_j^{k_j - 1} \prod_{l\neq j} U_l^{k_l}\right) =$$

$$= \frac{\Gamma\left(\sum\limits_{l\in A_i} k_l + q_j\right)}{\Gamma\left(\sum\limits_{l\in A_i\setminus\{j\}} k_l + q_j\right)} \frac{\Gamma\left(\sum\limits_{l\in A_i\setminus\{j\}} k_l + q_j + t_j\right)}{\Gamma\left(\sum\limits_{l\in A_i} k_l + q_j + t_j\right)} M\left(\bar{k}^{(j)}\right), \tag{3.24}$$

$$\forall j \in A_i, \quad k_j = 1,2,..., \quad k_l = 0,1,..., \quad l = 1,...,n, \quad l \neq j.$$

Observe that if $r < n$, then there exists a set A_i such that $\sharp A_i \geq 2$. Suppose that $1,2 \in A_1$ and write the above equation for $j = 1$ and then for $j = 2$, inserting each time $k_l = 0$ for

$l \neq 1,2$. This gives us

$$E\left(U_1^{k_1}U_2^{k_2}\right) = \frac{\Gamma(k_1+k_2+q_1)}{\Gamma(k_2+q_1)}\frac{\Gamma(k_2+q_1+t_1)}{\Gamma(k_1+k_2+q_1+t_1)}EU_2 = $$
$$= \frac{\Gamma(k_1+k_2+q_2)}{\Gamma(k_1+q_2)}\frac{\Gamma(k_1+q_2+t_2)}{\Gamma(k_1+k_2+q_2+t_2)}EU_1.$$

Taking $k_1 = k_2 = 1$ (observe that by the definition of q_j we have $q_j + t_j = p_{m_i}$ $\forall j \in A_i$) and using the fact that $U_j \sim \beta_{q_j,t_j}$ $\forall j \in A_i$ we obtain $q_1 = q_2$ and, consequently, $t_1 = t_2$. Generally, in the same way we can show that $q_j = q_{m_i}$ and $t_j = t_{m_i}$ $\forall j \in A_i$, which gives us $U_j \sim \beta_{q_{m_i},t_{m_i}}$ $\forall j \in A_i$. We can now proceed analogously to the case of $m(\bar{k})$ to obtain

$$M(\bar{k}) = \prod_{i=1}^{r} \frac{\Gamma\left(\sum_{l\in A_i} k_l + q_{m_i}\right)}{\Gamma(q_{m_i})} \frac{\Gamma(q_{m_i}+t_{m_i})}{\Gamma\left(\sum_{l\in A_i} k_l + q_{m_i} + t_{m_i}\right)}, \quad k_j = 0,1,... \tag{3.25}$$

Taking $k_l = 0$ for all $l \neq 1,2$ gives

$$E\left(U_1^{k_1}U_2^{k_2}\right) = \frac{\Gamma(k_1+k_2+q_{m_1})}{\Gamma(q_{m_1})}\frac{\Gamma(q_{m_1}+t_{m_1})}{\Gamma(k_1+k_2+q_{m_1}+t_{m_1})},$$

which, in particular, implies

$$E\left[(U_1-U_2)^2\right] = 0,$$

and hence we get $U_1 = U_2$. Similarly we can show $\forall i$ $U_j = U_{m_i}$ $\forall j \in A_i$. Thus the vectors $\bar{E}_1 = (U_l)_{l\in A_1}, ..., \bar{E}_r = (U_l)_{l\in A_r}$ have beta distributed components $U_j \sim \beta_{q_{m_i},t_{m_i}}$ $\forall j \in A_i$ such that $\forall i$ $U_j = U_{m_i}$ $\forall j \in A_i$. Moreover, note that the vectors $\bar{E}_1, ..., \bar{E}_r$ are independent which follows from inserting the above equalities into (3.25):

$$M(\bar{k}) = E\left(\prod_{i=1}^{r} U_{m_i}^{k_i}\right) = \prod_{i=1}^{r} \frac{\Gamma(k_i+q_{m_i})}{\Gamma(q_{m_i})}\frac{\Gamma(q_{m_i}+t_{m_i})}{\Gamma(k_i+q_{m_i}+t_{m_i})}, \quad k_i = 0,1,...$$

For $r = n$ the formula follows directly from the iteration (3.24).

Let $\bar{Z}_1 = (U_lV_l)_{l\in A_1} = (X_l)_{l\in A_1}, \dots, \bar{Z}_r = (U_lV_l)_{l\in A_r} = (X_l)_{l\in A_r}$. Then the vectors $\bar{Z}_1, ..., \bar{Z}_r$ are independent and have linearly dependent components: $\forall i$ $X_j = \frac{\alpha_j}{\alpha_{m_i}} X_{m_i}$ $\forall j \in A_i$ and (which follows from properties of univariate random variables) $X_{m_i} \sim \gamma_{\alpha_{m_i},q_{m_i}}$. Similarly, let $\bar{T}_1 = ((1-U_l)V_l)_{l\in A_1} = (Y_l)_{l\in A_1}, \dots, \bar{T}_r = ((1-U_l)V_l)_{l\in A_r} = (Y_l)_{l\in A_r}$. Then $\bar{T}_1, ..., \bar{T}_r$ are independent and $\forall i$ $Y_j = \frac{\alpha_j}{\alpha_{m_i}} Y_{m_i}$ $\forall j \in A_i$ and (which again follows from properties of univariate random variables) $Y_{m_i} \sim \gamma_{\alpha_{m_i},t_{m_i}}$. Hence $\bar{X}$ and $\bar{Y}$ have distributions $MG^*(\bar{A},\bar{\alpha},\bar{q})$ and $MG^*(\bar{A},\bar{\alpha},\bar{t})$, respectively, where $\bar{A} = (A_1, ..., A_r)$, $\bar{\alpha} = (\alpha_1, ..., \alpha_n)$, $\bar{q} = (q_{m_1}, ..., q_{m_r})$, $\bar{t} = (t_{m_1}, ..., t_{m_r})$. □

Similarly as in the case of univariate random variables, the analogous result with $\bar{X}$ and $\bar{Y}$ interchanged follows immediately.

Corollary 3.1. If we replace the conditions (3.1) in Theorem 1 by

$$E(\bar{X}|\bar{Y}) = \bar{c}, \quad E(\bar{X}^2|\bar{Y}) = \bar{d}, \tag{3.26}$$

then $\bar{X}$ and $\bar{Y}$ have distributions $MG^*(\bar{A},\bar{\alpha},\bar{t})$ and $MG^*(\bar{A},\bar{\alpha},\bar{q})$, respectively, with α_j, q_i, t_i, $j \in A_i$, $i = 1, ..., r$, defined as in Theorem 1.

Proof. The conditions (3.26) are equivalent to

$$E[V_jU_j|V_1(1-U_1), ..., V_n(1-U_n)] = c_j,$$
$$E[V_j^2U_j^2|V_1(1-U_1), ..., V_n(1-U_n)] = d_j, \quad j = 1, 2, ..., n.$$

Denote $\bar{W} = (W_1, ..., W_n) = (1 - U_1, ..., 1 - U_n)$. Then the above equations can be rewritten as

$$E[V_j(1-W_j)|V_1W_1, ..., V_nW_n] = c_j,$$
$$E[V_j^2(1-W_j)^2|V_1W_1, ..., V_nW_n] = d_j, \quad j = 1, 2, ..., n.$$

Observe that $\bar{W}$ and $\bar{V}$ are independent. Then, by Theorem 1, $\exists\ A_1, ..., A_r$ such that $\bigcup_{i=1}^{r} A_k = \{1, ..., n\}$, $A_i \cap A_j = \emptyset$, $i \neq j$ and the vectors $\bar{D}_1 = (V_l)_{l\in A_1}, ..., \bar{D}_r = (V_l)_{l\in A_r}$ are independent and $\forall i\ V_j = \frac{\alpha_j}{\alpha_{m_i}} V_{m_i}\quad \forall j \in A_i$ and $V_{m_i} \sim \gamma_{\alpha_{m_i}, p_{m_i}}$ and the vectors $\bar{F}_1 = (W_l)_{l\in A_1}, ..., \bar{F}_r = (W_l)_{l\in A_r}$ are independent and $\forall i\quad W_j = W_{m_i}\quad \forall j \in A_i$ and $W_{m_i} \sim \beta_{t_{m_i}, q_{m_i}}$. Repeating the argumentation from the final part of the proof of Theorem 1, we get our claim. □

Next result is an n - variate analogue of Theorem 2 in Bobecka and Wesołowski (2002b), i.e. it is the multivariate version of the dual theorem to Wesołowski (1990).

Theorem 2. *Let $\bar{U}$ and $\bar{V}$ be independent random vectors. Assume that*

$$E(\bar{Y}|\bar{X}) = \bar{c}, \quad E(\bar{Y}^{-1}|\bar{X}) = \bar{d}, \tag{3.27}$$

for some real vectors $\bar{c}$, $\bar{d}$. Then $c_jd_j > 1$, $j = 1, 2, ..., n$, and $\bar{X}$, $\bar{Y}$ have distributions $MG^*(\bar{A},\bar{\alpha},\bar{q})$, $MG^*(\bar{A},\bar{\alpha},\bar{t})$, respectively, for some $\bar{A}$, $\bar{\alpha}$, $\bar{q}$ and $\bar{t}$, where

$$\alpha_j = \frac{c_jd_j - 1}{d_j} > 0, \quad t_i = \frac{c_j}{\alpha_j} > 1, \quad j \in A_i, \quad i = 1, ..., r.$$

Proof. Conditions (3.27) can be rewritten as

$$E\left(V_j(1-U_j) \mid V_1U_1, ..., V_nU_n\right) = c_j, \tag{3.28}$$

$$E\left(\frac{1}{V_j(1-U_j)} \mid V_1U_1, ..., V_nU_n\right) = d_j, \quad j = 1, 2, ..., n. \tag{3.29}$$

Let us note that similarly as in Theorem 1 we have $E(U_1^{k_1}...U_n^{k_n}) < \infty$ and $E(V_1^{k_1}...V_n^{k_n}) < \infty$ for all $k_j = 1,2,...$ Observe also that

$$E\left(\frac{U_j^{k_j}}{1-U_j}\right) < E\left(\frac{1}{1-U_j}\right) < \infty, \quad j = 1,2,...,n.$$

From (3.29) it follows that

$$E\left(\frac{1}{V_j(1-U_j)}\prod_{l=1}^{n}(V_lU_l)^{k_l}\right) = d_jE\left(\prod_{l=1}^{n}(V_lU_l)^{k_l}\right), \qquad (3.30)$$
$$k_j = 0,1,..., \quad j = 1,2,...,n.$$

Since $\bar{U}$ and $\bar{V}$ are independent, by (3.30) we get

$$E(V_j^{k_j-1}\prod_{l\neq j}V_l^{k_l})E(\frac{U_j^{k_j}}{1-U_j}\prod_{l\neq j}U_l^{k_l}) = d_jE\left(\prod_{l=1}^{n}(V_lU_l)^{k_l}\right),$$
$$k_j = 0,1,..., \quad j = 1,2,...,n.$$

which can be written in the form

$$E(V_j^{k_j-1}\prod_{l\neq j}V_l^{k_l})\sum_{i=k_j}^{\infty}E(U_j^i\prod_{l\neq j}U_l^{k_l}) = d_jE\left(\prod_{l=1}^{n}(V_lU_l)^{k_l}\right), \qquad (3.31)$$
$$k_j = 0,1,..., \quad j = 1,2,...,n.$$

Denote

$$h_j(k_1,...,k_{j-1},k_j-1,k_{j+1},...,k_n) = \frac{E(V_j^{k_j}\prod_{l\neq j}V_l^{k_l})}{E(V_j^{k_j-1}\prod_{l\neq j}V_l^{k_l})},$$

$$G_j(\bar{k}) = \sum_{i=k_j}^{\infty}E(U_j^i\prod_{l\neq j}U_l^{k_l}), \quad k_j = 0,1,..., \quad j = 1,2,...,n.$$

(observe that $E(V_j^{-1}) < \infty, \ j = 1,2,...,n$). Then (3.31) can be written as

$$G_j(\bar{k}) = d_jh_j(k_1,...,k_{j-1},k_j-1,k_{j+1},...,k_n)\left[G_j(\bar{k}) - G_j(k_1,...,k_{j-1},k_j+1,k_{j+1},...,k_n)\right],$$
$$k_j = 0,1,..., \quad j = 1,2,...,n,$$

or as

$$P_j(\bar{k}) = 1 - \frac{1}{d_jh_j(k_1,...,k_{j-1},k_j-1,k_{j+1},...,k_n)}, \qquad (3.32)$$
$$k_j = 0,1,..., \quad j = 1,2,...,n,$$

where

$$P_j(\bar{k}) = \frac{G_j(k_1, ..., k_{j-1}, k_j + 1, k_{j+1}, ..., k_n)}{G_j(\bar{k})}, \quad k_j = 0, 1, ..., \quad j = 1, 2, ..., n. \tag{3.33}$$

The condition (3.28) is the same as in Theorem 1. Thus we get

$$h_j(\bar{k}) = c_j - g_j(\bar{k}) h_j(\bar{k}), \quad k_j = 0, 1, ..., \tag{3.34}$$

where

$$g_j(\bar{k}) = \frac{E\left(U_j^{k_j+1} \prod_{l \neq j} U_l^{k_l}\right)}{E\left(U_j^{k_j} \prod_{l \neq j} U_l^{k_l}\right)}, \quad k_j = 0, 1, ..., \quad j = 1, 2, ..., n.$$

On the other hand, the condition (3.28) can be written as

$$[h_j(\bar{k}) - c_j][1 - P_j(\bar{k})] = h_j(\bar{k}) P_j(\bar{k})[1 - P_j(k_1, ..., k_{j-1}, k_j + 1, k_{j+1}, ..., k_n)], \tag{3.35}$$
$$k_j = 0, 1, ..., \quad j = 1, 2, ..., n.$$

Substituting $P_j(\bar{k})$ and $P_j(k_1, ..., k_{j-1}, k_j + 1, k_{j+1}, ..., k_n)$ from (3.33) into (3.35) we obtain

$$h_j(\bar{k}) = h_j(k_1, ..., k_{j-1}, k_j - 1, k_{j+1}, ..., k_n) + c_j - \frac{1}{d_j}, \tag{3.36}$$
$$k_j = 0, 1, ..., \quad j = 1, 2, ..., n.$$

Denote $\alpha_j = c_j - 1/d_j$ and

$$\bar{k}^{(j)-1} = (k_1, ..., k_{j-1}, -1, k_{j+1}, ..., k_n).$$

Then by (3.36), iterating, we get

$$h_j(\bar{k}) = h_j(\bar{k}^{(j)-1}) + \alpha_j(k_j + 1) = h_j(\bar{k}^{(j)}) + \alpha_j k_j, \tag{3.37}$$
$$k_j = 0, 1, ..., \quad j = 1, 2, ..., n.$$

Observe that

$$\begin{aligned} c_j d_j &= E[V_j(1 - U_j)]\, E[V_j(1 - U_j)]^{-1} = \\ &= \left[EV_j EV_j^{-1}\right]\left[E(1 - U_j)E(1 - U_j)^{-1}\right] > 1, \quad j = 1, 2, ..., n. \end{aligned}$$

Hence $\alpha_j = \frac{c_j d_j - 1}{d_j} > 0$, $j = 1, 2, ..., n$. Let $p_j = \frac{1}{\alpha_j} EV_j = \frac{1}{\alpha_j} h_j(\bar{0}) = \frac{1}{\alpha_j} h_j(\bar{0}^{(j)-1}) + 1 > 1$. Inserting $k_i = 0$ for $i \neq j$ into (3.37) and by the definition of h_j we obtain

$$E\left(V_j^{k_j}\right) = \alpha_j^{k_j} \frac{\Gamma(k_j + p_j)}{\Gamma(p_j)}, \quad k_j = 1, 2, ..., \quad j = 1, 2, ..., n.$$

Hence, by the uniqueness of the moments sequence for the gamma distribution, we get $V_j \sim \gamma_{\alpha_j, p_j} \quad \forall j = 1, 2, ..., n$. Denote

$$m(\bar{k}) = E\left(\prod_{j=1}^{n} V_j^{k_j}\right), \quad k_j = 0, 1, ..., \quad j = 1, 2, ..., n.$$

From (3.37) it follows that

$$m(k_1, ..., k_{j-1}, k_j + 1, k_{j+1}, ..., k_n) = m(\bar{k})\left[\alpha_j k_j + h_j(\bar{k}^{(j)})\right], \qquad (3.38)$$
$$k_j = 0, 1, ..., \quad j = 1, 2, ..., n.$$

Replacing k_j by $k_j - 1$ in (3.38), and then iterating this operation, gives

$$m(\bar{k}) = m(\bar{k}^{(j)}) \prod_{l_j=0}^{k_j-1}\left[h_j(\bar{k}^{(j)}) + \alpha_j l_j\right], \quad k_j = 0, 1, ..., \quad j = 1, 2, ..., n. \qquad (3.39)$$

(we adopt the convention that $\prod_{l=0}^{-1} \equiv 1$). Thus, repeating the argumentation from the proof of the preceding theorem we get that the vectors $\bar{D}_1 = (V_l)_{l \in A_1}, ..., \bar{D}_r = (V_l)_{l \in A_r}$ are independent and have linearly dependent components which are gamma distributed, i.e. $\forall i$ $\exists m_i \in A_i$ such that $V_j = \frac{\alpha_j}{\alpha_{m_i}} V_{m_i} \quad \forall j \in A_i$ and $V_{m_i} \sim \gamma_{\alpha_{m_i}, p_{m_i}}$. Thus, from the definition of h_j and (3.37) it follows that

$$h_j(\bar{k}) = \frac{\alpha_j}{\alpha_{m_i}} \frac{E\left(V_{m_i} \prod_{l \in A_i} V_{m_i}^{k_l}\right)}{E\left(\prod_{l \in A_i} V_{m_i}^{k_l}\right)} = \alpha_j\left(\sum_{l \in A_i} k_l + p_{m_i}\right), \quad \forall j \in A_i, \quad k_l = 0, 1, ...$$

Hence by (3.34) we get

$$g_j(\bar{k}) = \frac{h_j(\bar{k}) - c_j}{h_j(\bar{k})} = \frac{\alpha_j\left(\sum_{l \in A_i} k_l + p_{m_i}\right) - c_j}{\alpha_j\left(\sum_{l \in A_i} k_l + p_{m_i}\right)}, \quad \forall j \in A_i, \quad k_l = 0, 1, ...,$$

which again as in the preceding proof leads us to the conclusion that the vectors $\bar{E}_1 = (U_l)_{l \in A_1}, ..., \bar{E}_r = (U_l)_{l \in A_r}$ are independent and have beta distributed components $U_j \sim \beta_{q_{m_i}, t_{m_i}} \quad \forall j \in A_i$ such that $\forall i \quad U_j = U_{m_i} \quad \forall j \in A_i$, where

$$q_{m_i} = p_{m_i} - \frac{c_j}{\alpha_j} = \frac{1}{\alpha_j}(EV_j - c_j) = \frac{1}{\alpha_j} E(V_j U_j) = \frac{1}{\alpha_j} EX_j > 0,$$
$$t_{m_i} = p_{m_i} - q_{m_i} = \frac{c_j}{\alpha_j} = 1 + \frac{1}{d_j \alpha_j} > 1, \quad \forall j \in A_i, \quad i = 1, ..., r.$$

Hence $\bar{X}$ and $\bar{Y}$ have distributions $MG^*(\bar{A}, \bar{\alpha}, \bar{q})$ and $MG^*(\bar{A}, \bar{\alpha}, \bar{t})$, respectively, where $\bar{A} = (A_1, ..., A_r)$, $\bar{\alpha} = (\alpha_1, ..., \alpha_n)$, $\bar{q} = (q_{m_1}, ..., q_{m_r})$, $\bar{t} = (t_{m_1}, ..., t_{m_r})$. □

And, again, we can exchange the roles of $\bar{X}$ and $\bar{Y}$ in the previous theorem.

Corollary 3.2. If we replace the conditions (3.27) in Theorem 2 by

$$E(\bar{X}|\bar{Y}) = \bar{c}, \quad E(\bar{X}^{-1}|\bar{Y}) = \bar{d},$$

then $\bar{X}$ and $\bar{Y}$ have distributions $MG^*(\bar{A}, \bar{\alpha}, \bar{t})$ and $MG^*(\bar{A}, \bar{\alpha}, \bar{q})$, respectively, with α_j, q_i, t_i, $j \in A_i$, $i = 1, ..., r$, defined as in Theorem 2.

The last theorem is a multivariate analogue of Theorem 3 in Bobecka and Wesołowski (2002b), i.e. it is the multivariate version of the dual result to the characterization obtained in Li, Huang and Huang (1994).

Theorem 3. *Let $\bar{U}$ and $\bar{V}$ be independent random vectors.*
Assume that

$$E(\bar{Y}^{-1}|\bar{X}) = \bar{c}, \qquad E(\bar{Y}^{-2}|\bar{X}) = \bar{d}, \tag{3.40}$$

for some real vectors $\bar{c}$, $\bar{d}$. Then $d_j > c_j^2$, $j = 1, 2, ..., n$, and $\bar{X}$, $\bar{Y}$ have distributions $MG^*(\bar{A}, \bar{\alpha}, \bar{q})$, $MG^*(\bar{A}, \bar{\alpha}, \bar{t})$, respectively, for some $\bar{A}$, $\bar{\alpha}$, $\bar{q}$ and $\bar{t}$, where

$$\alpha_j = \frac{d_j - c_j^2}{c_j d_j} > 0, \quad t_i = 1 + \frac{1}{\alpha_j c_j} > 2, \quad j \in A_i, \quad i = 1, ..., r.$$

Proof. Conditions (3.40) are equivalent to

$$E\left(\frac{1}{V_j(1-U_j)}|V_1U_1, ..., V_nU_n\right) = c_j, \tag{3.41}$$

$$E\left(\frac{1}{V_j^2(1-U_j)^2}|V_1U_1, ..., V_nU_n\right) = d_j, \quad j = 1, 2, ..., n. \tag{3.42}$$

Since $\bar{U}$ has values in $(0,1)^n$ we obtain $E(U_1^{k_1}...U_n^{k_n}) < \infty$ for $k_j = 1, 2, \ldots$ To show that also $E(V_1^{k_1}...V_n^{k_n}) < \infty$ for $k_j = -2, -1, 0, 1, ...$, it suffices to prove that $\forall j = 1, ..., n$ all the moments of V_j are finite. Fix $j = 1$ and apply induction with respect to $k_1 = k$. The regression conditions imply $E(1-U_1)^{-2} < \infty$ and $EV_1^{-2} < \infty$. Assume that $E(V_1^k) < \infty$ for some k. By (3.41) we have

$$E\left(V_1^k \frac{U_1^{k+1}}{(1-U_1)}|V_1U_1, ..., V_nU_n\right) = V_1^{k+1}U_1^{k+1}c_1,$$

and hence

$$\infty > E(V_1^k)E\left(\frac{1}{1-U_k}\right) > E(V_1^k)E\left(\frac{U_1^{k+1}}{(1-U_1)}\right) = E(V_1^{k+1})E(U_1^{k+1})c_1.$$

Thus $E(V_1^{k+1}) < \infty$ and by the induction argument we conclude that all the moments of V_1 are finite.

Note that the condition (3.41) is as the one in Theorem 2. Hence

$$P_j(\bar{k}) = 1 - \frac{1}{c_j h_j(k_1, ..., k_{j-1}, k_j - 1, k_{j+1}, ..., k_n)}, \quad k_j = 0, 1, ..., \quad j = 1, 2, ..., n, \tag{3.43}$$

where

$$P_j(\bar{k}) = \frac{G_j(k_1, ..., k_{j-1}, k_j + 1, k_{j+1}, ..., k_n)}{G_j(\bar{k})}, \tag{3.44}$$

$$G_j(\bar{k}) = \sum_{i=k_j}^{\infty} E(U_j^i \prod_{l \neq j} U_l^{k_l}),$$

$$h_j(k_1, ..., k_{j-1}, k_j - 2, k_{j+1}, ..., k_n) = \frac{E(V_j^{k_j - 1} \prod_{l \neq j} V_l^{k_l})}{E(V_j^{k_j - 2} \prod_{l \neq j} V_l^{k_l})},$$

$$k_j = 0, 1, ..., \quad j, l = 1, 2, ..., n.$$

By (3.42) and independence of $\bar{U}$ and $\bar{V}$ we obtain

$$E\left(V_j^{k_j - 2} \prod_{l \neq j} V_l^{k_l}\right) E\left(\frac{U_j^{k_j}}{(1 - U_j)^2} \prod_{l \neq j} U_l^{k_l}\right) = d_j E\left(\prod_{l=1}^{n} (V_l U_l)^{k_l}\right),$$
$$k_j = 0, 1, ..., \quad j = 1, 2, ..., n.$$

which can be rewritten as

$$E\left(\frac{U_j^{k_j}}{(1 - U_j)^2} \prod_{l \neq j} U_l^{k_l}\right) = d_j \left[G_j(\bar{k}) - G_j(k_1, ..., k_{j-1}, k_j + 1, k_{j+1}, ..., k_n)\right] \cdot \tag{3.45}$$
$$\cdot h_j(k_1, ..., k_{j-1}, k_j - 1, k_{j+1}, ..., k_n) h_j(k_1, ..., k_{j-1}, k_j - 2, k_{j+1}, ..., k_n),$$
$$k_j = 0, 1, ..., \quad j = 1, 2, ..., n.$$

Observe that ($'$ denotes differentiation with respect to U_j)

$$E\left(\frac{U_j^{k_j}}{(1-U_j)^2}\prod_{l\neq j}U_l^{k_l}\right) = E\left[U_j^{k_j}\left(\frac{1}{1-U_j}\right)'\prod_{l\neq j}U_l^{k_l}\right] =$$

$$= E\left[U_j^{k_j}\left(\sum_{i=0}^{\infty}U_j^i\right)'\prod_{l\neq j}U_l^{k_l}\right] = \sum_{i=k_j}^{\infty}(i-k_j+1)E(U_j^i\prod_{l\neq j}U_l^{k_l}) =$$

$$= \sum_{i=k_j}^{\infty}G_j(k_1,...,k_{j-1},i,k_{j+1},...,k_n),$$

$$k_j = 0,1,..., \quad j=1,2,...,n.$$

Thus (3.45) takes the form

$$\sum_{i=k_j}^{\infty}G_j(k_1,...,k_{j-1},i,k_{j+1},...,k_n) = d_j\left[G_j(\bar{k}) - G_j(k_1,...,k_{j-1},k_j+1,k_{j+1},...,k_n)\right]\cdot$$

$$\cdot h_j(k_1,...,k_{j-1},k_j-1,k_{j+1},...,k_n)h_j(k_1,...,k_{j-1},k_j-2,k_{j+1},...,k_n), \tag{3.46}$$

$$k_j = 0,1,..., \quad j=1,2,...,n.$$

Writing (3.46) for k_j+1 and then subtracting the obtained equation from (3.46) gives

$$G_j(\bar{k}) = d_j\left[G_j(\bar{k}) - G_j(k_1,...,k_{j-1},k_j+1,k_{j+1},...,k_n)\right]\cdot$$

$$\cdot h_j(k_1,...,k_{j-1},k_j-1,k_{j+1},...,k_n)h_j(k_1,...,k_{j-1},k_j-2,k_{j+1},...,k_n)-$$

$$-d_j\left[G_j(k_1,...,k_{j-1},k_j+1,k_{j+1},...,k_n) - G_j(k_1,...,k_{j-1},k_j+2,k_{j+1},...,k_n)\right]\cdot$$

$$\cdot h_j(\bar{k})h_j(k_1,...,k_{j-1},k_j-1,k_{j+1},...,k_n),$$

$$k_j = 0,1,..., \quad j=1,2,...,n,$$

which can be written in the form

$$d_jh_j(k_1,...,k_{j-1},k_j-1,k_{j+1},...,k_n)\{\left[1-P_j(\bar{k})\right]h_j(k_1,...,k_{j-1},k_j-2,k_{j+1},...,k_n)-$$

$$-P_j(\bar{k})\left[1-P_j(k_1,...,k_{j-1},k_j+1,k_{j+1},...,k_n)\right]h_j(k_1,...,k_{j-1},k_j-1,k_{j+1},...,k_n)\} = 1,$$

$$k_j = 0,1,..., \quad j=1,2,...,n. \tag{3.47}$$

Substituting $P_j(\bar{k})$ and $P_j(k_1,...,k_{j-1},k_j+1,k_{j+1},...,k_n)$ from (3.43) into (3.47) we get

$$h_j(k_1,...,k_{j-1},k_j-1,k_{j+1},...,k_n) = h_j(k_1,...,k_{j-1},k_j-2,k_{j+1},...,k_n)+\alpha_j, \tag{3.48}$$

$$k_j = 0,1,..., \quad j=1,2,...,n,$$

where

$$\alpha_j = \frac{1}{c_j} - \frac{c_j}{d_j} = \frac{d_j - c_j^2}{c_jd_j} = \frac{Var\left(Y_j^{-1}\right)}{E\left(Y_j^{-1}\right)E\left(Y_j^{-2}\right)} > 0, \quad j=1,2,...,n.$$

Denote

$$\bar{k}^{(j)-2} = (k_1, ..., k_{j-1}, -2, k_{j+1}, ..., k_n).$$

Then by (3.48) we obtain, iterating,

$$h_j(k_1, ..., k_{j-1}, k_j - 1, k_{j+1}, ..., k_n) = h_j(\bar{k}^{(j)-2}) + \alpha_j\,(k_j + 1) = \qquad (3.49)$$
$$= h_j(\bar{k}^{(j)}) + \alpha_j\,(k_j - 1)\,, \qquad k_j = 0, 1, ..., \quad j = 1, 2, ..., n.$$

Let $p_j = \frac{1}{\alpha_j} EV_j = \frac{1}{\alpha_j} h_j(\bar{0}) = \frac{1}{\alpha_j} h_j(\bar{0}^{(j)-2}) + 2 > 2$. Insert $k_i = 0$ for $i \neq j$ into (3.49). Then, as in previous proofs, we have $V_j \sim \gamma_{\alpha_j, p_j}$ $\forall j = 1, ..., n$. Denote

$$m(\bar{k}) = E\left(\prod_{j=1}^{n} V_j^{k_j}\right), \quad k_j = 0, 1, ..., \quad j = 1, 2, ..., n,$$

From (3.49) it follows that

$$m(\bar{k}) = \left[\alpha_j\,(k_j - 1) + h_j(\bar{k}^{(j)})\right] m\,(k_1, ..., k_{j-1}, k_j - 1, k_{j+1}, ..., k_n)\,, \qquad (3.50)$$
$$k_j = 0, 1, ..., \quad j = 1, 2, ..., n,$$

and, iterating, gives

$$m(\bar{k}) = m(\bar{k}^{(j)}) \prod_{l_j=0}^{k_j-1} \left[h_j(\bar{k}^{(j)}) + \alpha_j l_j\right], \quad k_j = 0, 1, ..., \quad j = 1, 2, ..., n.$$

Thus, repeating the argumentation from the proof of Theorem 1, we get that the vectors $\bar{D}_1 = (V_l)_{l \in A_1}, ..., \bar{D}_r = (V_l)_{l \in A_r}$ are independent and have linearly dependent gamma distributed components, i.e. $\forall i\ \exists m_i \in A_i$ such that $V_j = \frac{\alpha_j}{\alpha_{m_i}} V_{m_i}$ $\forall j \in A_i$ and $V_{m_i} \sim \gamma_{\alpha_{m_i}, p_{m_i}}$. Thus, by the definition of h_j and (3.49), we get

$$h_j(k_1, ..., k_{j-1}, k_j - 1, k_{j+1}, ..., k_n) = \frac{\alpha_j}{\alpha_{m_i}} \frac{E\left(\prod_{l \in A_i} V_{m_i}^{k_l}\right)}{E\left(V_{m_i}^{-1} \prod_{l \in A_i} V_{m_i}^{k_l}\right)} =$$
$$= \alpha_j \left(\sum_{l \in A_i} k_l + p_{m_i} - 1\right), \quad \forall j \in A_i, \quad k_l = 0, 1, ...$$

Hence, by (3.43),

$$P_j(\bar{k}) = \frac{c_j h_j(k_1, ..., k_{j-1}, k_j - 1, k_{j+1}, ..., k_n) - 1}{c_j h_j(k_1, ..., k_{j-1}, k_j - 1, k_{j+1}, ..., k_n)} = \tag{3.51}$$

$$= \frac{c_j \alpha_j \left(\sum\limits_{l \in A_i} k_l + p_{m_i} - 1 \right) - 1}{c_j \alpha_j \left(\sum\limits_{l \in A_i} k_l + p_{m_i} - 1 \right)} =$$

$$= \frac{\sum\limits_{l \in A_i} k_l + q_j}{\sum\limits_{l \in A_i} k_l + q_j + t_j - 1}, \qquad \forall j \in A_i, \quad k_l = 0, 1, ...$$

where

$$q_j = p_{m_i} - 1 - \frac{1}{c_j \alpha_j} = \frac{1}{\alpha_j} \left[h_j \left(\bar{0}^{(j)-1} \right) - \frac{1}{c_j} \right] = \frac{1}{\alpha_j} \frac{1}{E\left(V_j^{-1}\right)} \left[\frac{E\left(1 - U_j\right)^{-1} - 1}{E\left(1 - U_j\right)^{-1}} \right] > 0,$$

$$t_j = p_{m_i} - q_j = 1 + \frac{1}{c_j \alpha_j} = 2 + \frac{c_j^2}{d_j - c_j^2} > 2, \qquad j = 1, 2, ..., n.$$

On the other hand, from (3.44) it follows that

$$P_j(\bar{k}) = \frac{E\left(\frac{U_j^{k_j+1}}{1 - U_j} \prod\limits_{l \neq j} U_l^{k_l} \right)}{E\left(\frac{U_j^{k_j}}{1 - U_j} \prod\limits_{l \neq j} U_l^{k_l} \right)}, \quad k_j = 0, 1, ..., \quad j, l = 1, 2, ..., n.$$

Therefore (3.51) takes the form

$$E\left(\frac{U_j^{k_j+1}}{1 - U_j} \prod_{l \neq j} U_l^{k_l} \right) = \frac{\sum\limits_{l \in A_i} k_l + q_j}{\sum\limits_{l \in A_i} k_l + q_j + t_j - 1} E\left(\frac{U_j^{k_j}}{1 - U_j} \prod_{l \neq j} U_l^{k_l} \right), \quad \forall j \in A_i, \quad k_l = 0, 1, ... \tag{3.52}$$

Moreover, observe that

$$E\left(\frac{U_j^{k_j+1}}{1 - U_j} \prod_{l \neq j} U_l^{k_l} \right) = E\left(\frac{U_j^{k_j}}{1 - U_j} \prod_{l \neq j} U_l^{k_l} \right) - E\left(U_j^{k_j} \prod_{l \neq j} U_l^{k_l} \right), \quad \forall j \in A_i, \quad k_l = 0, 1, ... \tag{3.53}$$

By(3.52) and (3.53) we get

$$E\left(U_j^{k_j} \prod_{l \neq j} U_l^{k_l} \right) = \frac{t_j - 1}{\sum\limits_{l \in A_i} k_l + q_j + t_j - 1} E\left(\frac{U_j^{k_j}}{1 - U_j} \prod_{l \neq j} U_l^{k_l} \right), \quad \forall j \in A_i, \quad k_j = 0, 1, ... \tag{3.54}$$

and

$$E\left(U_j^{k_j-1}\prod_{l\neq j}U_l^{k_l}\right)=\frac{t_j-1}{\sum\limits_{l\in A_i}k_l+q_j-1}E\left(\frac{U_j^{k_j}}{1-U_j}\prod_{l\neq j}U_l^{k_l}\right),\quad \forall j\in A_i,\quad k_j=1,2,... \tag{3.55}$$

Denote

$$M(\bar{k})=E\left(\prod_{j=1}^{n}U_j^{k_j}\right),\quad k_j=0,1,...,\quad j=1,2,...,n.$$

Combining (3.54) with (3.55) we obtain

$$M(\bar{k})=\frac{\sum\limits_{l\in A_i}k_l+q_j-1}{\sum\limits_{l\in A_i}k_l+q_j+t_j-1}E\left(U_j^{k_j-1}\prod_{l\neq j}U_l^{k_l}\right)\quad \forall j\in A_i,$$
$$k_j=1,2,...,\quad k_l=0,1,...,\quad l=1,2,...,n,\quad l\neq j.$$

And again the argumentation analogous to the one from the proof of Theorem 1 leads us to the conclusion that the vectors $\bar{E}_1=(U_l)_{l\in A_1},...,\bar{E}_r=(U_l)_{l\in A_r}$ are independent and have beta distributed components $U_j\sim\beta_{q_{m_i},t_{m_i}}$ $\forall j\in A_i$ such that $\forall i$ $U_j=U_{m_i}$ $\forall j\in A_i$. Hence $\bar{X}$ and $\bar{Y}$ have distributions $MG^*(\bar{A},\bar{\alpha},\bar{q})$ and $MG^*(\bar{A},\bar{\alpha},\bar{t})$, respectively, where $\bar{A}=(A_1,...,A_r)$, $\bar{\alpha}=(\alpha_1,...,\alpha_n)$, $\bar{q}=(q_{m_1},...,q_{m_r})$, $\bar{t}=(t_{m_1},...,t_{m_r})$. □

As in the previous cases the analogous result with $\bar{X}$ and $\bar{Y}$ interchanged follows easily.

Corollary 3.3. If we replace the conditions (3.27) in Theorem 3 by

$$E(\bar{X}^{-1}|\bar{Y})=\bar{c},\quad E(\bar{X}^{-2}|\bar{Y})=\bar{d},$$

then $\bar{X}$ and $\bar{Y}$ have distributions $MG^*(\bar{A},\bar{\alpha},\bar{t})$ and $MG^*(\bar{A},\bar{\alpha},\bar{q})$, respectively, with α_j, q_i, t_i, $j\in A_i$, $i=1,...,r$, defined as in Theorem 3.

Acknowledgement

The authors are grateful to W.-J. Huang for sending them preprints of papers Chou and Huang (2002a,b).

References

[1] Bobecka, K. (2002) Regression versions of Lukacs type characterizations for the bivariate gamma distribution. J. Appl. Statist. Sci. 11, 213-233.

[2] Bobecka, K., Wesołowski, J. (2001) Multivariate Lukacs theorem. Preprint, 1-16.

[3] Bobecka, K., Wesołowski, J. (2002a) The Lukacs-Olkin-Rubin theorem without invariance of the "quotient". Studia Math. 152, 147-160.

[4] Bobecka, K., Wesołowski, J. (2002b) Three dual regression schemes for the Lukacs theorem. Metrika 56, 43-54.

[5] Bolger, E. M., Harkness, W.L. (1965) A characterization of some distributions by conditional moments. Ann. Math. Statist. 36, 703-705.

[6] Casalis, M., Letac, G. (1996) The Lukacs-Olkin-Rubin characterization of Wishart distributions on symmetric cones. Ann. Statist. 24, 763-786.

[7] Chou, C.-W., Huang, W.-J. (2002a) Characterizations of the gamma distribution via conditional moments. Sankhyā A - to appear.

[8] Chou, C.-W., Huang, W.-J. (2002b) A note on characterizations of the bivariate gamma distribution. Preprint, 1-21.

[9] Hall, W.J., Simons, G. (1969) On characterizations of the gamma distributions. Sankhyā, A 31, 385-390.

[10] Huang, W.-J., Su, J.-C. (1997) On a study of renewal process connected with certain conditional moments. Sankhyā A 59, 28-41.

[11] Letac, G., Massam, H. (1998) Quadratic and inverse regressions for Wishart distributions. Ann. Statist. 26, 573-595.

[12] Li, S.-H., Huang, W.-J., Huang, M.-N.L. (1994) Characterizations of the Poisson process as a renewal process via two conditional moments. Ann. Inst. Statist. Math. 46, 351-360.

[13] Lukacs, E. (1955) A characterization of the gamma distribution. Ann. Math. Statist. 26, 319-324.

[14] Olkin, I., Rubin, H. (1962) A characterization of the Wishart distribution. Ann. Math. Statist. 33, 1272-1280.

[15] Wang, Y. (1981) Extensions of Lukacs' characterization of the gamma distribution. Analytic Methods in Probability Theory, Lecture Notes in Math. 861, 166-177. Springer, New York.

[16] Wesołowski, J. (1990) A constant regression characterization of the gamma law. Adv. Appl. Probab. 22, 488-490.

[17] Wesołowski, J. (1993) Stochastic processes with linear conditional expectation and quadratic conditional variance. Probab. Math. Statist. 14, 33-44.